Segregated Nodal Domains of Two-Dimensional Multispecies Bose-Einstein Condensates

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Abstract

In this paper, we study the distribution of m segregated nodal domains of the m-mixture of Bose-Einstein condensates under positive and large repulsive scattering lengths. It is shown that components of positive bound states may repel each other and form segregated nodal domains as the repulsive scattering lengths go to infinity. Efficient numerical schemes are created to confirm our theoretical results and discover a new phenomenon called verticillate multiplying, i.e., the generation of multiple verticillate structures. In addition, our proposed Gauss-Seidel-type iteration method is very effective in that it converges linearly in 10 to 20 steps.

1 Introduction

In an ultracold dilute Bose gas, two different hyperfine spin states may repel each other and form segregated domains like the mixture of oil and water. Such a phenomenon is called phase separation of a binary mixture of Bose-Einstein condensates (BECs) and has been investigated extensively by experimental and theoretical physicists ([15], [22], [25]). Recently, Bose-Einstein condensation of the triplet states has been observed [24]. It is possible to observe multispecies Bose-Einstein condensates with more spin states. This motivates us to study phase separation of general m-mixture of BECs both mathematically and numerically. As the number m becomes larger and larger, due to phase separation, more and more segregated domains may occur. It is natural to ask how these segregated domains distribute. Are there any rules for the distribution of segregated domains? We will answer such a question by studying the distribution of nodal domains of the m-component ground state when the number m increases from two to thirty-three. The limit on m is merely due to the huge computational resources.

The coupled Gross-Pitaevskii equations ([15], [23]), i.e., the coupled nonlinear Schrödinger equations,

$$\begin{cases} -\boldsymbol{\iota}\frac{\partial}{\partial t}\Phi_j = \Delta\Phi_j - V_j(x)\Phi_j - \mu_j|\Phi_j|^2\Phi_j - \sum\limits_{i\neq j}\beta_{ij}|\Phi_i|^2\Phi_j \text{ for } \ x\in\Omega, t>0, \\ \Phi_j = \Phi_j(x,t)\in\mathbb{C}\,, \quad \boldsymbol{\iota}=\sqrt{-1}, \quad j = \ 1,\ldots,m, \\ \Phi_j(x,t)=0 \text{ for } \ x\in\partial\Omega, t>0, \end{cases} \tag{1.1}$$

can be used as a mathematical model for multispecies Bose-Einstein condensates in m different hyperfine spin states on the corresponding condensate wave functions Φ_j's. Here Ω is a bounded smooth domain in $\mathbb{R}^d, d = 2, 3$, and the nonnegative constants μ_j's and β_{ij}'s are the intraspecies and interspecies scattering lengths which represents the interactions between like and unlike particles, respectively. Hereafter, it is natural to assume that β_{ij}'s are symmetric i.e. $\beta_{ij} = \beta_{ji}$ for $i \neq j$. For simplicity, we may choose suitable scales for the Planck constant, atom mass and mean number of atoms in hyperfine states to make the system (1.1) consistent with the physical model. The functions $V_j, j = 1,\ldots,m$ represent the magnetic trapping potentials.

From(cf. [25]), we learned that there are two distinct types of spatial separation: (i) potential separation, caused by the external trapping potentials in much the same way that gravity can separate fluids of different specific weight. (ii) phase separation, which persists in the absence of external potentials. In the fluid analogy, phase separated condensates can be compared to a system of two immiscible fluids, such as oil and water. The main purpose of this paper is to study phase separation in the coupled nonlinear Schrödinger equations. Hence we may assume $V_j \equiv 0$ for $j = 1,\ldots,m$ in the rest of this paper.

As $m > 3$ and $V_j \equiv 0, j = 1,\ldots,m$, the coupled nonlinear Schrödinger equations of the system (1.1) are of physical relevance in the theory of multichannel bitparallel- wavelength optical fiber networks(cf. [27]) and photorefractive media in nonlinear optics(cf. [1]). Generically, the spatial dimension d can be one, two and three for different physical situations. However, until now, most results on the coupled nonlinear Schrödinger equations are of only one spatial dimension(cf. [9], [16], [17], [18], etc). Here we may provide some results in high spatial dimensions, especially in two spatial dimension for the coupled nonlinear Schrödinger equations.

To find solitary wave solutions of the system (1.1) , we set

$$\Phi_j = e^{-\iota \lambda_j t} u_j(x)\,, j = 1, \ldots, m\,.$$

Then we may transform the system (1.1) into a m-component system of semilinear elliptic equations given by

$$-\Delta u_j + \mu_j u_j^3 + \Lambda \sum_{i \neq j} \tilde{\beta}_{ij} u_i^2\, u_j = \lambda_j u_j \quad \text{in } \Omega\,, j = 1, \ldots, m\,, \tag{1.2}$$

which are time independent vector Gross-Pitaevskii Hartree-Fock equations (cf. [11], [12]) for the condensate wave functions u_j's, where $\beta_{ij} = \Lambda \tilde{\beta}_{ij}$, Λ is a parameter, and $\tilde{\beta}_{ij}$'s are positive constants. The standard conservation law of mass on the coupled nonlinear Schrödinger equations of the system (1.1) may give $\int_\Omega u_j^2 = m_j$ for $j = 1, \ldots, m$, where m_j's are constants. For simplicity, we may set $m_j = 1$ for $j = 1, \ldots, m$ and assume

