Chapter 5

Entire Functions

5.1 Jensen's formula

Theorem. *Let Ω be an open set that contains $\overline{D_R}$ for some disc $D_R = \{z : |z| < R\}$ and suppose f is holomorphic in Ω, $f(0) \neq 0$ and $f(z) \neq 0$ for z on the circle C_R. If $z_1, z_2, ..., z_N$ are the zeros of f inside D_R (counted with multiplicities), then*

$$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\theta.$$

Proof. First, we observe the simple fact that if f_1 and f_2 are two functions satisfying the hypothesis and the conclusion of the theorem, then so is $f_1 f_2$.

Step 1. The function

$$g(z) = \frac{f(z)}{(z - z_1)\cdots(z - z_N)}$$

is bounded near each z_k and hence each z_k is removable. Therefore, we can write

$$f(z) = (z - z_1)\cdots(z - z_N)g(z)$$

where g is holomorphic in Ω and nowhere vanishing on $\overline{D_R}$.

Step 2 Assume g is holomorphic in Ω and nowhere vanishing on $\overline{D_R}$. Then is a slightly largge disc, we have $g(z) = e^{h(z)}$ for some holomorphic function h on that disc. Now we observe

$$|g(z)| = |e^{h(z)}| = e^{\mathrm{Re}(h(z))} \quad \text{so that} \quad \log|g(z)| = \mathrm{Re}(h(z)).$$

The desired formula for g follows by the mean value property for harmonic functions.

Step 3. Now assume $f(z) = z - w$, where $w \in D_R$. We must show that

$$\log|w| = \log\frac{|w|}{R} + \frac{1}{2\pi}\int_0^{2\pi} \log|Re^{i\theta} - w| d\theta.$$

Since $\log\frac{|w|}{R} = \log|w| - \log R$ and $\log|Re^{i\theta} - w| = \log R + \log|e^{i\theta} - \frac{w}{R}|$, it suffices to show that

$$\int_0^{2\pi} \log|1 - ae^{i\theta}| = 0, \quad \text{whenever } |a| < 1.$$

To prove this, set $F(z) = 1 - az$. Then $F(z) = e^{G(z)}$ for some holomorphic functions on some D_r, $r < 1$. Therefore

$$\log|F(z)| = \log|e^{G(z)}| = \mathrm{Re}(G(z)).$$

Since $F(0) = 1$, we have $\log |F(0)| = 0$ and an application of mean value property to the harmonic function $\log |F(z)|$ concludes the proof of the theorem. □

We denote by $n(r)$ the number of zeros of f inside D_r.

Lemma. If $z_1, ..., z_N$ are the zeros of f inside the disc D_R, then

$$\int_0^R n(r)\frac{dr}{r} = \sum_{k=1}^{N} \log\left|\frac{R}{z_k}\right|.$$

Proof. For $1 \le k \le N$, define the function

$$\eta_k(r) = \begin{cases} 1, & \text{if } r > |z_k| \\ 0, & \text{if } r \le |z_k| \end{cases}$$

Then $\sum_{k=1}^{N} \eta_k(r) = n(r)$ and so

$$\sum_{k=1}^{N} \log\left|\frac{R}{z_k}\right| = \sum_{k=1}^{N} \int_{|z_k|}^{R} \frac{dr}{r} = \sum_{k=1}^{N} \int_0^R \eta_k(r)\frac{dr}{r} = \int_0^R \sum_{k=1}^{N} \eta_k(r)\frac{dr}{r} = \int_0^R n(r)\frac{dr}{r}.$$

□

From Jensen's equality and the previous lemma, we have the formula

$$\int_0^R n(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|.$$

5.2 Functions of finite order

We say that an entire function f has an order of growth $\le \rho$ if there exist constants A and $B > 0$ such that

$$|f(z)| \le Ae^{B|z|^\rho} \quad \text{for all} \quad z \in \mathbb{C}.$$

We define the oreder of growth of f as

$$\rho_f = \inf \rho$$

where the infimum is over all $\rho > 0$ wuch that f has an order of growth $\le \rho$.