$$\int_\Omega u_j^2 = 1\,, \quad j = 1, \ldots, m\,. \tag{1.3}$$

Moreover, by the boundary conditions of the system (1.1), we obtain the Dirichlet boundary conditions:

$$u_j|_{\partial\Omega} = 0\,, \quad j = 1, \ldots, m\,. \tag{1.4}$$

From [2] and [12], a large interspecies scattering length may set in spontaneous symmetry breaking inducing phase separation. Furthermore, due to Feshbach resonance, interspecies scattering lengths can be positive and large by adjusting the externally applied magnetic field [14]. Hence we may assume the parameter Λ as a large parameter. Actually, in a binary mixture of Bose-Einstein condensates i.e. $m = 2$, spontaneous symmetry breaking may occur when $\Lambda^2 \tilde{\beta}_{12}^2 > \mu_1 \mu_2$(cf. [2], [25] and [26]). To fulfill such a condition, we may assume intraspecies scattering lengths μ_j's are constants and the parameter Λ as a large parameter. However, when the parameter Λ is large but finite, it is easy to show that each component u_j of the solution $(u_1, \cdots, u_m)$ of the system (1.2) cannot be zero in a nonempty domain by the standard maximum principle of elliptic partial differential equations(cf. [13]). Hence the segregated nodal domains are not clear to figure out as the parameter Λ is large but finite. On the other hand, it is expected that repelling condensates would separate into single condensate regions if the repulsive interaction is sufficiently large(cf. [25]). Therefore we let the parameter Λ tend to infinity to find well separated nodal domains.

As the parameter Λ goes to infinity, some basic questions need to be asked as follows:

1. What are the governing equations of the limiting functions of the bound state solutions of the system (1.2)?

2. What the nodal domains of the limiting functions look like?

To answer these questions, we state the following theorem:

Theorem 1.1 *Assume Ω is a bounded smooth domain in $\mathbb{R}^2$. Let $(u_{1,\Lambda}, \ldots, u_{m,\Lambda})$ be a positive solution of the system (1.2) satisfying (1.3) and (1.4), where λ_j's are bounded quantities as $\Lambda \to \infty$. Then*

(i) $u_{j,\Lambda} \rightharpoonup u_{j,0}\,,\quad$ *in* $H_0^1(\Omega;\mathbb{R})\,,\;$ *as* $\Lambda \to \infty\,,$ *(up to a subsequence)*

(ii) *Assume the nodal domains $\Omega_j \equiv \{x \in \Omega : u_{j,0}(x) > 0\}\,, j = 1, \ldots, m$ are open. Then the limiting functions $u_{j,0}$'s satisfy*

$$-\Delta u_{j,0} + \mu_j u_{j,0}^3 = \tilde{\lambda}_j u_{j,0} \quad \text{in } \Omega_j\,, \tag{1.5}$$

where $\tilde{\lambda}_j$'s are the limits of λ_j's as $\Lambda \to \infty$ (up to a subsequence). Moreover, $u_{j,0}$ is smooth in Ω_j for $j = 1, \ldots, m$.

(iii) *The nodal domains $\Omega_j \equiv \{x \in \Omega : u_{j,0}(x) > 0\}\,, j = 1, \ldots, m$ are finitely union of disjoint domains with positive Lebesgue measure.*

Theorem 1.1 is the main result of this paper which shows that phase separation may occur for all positive bound state solutions as the parameter $\Lambda \to \infty$. The main difficulty in proving Theorem 1.1 is to show that m components $u_{j,\Lambda}$'s of the solution repel each other and form separate domains Ω_j's, as Λ goes to infinity. Moreover, $\Lambda u_{i,\Lambda}^2 u_{j,\Lambda}, \forall i \neq j$, tend to zero pointwise in Ω respectively, as Λ goes to infinity. This is essential to derive the system (1.5) as the governing equations of the limiting functions $u_{j,0}$'s. One may read Proposition 2.1 and 2.2 in Section 2 for detail.

To investigate ground state solutions of the system (1.2), we may study the energy minimization problem given by

$$\text{Minimize } E_\Lambda(u) \quad \text{for } u = (u_1, \dots, u_m) \in \left(H_0^1(\Omega;\mathbb{R})\right)^m , \int_\Omega u_j^2 = 1 , \tag{1.6}$$

where Ω is a bounded smooth domain in $\mathbb{R}^d$, $d = 2, 3$, and the energy functional E_Λ is defined by

$$E_\Lambda(u) = \int_\Omega \sum_{j=1}^m \frac{1}{2}|\nabla u_j|^2 + \frac{\mu_j}{4}\, u_j^4 + \frac{1}{4}\Lambda \sum_{\substack{i,j=1,\\ i\neq j}}^m \tilde{\beta}_{ij} \int_\Omega u_i^2 u_j^2 . \tag{1.7}$$

Here μ_j's and $\tilde{\beta}_{ij}$'s are nonnegative constants independent of Λ, and Λ is a large parameter. The Euler-Lagrange equation of (1.7) is the system (1.2) with λ_j's the associated Lagrange multipliers. For ground state solutions, we prove

Theorem 1.2 *Assume Ω is a bounded smooth domain in $\mathbb{R}^d$, $d = 2, 3$. Then there exists $u_\Lambda = (u_{1,\Lambda}, \dots, u_{m,\Lambda})$ the energy minimizer of (1.6) such that u_Λ is a positive solution of the system (1.2), and satisfy*

$$\Lambda \int_\Omega u_{i,\Lambda}^2\, u_{j,\Lambda}^2 \to 0 \,, \forall i \neq j \quad \textit{as } \Lambda \to \infty \quad (\textit{up to a subsequence}) , \tag{1.8}$$

and

$$u_{j,\Lambda} \to u_{j,\infty} \quad \textit{in } H_0^1(\Omega;\mathbb{R}) \ \textit{as } \Lambda \to \infty \quad (\textit{up to a subsequence}) . \tag{1.9}$$

The multipliers λ_j's are positive constants and are bounded quantities as $\Lambda \to \infty$. Assume the nodal domains $\Omega_j \equiv \{x \in \Omega : u_{j,\infty}(x) > 0\}$, $j = 1, \dots, m$ are open. Then the nodal domains Ω_j's are separated by the nodal line $\{x \in \Omega : u_{j,\infty}(x) = 0,\ j = 1, \dots, m\}$ which has no interior point. Furthermore, if $u_{j,\infty}$ depends on μ_j continuously for $j = 1, \dots, m$, then the nodal domains Ω_j's are m disjoint domains.

From Theorem 1.2, m nodal domains Ω_j's can be determined by finding an optimal partition of the domain Ω that achieves

$$\min \left\{ \sum_{j=1}^m \xi_j(\omega_j) : \omega_j \in \mathcal{A}(\Omega) \,, \cup_{j=1}^m \bar{\omega}_j = \bar{\Omega} \,, \omega_i \cap \omega_j = \emptyset, \forall i \neq j \right\} , \tag{1.10}$$

where $\mathcal{A}(\Omega)$ is the class of all admissible domains, and $\xi_j(\omega_j)$ denotes the first Dirichlet eigenvalue defined by

$$\xi_j(\omega_j) = \min_{\substack{u\in H_0^1(\omega_j), \\ \|u\|_{L^2(\omega_j)}=1}} \int_{\omega_j} \frac{1}{2}|\nabla u|^2 + \frac{\mu_j}{4}u^4 \,. \tag{1.11}$$

The problem (1.10) is complicated but may have some geometric structures for the distribution of nodal domains. For instance, if $\mu_j = 0\,, \forall j$ and $m = 2$, the problem (1.10) can be reduced to

$$\min\left\{\lambda(A) + \lambda(B) : A, B \in \mathcal{A}(\Omega)\,, A \cap B = \emptyset\right\}\,, \tag{1.12}$$

where λ denotes the first Dirichlet eigenvalue for the operator $-\Delta$, and $\mathcal{A}(\Omega)$ is the class of all admissible domains. About the problem (1.12), only few results are known which may depend on the geometric restriction of the domain Ω (cf. [7]). Generically, if the domain Ω is assumed to be convex, then it is conjectured that the minimum of (1.12) is achieved when A, B are two nodal domains of the second Dirichlet eigenfunction for the operator $-\Delta$. A remark by Kawohl [19] may support such a conjecture. However, such a conjecture has not yet been proved.

As m becomes larger and larger, it is natural to believe that the distribution of m nodal domains may become more and more complicated. To study the distribution of m nodal domains Ω_j's, we design efficient numerical schemes by Gauss-Seidel-type iteration method to do numerical computation. When the domain Ω is a unit disk, and the number m varies from two to thirty-three, we may observe multiple verticillate structures of m nodal domains. For $m = 2, \ldots, 5$, m equal nodal domains Ω_j's with centers at vertices of m-polygon form (m)-verticillate structures. As $m = 6, 7, 8$, one nodal domain Ω_{j_0} occupies the center of Ω and the rest $m-1$ nodal domains equally distribute around the outside of Ω_{j_0}. As $m = 9, 10, 11$, two nodal domains Ω_{j_1} and Ω_{j_2} locate near the center of Ω and the rest $m-2$ nodal domains equally distribute around the outside of Ω_{j_1} and Ω_{j_2}. As m increases from 12 to 16, three, four, and five nodal domains may occur near the center of Ω and the rest nodal domains equally distribute the rest of domain Ω. Basically, centers of nodal domains are located at vertices of two eccentric polygons. Such new structures of nodal domains called verticillate doubling can be observed in Figure 4.1 $(a) - (c)$. It is naturally expected that we should have verticillate tripling or quadrupling for structures of m nodal domains when

m increases. In Figure 4.1 (c) and (e), we observe verticillate tripling at $m = 17$ and quadrupling at $m = 32$.