Theorem. *Let f be an entire function that has an order of growth $\le \rho$.*

(a) $n(r) \le Cr^\rho$ for some $C > 0$ and all sufficiently large r.

(b) If $z_1, z_2, \ldots$ are zeros of f, with $z_k \neq 0$, then, for all $s > \rho$, we have

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty.$$

Proof. It suffices to prove the estimate for $n(r)$ when $f(0) \neq 0$ (otherwise consider $F(z) = f(z)/z^\ell$, $\ell =$ multiplicity of the zero of f at 0). For $R = 2r$, we have

$$\int_r^{2r} n(x)\frac{dx}{x} \leq \int_0^{2r} n(x)\frac{dx}{x} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|.$$

Note that

$$\int_r^{2r} n(x)\frac{dx}{x} \geq n(r)\int_r^{2r}\frac{dx}{x} = n(r)\log 2$$

and

$$\begin{aligned}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta \leq &\int_0^{2\pi} \log(Ae^{BR^\rho})d\theta \\ \leq &C' r^\rho\end{aligned}$$

for sime constant C' and all sufficiently large r.

For (b), we have, if $s > \rho$,

$$\begin{aligned}\sum_{|z_k|\geq 1} |z_k|^{-s} &= \sum_{\delta=0}^{\infty}\left(\sum_{2^\delta \leq |z_k| < 2^{\delta+1}} |z_k|^{-s}\right) \\ &\leq \sum_{\delta=0}^{\infty} 2^{-\delta s} n(2^{\delta+1}) \\ &\leq C\sum_{\delta=0}^{\infty} 2^{-\delta s}\, 2^{(\delta+1)\rho} < \infty.\end{aligned}$$

□

5.3 Infinite products

Proposition. *If $\sum |a_n| < \infty$, then the product $\prod_{n=1}^{\infty}(1+a_n)$ converges. Moreover, the product converges to 0 if and only if one of its factor is 0.*

Proof. Without loss of generality, we may assume $|a_n| \leq \frac{1}{2}$ for all n and $\sum |a_n| < \infty$. We then define the branch of the logarithm so that $1+z = e^{\log(1+z)}$ whenever $|z| < 1$. Hence we have

$$\prod_{n=1}^{N}(1+a_n) = e^{B_N}$$

where $B_N = \sum_{n=1}^{N} \log(1+a_n) = \sum_{n=1}^{N} b_n$ with $b_n = \log(1+a_n)$. By the power series expansion we get $|\log(1+z)| \leq 2|z|$, if $|z| < \frac{1}{2}$. Hence $|b_n| \leq 2a_n$, so B_N converges to some $B \in \mathbb{C}$, as $N \to \infty$. We conclude that e^{B_N} converges to e^B. Observe also that if $1 + a_n \neq 0$ for all n, then the product converges to a non-zero limit e^B. □

Proposition. *Suppose $\{F_n\}$ is a sequence of holomorphic functions on the open set Ω. If there exist constants $c_n > 0$ such that*

$$\sum_n c_n < \infty \quad \textit{and} \quad |F_n(z) - 1| \leq c_n \quad \textit{for all } n \textit{ and } z \in \Omega,$$

then

(a) The product $\prod_{n=1}^{\infty} F_n(z)$ converges uniformly in Ω to a holomorphic function $F(z)$.

(b) If $F_n(z)$ does not vanish for any n, then $\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}$.

Proof. As before, we assume $\sum c_n < \infty$ and $|c_n| \leq \frac{1}{2}$, $\forall n$. Set $H_n(z) = F_n(z) - 1$. Then $|H_n(z)| \leq c_n \leq \frac{1}{2}$, $\forall n$, $z \in \Omega$ and

$$\begin{aligned}\prod_{n=1}^{\infty} F_n(z) &= \prod_{n=1}^{\infty}(1 + H_n(z)) = \lim_{N\to\infty} \prod_{n=1}^{N}(1 + H_n(z)) \\ &= \lim_{N\to\infty} \prod_{n=1}^{N} e^{\log(1+H_n(z))} = \lim_{N\to\infty} e^{\sum_{n=1}^{N} b_n(z)}.\end{aligned}$$

where $b_n(z) = \log(1 + H_n(z))$. Note that

$$|b_n(z)| = |\log(1 + H_n(z))| \leq \frac{|H_n(z)|}{2} \leq \frac{C_n}{2}, \quad \forall n \text{ and } z \in \Omega.$$

Therefore $\sum_{n=1}^{\infty} b_n(z)$ converge uniformly to some holomorphic function B on Ω. Hence $\prod_{n=1}^{\infty} F_n(z)$ converges uniformly to the holomorphic function $F = e^B$ on Ω.