The rest of this paper is organized as follows: We prove Theorem 1.1 and 1.2 in Section 2 and 3, respectively. In Section 4, we demonstrate our numerical results for multiple verticillate structures.

2 Phase Separation on Positive Bound States

In this section, we shall prove Theorem 1.1 as follows: Without loss of generality, we may assume $m = 2$, $u_1 \equiv u\,, u_2 \equiv v\,, \mu_1 = \alpha\,, \mu_2 = \beta\,, \tilde{\beta}_{ij} = 1\,$, and rewrite the system (1.2) as

$$\begin{aligned} -\Delta u + \alpha u^3 + \Lambda v^2\, u &= \lambda_1 u \quad \text{in } \Omega\,, && (2.1)\\ -\Delta v + \beta v^3 + \Lambda u^2\, v &= \lambda_2 v \quad \text{in } \Omega\,, && (2.2)\end{aligned}$$

Let (u_Λ, v_Λ) be a positive solution of equations (2.1) and (2.2), and satisfy (1.3) and (1.4). We may multiply both sides of the equation (2.1) by u_Λ and integrate over Ω. Then by (1.3) and (1.4), we have

$$\int_\Omega |\nabla u_\Lambda|^2 + \alpha\, u_\Lambda^4 + \Lambda\, u_\Lambda^2 v_\Lambda^2 = \lambda_1\,. \tag{2.3}$$

Similarly, by the equation (2.2), (1.3) and (1.4), we obtain

$$\int_\Omega |\nabla v_\Lambda|^2 + \beta\, v_\Lambda^4 + \Lambda\, u_\Lambda^2 v_\Lambda^2 = \lambda_2\,. \tag{2.4}$$

Since λ_1 and λ_2 are bounded quantities as $\Lambda \to \infty$, then by (2.3) and (2.4), we have

$$u_\Lambda \rightharpoonup u_0\,, \quad v_\Lambda \rightharpoonup v_0 \quad \text{in } H_0^1(\Omega;\mathbb{R}) \quad (\text{up to a subsequence})\,, \tag{2.5}$$

and

$$u_0\, v_0 = 0 \quad \text{almost everywhere in}\, \Omega\,. \tag{2.6}$$

Moreover, by (1.3) and (2.5), we obtain

$$u_\Lambda \to u_0\,, \quad v_\Lambda \to v_0 \quad \text{almost everywhere in } \Omega \quad (\text{up to a subsequence})\,, \tag{2.7}$$

and

$$|\{x : u_0(x) > 0\}| > 0 \quad \text{and} \quad |\{x : v_0(x) > 0\}| > 0\,, \tag{2.8}$$

where $|\cdot|$ denotes the Lebesgue measure. Moreover, the equation (2.6) implies that the sets $\{x : u_0(x) > 0\}$ and $\{x : v_0(x) > 0\}$ are disjoint. Hence we complete the proof of Theorem 1.1 (i). For (ii) of Theorem 1.1, we need two crucial lemmas to obtain an L^∞ estimate and a gradient estimate. Now we state these two lemmas as follows:

Lemma 2.1 (L^∞ estimate) *There exists a positive constant C_0 independent of Λ such that*

$$\|u_\Lambda\|_{L^\infty(\Omega)} \leq C_0\,, \quad \|v_\Lambda\|_{L^\infty(\Omega)} \leq C_0\,.$$

Lemma 2.2 (interior gradient estimate) *Let $x_0 \in \Omega$ and R_1 be a positive constant such that the disk $B_{R_1}(x_0)$ is in the interior of Ω. Then there exists a positive constant C_1 depending only on C_0 which is defined in Lemma 2.1 such that*

$$\|\nabla u_\Lambda\|_{L^\infty\left(B_{R_2}(x_0)\right)} \leq C_1\sqrt{\Lambda}\,, \quad \|\nabla v_\Lambda\|_{L^\infty\left(B_{R_2}(x_0)\right)} \leq C_1\sqrt{\Lambda}\,,$$

where $R_2 = R_1 - \Lambda^{-\frac{1}{2}}$.

Proof of Lemma 2.1: For simplicity, we write u and v instead of u_Λ and v_Λ, respectively. We may multiply both sides of the equation (2.1) by u^{2s-1} ($s \geq 1$) and integrate over Ω. Then by (1.4), we have

$$s^{-2}(2s-1)\int_\Omega |\nabla u^s|^2 = \lambda_1\int_\Omega u^{2s} - \alpha\int_\Omega u^{2s+2} - \Lambda\int_\Omega u^{2s}v^2\,. \tag{2.9}$$

Similarly, by (2.2) and (1.4), we have

$$s^{-2}(2s-1)\int_\Omega |\nabla v^s|^2 = \lambda_2\int_\Omega v^{2s} - \beta\int_\Omega v^{2s+2} - \Lambda\int_\Omega v^{2s}u^2\,. \tag{2.10}$$

Hence

$$s^{-2}(2s-1)\int_\Omega |\nabla u^s|^2 \leq \lambda_1\int_\Omega u^{2s}\,, \tag{2.11}$$

and

$$s^{-2}(2s-1)\int_\Omega |\nabla v^s|^2 \leq \lambda_2\int_\Omega v^{2s}\,. \tag{2.12}$$

By (2.11), (2.12) and $u, v \in H_0^1(\Omega;\mathbb{R})$, we obtain $u^s, v^s \in H_0^1(\Omega;\mathbb{R})$. Then by a Sobolev imbedding , we have

$$\left(\int_\Omega u^{s\nu}\right)^{2/\nu} \le C_2 \int_\Omega |\nabla u^s|^2 \,, \tag{2.13}$$

and

$$\left(\int_\Omega v^{s\nu}\right)^{2/\nu} \le C_2 \int_\Omega |\nabla v^s|^2 \,, \tag{2.14}$$

for $2 < \nu < \infty$, where $C_2 = C_2(\Omega)$ is the imbedding constant. Moreover, by (2.11)-(2.14), we have

$$\left(\int_\Omega u^{s\nu}\right)^{2/\nu} \le \lambda_1 C_2 s \int_\Omega u^{2s} \,, \tag{2.15}$$

and

$$\left(\int_\Omega v^{s\nu}\right)^{2/\nu} \le \lambda_2 C_2 s \int_\Omega v^{2s} \,, \tag{2.16}$$

for $2 < \nu < \infty$. Here we have used the fact that $s \ge 1$, i.e., $2s - 1 \ge s$.

We define sequences $\{s_j\}$ and $\{M_j\}$ by

$$2s_0 = \nu \,, \quad 2s_{j+1} = \nu s_j \,, \quad \text{ for } j \ge 0 \,,$$

and

$$M_0 = (\lambda_1\, C_2)^{\nu/2} \,, M_{j+1} = (\lambda_1 C_2 s_j M_j)^{\nu/2} \,,$$

where $\nu > 2$ is a constant. Then $s_j = (\nu/2)^{j+1}$. Now we claim that

$$\begin{aligned} \int_\Omega u^{2s_j} &\le M_j \,, \quad \text{ for } j \ge 0 \,, && (2.17) \\ M_j &\le e^{m\, s_{j-1}} \,, && (2.18) \end{aligned}$$

for some constant $m > 0$. We may prove (2.17) by induction as follows: As $j = 0$, we may use (1.3) and (2.15) with $s = 1$ to obtain (2.17). Suppose (2.17) holds as $j = k$. Then by (2.15), we have

$$\begin{aligned} \int_\Omega u^{2s_{k+1}} &= \int_\Omega u^{\nu s_k} \\ &\le \left(\lambda_1 C_2 s_k \int_\Omega u^{2s_k}\right)^{\nu/2} \quad (\text{ by } (2.15)) \\ &\le (\lambda_1 C_2 s_k M_k)^{\nu/2} = M_{k+1} \,(\text{ by induction hypothesis}) \,. \end{aligned}$$

Hence (2.17) is true. Now we prove (2.18) as follows: Let $\mu_j = \log M_j$. Then $\mu_{j+1} = \frac{\nu}{2}\mu_j + \sigma_j$, where $\sigma_j = \frac{\nu}{2}\log(\lambda_1 C_2 s_j)$. Hence

$$\sigma_j = \frac{\nu}{2}\left[\log(\lambda_1 C_2) + (j+1)\log\frac{\nu}{2}\right] \le C^*(j+1)\,,$$

where

$$C^* = \nu \max\{\log(\lambda_1 C_2)\ , \log\frac{\nu}{2}\}\,. \tag{2.19}$$

We may define a sequence $\{\tau_j\}$ by $\tau_0 = \mu_0$ and $\tau_{j+1} = \frac{\nu}{2}\tau_j + C^*(j+1)$ for $j \ge 0$. Clearly, $\mu_j \le \tau_j$ for $j \ge 0$. Moreover, since

$$\tau_j = (\nu/2)^j \left(\mu_0 + 2C^*\nu(\nu-2)^{-2}\right) - 2C^*(\nu-2)^{-1}\left[j + \nu/(\nu-2)\right]\,,$$

then by $s_j = (\nu/2)^{j+1}$, we have $\tau_j \le m s_{j-1}$, where

$$m = \mu_0 + 2C^*\nu(\nu-2)^{-2} = \frac{\nu}{2}\log(\lambda_1 C_2) + 2C^*\nu(\nu-2)^{-2}\,. \tag{2.20}$$

By (2.19) and (2.20), the constant m is a positive constant depending only on ν, λ_1 and C_2. Hence $\log M_j \le m s_{j-1}$ and we obtain (2.18). By (2.17) and (2.18), we have

$$\|u\|_{L^{2s_j}(\Omega)} \le e^{m/\nu} \quad \forall j \ge 0\,,$$

and hence letting $j \to \infty$, we obtain $\|u\|_{L^\infty(\Omega)} \le e^{m/\nu}$. Similarly, by (2.16), we may obtain $\|v\|_{L^\infty(\Omega)} \le e^{m^*/\nu}$, where m^* is a positive constant independent of Λ. Therefore, we may complete the proof of Lemma 2.1.