To prove the second statement, we set

$$G_n(z) = \prod_{n=1}^{N} F_n(z).$$

Now we have $G_N \to F$ uniformly on Ω as $N \to \infty$, annd so the sequence $G_N' \to F'$ converges uniformly on every compact subset of Ω. This implies in particular

$$\frac{G_N'(z)}{G_N(z)} \to \frac{F'}{F}(z) \quad \text{for all } z \in \Omega.$$

Therefore

$$\frac{F'(z)}{F(z)} = \lim_{N\to\infty} \frac{G_N'(z)}{G_N(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}, \quad \forall z \in \Omega.$$

□

5.4 Weierstrass infinite products

For each integer $k \geq 0$, the canonical factor of the degree k is given by the formula

$$E_0(z) = 1 - z \text{ and } E_k(z) = (1-z)e^{z+\frac{z^2}{z}+\cdots+\frac{z^k}{k}}, \text{ for } k \geq 1.$$

Lemma. *There exists a constant c independent of k so that $|1 - E_k(z)| \leq c|z|^{k+1}$, for all $|z| \leq \frac{1}{2}$.*

Proof. If $|z| < \frac{1}{2}$, with the logarithm defined in terms of power series, we have $1 - z = e^{\log(1-z)}$, and therefore

$$E_k(z) = e^{\log(1-z)+z+\cdots+\frac{z^k}{k}} = e^w$$

where $w = -\sum_{n=k+1}^{\infty} \frac{z^n}{n}$. Since $|z| \leq \frac{1}{2}$, we have

$$|w| \leq |z|^{k+1} \sum_{n=k+1}^{\infty} \frac{|z|^{n-k-1}}{n} \leq |z|^{k+1} \sum_{d=0}^{\infty} \frac{1}{2^d} = 2|z|^{k+1}.$$

In particular, we have $|w| \leq 1$ and this implies that

$$|1 - E_k(z)| = |1 - e^w| \leq e|w| \leq c|z|^{k+1}$$

with $c = 2e$. □

Theorem. *Given any sequence $\{a_n\}$ of complex numbers with $|a_n| \to \infty$ as $n \to \infty$, there exists an entire function f that vanish at all $z = a_n$ and nowhere else. Any other such function is of the form $f(z)e^{g(z)}$, where g is entire.*

Proof. *Uniqueness.* Assume f_1 and f_2 are such functions. Then f_1/f_2 is entire and vanishes nowhere, so there exists an entire function g so that $e^g = f_1/f_2$. Therefore, $f_1 = f_2 e^g$.

Suppose that we are given a zero of order m at $z = 0$, and that $a_1, a_2, \ldots$ are all non-zero. Then we define the Weierstrass product by

$$f(z) = z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right).$$

Fix any $R > 0$. For a_n with $|z_n > 2R|$, we have, on $|z| < R$,

$$|1 - E_n(\frac{z}{a_n})| \leq c \left|\frac{z}{a_n}\right|^{n+1} \leq c\frac{1}{2^{n+1}}.$$

Therefore $f(z)$ is holomorphic on the disc $|z| < R$ and has the desired proposition on this disc. Since R is arbitrary, hence f is the required function. $\square$

5.5 Hadamard's factorization theorem

Theorem. *Suppose f is entire and has growth order ρ_0. Let k be the integer so that $k \leq \rho_0 < k+1$. If $a_1, a_2, \ldots$ denote the (non-zero) zeros of f, then*

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right),$$

where P is a polynomial of degree $\leq k$, and m is the order of the zero of f at $z = 0$.

To prove this theorem, we need the following lemmas.