Proof of Lemma 2.2: Without loss of generality, we may assume x_0 is at the origin. Let $\tilde{u}(x) = u_\Lambda\left(x/\sqrt{\Lambda}\right)$, $\tilde{v}(x) = v_\Lambda\left(x/\sqrt{\Lambda}\right)$, for $x \in B_{R_1\sqrt{\Lambda}}(0)$. Then $\tilde{u}$ and $\tilde{v}$ satisfy

$$-\Delta\tilde{u} + \alpha\Lambda^{-1}\tilde{u}^3 + \tilde{v}^2\,\tilde{u} = \lambda_1\Lambda^{-1}\tilde{u} \quad \text{in } B_{R_1\sqrt{\Lambda}}(0)\,, \tag{2.21}$$

$$-\Delta\tilde{v} + \beta\Lambda^{-1}\tilde{v}^3 + \tilde{u}^2\,\tilde{v} = \lambda_2\Lambda^{-1}\tilde{v} \quad \text{in } B_{R_1\sqrt{\Lambda}}(0)\,, \tag{2.22}$$

Hence by (2.21), (2.22), Lemma 2.1 and the standard theorem of interior gradient estimates (cf. Theorem 8.32 of [13]), we have

$$\|\nabla\tilde{u}\|_{L^\infty\left(B_{R_2\sqrt{\Lambda}}(0)\right)} \le C_1\,, \quad \|\nabla\tilde{v}\|_{L^\infty\left(B_{R_2\sqrt{\Lambda}}(0)\right)} \le C_1\,,$$

where $R_2 = R_1 - \Lambda^{-\frac{1}{2}}$, and C_1 is a positive constant depending only on C_0 which is defined in Lemma 2.1. Here we have used the fact that α and β are nonnegative constants independent of Λ, and λ_j's are bounded quantities as $\Lambda \to \infty$. Thus we have

$$\|\nabla u_\Lambda\|_{L^\infty\left(B_{R_2}(0)\right)} \leq C_1\sqrt{\Lambda}\,, \quad \|\nabla v_\Lambda\|_{L^\infty\left(B_{R_2}(0)\right)} \leq C_1\sqrt{\Lambda}\,.$$

Therefore we complete the proof of Lemma 2.2.

By Lemma 2.1 and Lemma 2.2, we may obtain

Proposition 2.1 *Assume* $x_0 \in \Omega$ *such that* $u_\Lambda(x_0) \to u_0(x_0) \geq 2\epsilon_0 > 0\,,$ *as* $\Lambda \to \infty\,,$ *where* ϵ_0 *is any positive constant independent of* Λ. *Then* $\forall \eta > 1\,,$ $v_\Lambda(x_0) \leq 2C_0\,\Lambda^{-\eta}\,,$ *as* $\Lambda \geq \Lambda_0\,,$ *where* C_0 *is the positive constant defined in Lemma 2.1, and* Λ_0 *is a positive constant depending only on* $x_0\,, \epsilon_0\,, \eta\,, C_0\,,$ *and the upper bound of* λ_1.

As for the proof of Proposition 2.1, we have

Proposition 2.2 *Assume* $x_1 \in \Omega$ *such that* $v_\Lambda(x_1) \to v_0(x_1) \geq 2\epsilon_1 > 0\,,$ *as* $\Lambda \to \infty$ *where* ϵ_1 *is any positive constant independent of* Λ. *Then* $\forall \eta > 1\,,$ $u_\Lambda(x_1) \leq 2C_0\,\Lambda^{-\eta}\,,$ *as* $\Lambda \geq \Lambda_1\,,$ *where* C_0 *is the positive constant defined in Lemma 2.1, and* Λ_1 *is a positive constant depending only on* $x_1\,, \epsilon_1\,, \eta\,, C_0\,,$ *and the upper bound of* λ_2.

We shall prove Proposition 2.1 and 2.2 later. Now we want to prove Theorem 1.1 (ii) and (iii) as follows. By Lebesgue dominated convergence theorem, Proposition 2.1 and 2.2, it is easy to prove Theorem 1.1 (ii). Now we want to prove Theorem 1.1 (iii) by contradiction. Suppose that Ω_u can be decomposed into infinitely many disjoint subdomains $\Omega_j\,, j = 1, 2, 3, \ldots$. Then without loss of generality, we may assume

$$\lambda(\Omega_j) \to \infty \text{ as } j \to \infty\,, \tag{2.23}$$

where $\lambda(\Omega_j)$ is the first eigenvalue of $-\Delta$ on the space $H_0^1(\Omega_j)$. Moreover, u_0 satisfies

$$-\Delta u_0 + \alpha u_0^3 = \tilde{\lambda}_1 u_0 \quad \text{ in } \Omega_j\,,\; j = 1, 2, 3, \ldots\,,$$

and

$$u_0 = 0 \quad \text{ on } \partial\Omega_j\,,\; j = 1, 2, 3, \ldots\,.$$

In each Ω_j, we may define $U_j = u_0/\|u_0\|_{L^2(\Omega_j)}$. Then $U_j \in H_0^1(\Omega_j)\,, \|U_j\|_{L^2(\Omega_j)} = 1\,,$ for $j = 1, 2, 3, \ldots$. Moreover, U_j satisfies

$$-\Delta U_j + \alpha\|u_0\|^2_{L^2(\Omega_j)}\,U_j^3 = \tilde{\lambda}_1 U_j \quad \text{ in } \Omega_j\,, j = 1, 2, 3, \ldots\,. \tag{2.24}$$

We may multiply both sides of (2.24) by U_j and integrate over Ω_j. Then we have

$$\int_{\Omega_j} |\nabla U_j|^2 \leq \tilde{\lambda}_1 \int_{\Omega_j} U_j^2 = \tilde{\lambda}_1 \,.$$

Consequently, we have $\lambda(\Omega_j) \leq \tilde{\lambda}_1 < \infty$. This contradict with (2.23) and the proof of Theorem 1.1 (iii) is completed.

The main ideas of the proof of Proposition 2.1 are as follows: (i) rescale spatial variables of the solution (u_Λ, v_Λ) by $\sqrt{\log\Lambda/\Lambda}$ and show that u_Λ is positive in a suitable neighborhood of x_0. (ii) find the comparison function and apply maximum principle to force the function v_Λ tending to zero near x_0(see the formulation of the statements right after (2.36)). By Lemma 2.2, the solution u_Λ may be positive in a ball $B_\rho(x_0)\,, \rho = \sqrt{a/\Lambda}$ for some constant $a > 0$. It is natural to rescale spatial variables by $\sqrt{1/\Lambda}$. However, when we rescale spatial variables of the solution (u_Λ, v_Λ) by $\sqrt{1/\Lambda}$, the nonlinear terms $\Lambda u^2 v$ and $\Lambda v^2 u$ become $u^2 v$ and $v^2 u$, respectively. Consequently, the large parameter Λ disappears and we cannot find the comparison function to force the function v_Λ tending to zero near x_0. We need to enlarge the scale $\sqrt{1/\Lambda}$ but due to Lemma 2.2, the scale $\sqrt{1/\Lambda}$ cannot be enlarged arbitrarily. So one may enlarge the scale $\sqrt{1/\Lambda}$ as $\sqrt{\log\Lambda/\Lambda}$. Then the nonlinear terms $\Lambda u^2 v$ and $\Lambda v^2 u$ become $(\log\Lambda) u^2 v$ and $(\log\Lambda) v^2 u$, respectively. Hence we may find the comparison function to force the function v_Λ tending to zero near x_0.

Now we demonstrate the proof of Proposition 2.1 as follows:

Proof of Proposition 2.1: Without loss of generality, we may assume $x_0 = 0\,, u_\Lambda(0) \geq 2\epsilon_0$ for $\Lambda > 0$, where the origin 0 is in the interior of Ω, and ϵ_0 is a positive constant independent of Λ. Let $\hat{u}(x) = u_\Lambda(\tilde{\Lambda}x)\,,$ and $\hat{v}(x) = v_\Lambda(\tilde{\Lambda}x)\,,$ for $x \in B_{r_0\tilde{\Lambda}^{-1}}(0)\,,$ where $\tilde{\Lambda} = \sqrt{\log\Lambda/\Lambda}\,,$ and r_0 is a positive constant independent of Λ such that the disk $B_{r_0}(0)$ with radius r_0 and center at the origin is in the interior of Ω. Then the equations of $\hat{u}$ and $\hat{v}$ are

$$-\Delta\hat{u} + \alpha\tilde{\Lambda}^2\,\hat{u}^3 + (\log\Lambda)\,\hat{v}^2\,\hat{u} \;=\; \lambda_1\tilde{\Lambda}^2\,\hat{u} \quad \text{in } B_{r_0\tilde{\Lambda}^{-1}}(0)\,, \tag{2.25}$$

$$-\Delta\hat{v} + \beta\tilde{\Lambda}^2\,\hat{v}^3 + (\log\Lambda)\,\hat{u}^2\,\hat{v} \;=\; \lambda_2\tilde{\Lambda}^2\,\hat{v} \quad \text{in } B_{r_0\tilde{\Lambda}^{-1}}(0)\,, \tag{2.26}$$

Let $f_\Lambda(r) = \frac{1}{2\pi r}\int_{\partial B_r(0)} \hat{u}^2\,dS\,,$ for $0 < r \leq r_0\tilde{\Lambda}^{-1}\,, \Lambda > 0\,.$ Fix ϵ_1 as a positive constant. Let $\{\Lambda_i\}$ be any increasing sequence of positive numbers such that $\Lambda_i \to \infty$ as $i \to \infty\,.$ Let $0 < \delta < \frac{1}{2}$ be a positive constant such that $\epsilon_1 \log\left(\frac{1}{2}/\delta\right) \geq 3C_0^2$, where C_0 is the positive constant in Lemma 2.1. Now we replace Λ by the sequence $\{\Lambda_i\}$ and $\hat{u}(x) = u_{\Lambda_i}(\tilde{\Lambda}_i x)$, where $\tilde{\Lambda}_i = \sqrt{\log\Lambda_i/\Lambda_i}$. We claim that there exists a sequence $\{r_i\}$ such that

$$r_i \in [\Lambda_i^\delta, \Lambda_i^{\frac{1}{2}}/\log\Lambda_i] \quad \text{and} \quad |f'_{\Lambda_i}(r_i)| \leq \epsilon_1 r_i^{-1} \left(\log r_i\right)^{-1}. \tag{2.27}$$