Lemma. *Suppose g is entire and $u = Re(g)$ satisfies*

$$u(z) \leq cr^s \text{ whenever } |z| = r$$

for a sequence of positive number r that tends to infinity. Then g is a polynomial of degree $\leq s$.

Proof. Write $g(z) = \sum_{n=0}^{\infty} a_n z^n$. Then we have, by Cauchy's integral formulas, that

$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})e^{-in\theta}d\theta = \begin{cases} a_n r^n, & \text{if } n \geq 0 \\ 0, & \text{if } n < 0 \end{cases}$$

which implies that

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{g(re^{i\theta})}e^{in\theta}d\theta = 0, \quad \text{if } n < 0.$$

Since $u = \frac{1}{2}(g + \overline{g})$, we have

$$a_n r^n = \frac{1}{\pi}\int_0^{2\pi} u(re^{i\theta})e^{-in\theta}d\theta, \quad \text{for } n > 0.$$

Also we have

$$2\text{Re}(a_0) = \frac{1}{\pi}\int_0^{2\pi} u(re^{i\theta})d\theta.$$

Now, when $n > 0$

$$a_n = \frac{1}{\pi r^n} \int_0^{2\pi} [u(re^{i\theta}) - Cr^s] e^{-in\theta} d\theta$$

and so

$$\begin{aligned} |a_n| &\leq \frac{1}{\pi r^n} \int_0^{2\pi} (Cr^s - u(re^{i\theta})) d\theta \\ &\leq 2Cr^{s-n} - 2\text{Re}(a_0) r^{-n}. \end{aligned}$$

Letting $r \uparrow \infty$ gives $a_n = 0$ for $n > s$. This completes the proof of the lemma. $\square$

Lemma. *The canonical products satisfy*

$$|E_k(z)| \geq e^{-c|z|^{k+1}} \quad \textit{if } |z| \leq \frac{1}{2}$$

and

$$|E_k(z)| \geq |1 - z| e^{-c'|z|^k} \quad \textit{if } |z| \geq \frac{1}{2}.$$

Proof. If $|z| \leq \frac{1}{2}$, we have

$$E_k(z) = e^w$$

where $w = -\sum_{n=k+1}^{\infty} \frac{z^n}{n}$. Since $|e^w| \geq e^{-|w|}$ and $|w| \leq c|z|^{k+1}$, the first part of the lemma follows. For $|z| > \frac{1}{2}$, there exists c' so that

$$|z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}| \leq c'|z|^k$$

and so

$$|E_k(z)| = |1 - z||e^w| \geq |1 - z| e^{-|w|} \geq |1 - z| e^{-c'|z|^k}.$$

$\square$

Lemma. *For any s with $\rho_0 < s < k + 1$, we have*

$$\left| \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-c|z|^s}$$

whenever $z \neq \mathbb{D}$, for $n = 1, 2, 3, ...$

Corollary. *There exists a sequence of radii, $r_1, r_2, ...$ with $r_m \to \infty$, such that*

$$\left| \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-c|z|^s} \quad \textit{for } |z| = r_m.$$

Proof. Since $\sum_n |a_n|^{-k-1} < \infty$, there exists an integer N so that

$$\sum_{n=N}^{\infty} |a_n|^{-k-1} < \frac{1}{10}.$$

Therefore, given any two consecutive large integers L and $L+1$, we can finda positive number r with $| \leq r \leq L+1$, such that the circle of radius r centered at the origin does not intersect the forbidden discs of the previous lemma. We can then apply the previous lemma with $|z| = r$ to conclude the proof of the corollary. □

Proof of Hadamard's theorem. Let $E(z) = z^m \prod_{n=1}^{\infty} E_k(z/a_n)$. Then repeating the argument as before gives that E is entire and has the same zeros of f. Therefore f/E is entire and nowhere vanishing. Hence $f/E = e^g$ for some entire function g. Note that

$$|e^g| = e^{\text{Re}(g(z))} = \left| \frac{f(z)}{E(z)} \right| \leq c' e^{c|z|^s},$$

for $|z| = r_m$. This proves that $\text{Re}(g(z)) \leq c|z|^s$ for $|z| = r_m$ and so g is a polynomial with degree $< k+1$. □