We may prove (2.27) by contradiction. Suppose $f'_{\Lambda_i}(r) > \epsilon_1 r^{-1} \left(\log r\right)^{-1}$, for $r \in [\Lambda_i^\delta, \Lambda_i^{\frac{1}{2}}/\log\Lambda_i]$. Then

$$\begin{aligned} f_{\Lambda_i}(\Lambda_i^{\frac{1}{2}}/\log\Lambda_i) - f_{\Lambda_i}(\Lambda_i^\delta) &= \int_{\Lambda_i^\delta}^{\Lambda_i^{\frac{1}{2}}/\log\Lambda_i} f'(r)\,dr \\ &\geq \int_{\Lambda_i^\delta}^{\Lambda_i^{\frac{1}{2}}/\log\Lambda_i} \epsilon_1 r^{-1}\left(\log r\right)^{-1}\,dr \\ &= \epsilon_1 \log\left[\frac{\log\left(\Lambda_i^{\frac{1}{2}}/\log\Lambda_i\right)}{\log\Lambda_i^\delta}\right] \to \epsilon_1 \log\left(\frac{1}{2}/\delta\right) \geq 3C_0^2. \end{aligned}$$

However, by Lemma 2.1, we have $f_{\Lambda_i}(\Lambda_i^{\frac{1}{2}}/\log\Lambda_i) - f_{\Lambda_i}(\Lambda_i^\delta) \leq 2C_0^2$. Hence we obtain contradiction and complete the proof of (2.27).

By (2.27), we have

$$\left|\int_{\partial B_{r_i}(0)} \hat{u}\,\partial_n\,\hat{u}\,dS\right| \leq \pi\epsilon_1 \left(\log r_i\right)^{-1},$$

i.e.

$$\left|\int_{\partial B_{r_i}(0)} \hat{u}\,\partial_n\,\hat{u}\,dS\right| \leq K_0/\log\Lambda_i, \tag{2.28}$$

where K_0 is a positive constant depending only on δ and ϵ_1, and ∂_n is the standard normal derivative on the boundary. Now we multiply both sides of (2.25) by $\hat{u}$ and integrate over $B_{r_i}(0)$. Then we obtain

$$\begin{aligned} \int_{B_{r_i}(0)} |\nabla\hat{u}|^2 &= \int_{\partial B_{r_i}(0)} \hat{u}\,\partial_n\,\hat{u}\,dS - \alpha\tilde{\Lambda}_i^2 \int_{B_{r_i}(0)} \hat{u}^4 \\ &\quad - \left(\log\Lambda_i\right) \int_{B_{r_i}(0)} \hat{u}^2\hat{v}^2 + \lambda_1\tilde{\Lambda}_i^2 \int_{B_{r_i}(0)} \hat{u}^2, \end{aligned} \tag{2.29}$$

where $\tilde{\Lambda}_i = \sqrt{\log \Lambda_i / \Lambda_i}$. Hence by (2.28), (2.29) and Lemma 2.1, we have

$$\int_{B_{r_i}(0)} |\nabla \hat{u}|^2 \leq K_1 / \log \Lambda_i \,, \tag{2.30}$$

where K_1 is a positive constant depending only on $C_0\,, \lambda_1\,, \delta$, and ϵ_1.

From Lemma 2.2, we have

$$\|\nabla \hat{u}\|_{L^\infty\left(B_{r_i}(0)\right)} \leq C_1 \sqrt{\log \Lambda_i}\,, \tag{2.31}$$

where C_1 is the positive constant defined in Lemma 2.2. Hence by (2.30), (2.31) and the imbedding theorem of Morrey (cf. Theorem 7.17 of [13]), we have

$$\begin{aligned}
\operatorname{osc}_{B_R(0)} \hat{u} &\leq C_2\, R^\gamma\, \|\nabla \hat{u}\|_{L^p(B_R(0))} \\
&= C_2\, R^\gamma \left(\int_{B_R(0)} |\nabla \hat{u}|^{p-2}\, |\nabla \hat{u}|^2 \right)^{1/p} \\
&\leq C_2\, R^\gamma \left(C_1 \sqrt{\log \Lambda_i} \right)^\gamma \left(\int_{B_R(0)} |\nabla \hat{u}|^2 \right)^{1/p} \qquad (\text{ by } (2.31)) \\
&\leq C_2\, R^\gamma \left(C_1 \sqrt{\log \Lambda_i} \right)^\gamma \left(K_1 / \log \Lambda_i \right)^{1/p} \qquad (\text{ by } (2.30)) \\
&= K_2\, R^\gamma\, (\log \Lambda_i)^{\gamma_*}\,,
\end{aligned}$$

for $0 < R \leq r_i$, and $p > 2$, where $\gamma = 1 - \frac{2}{p}\,, \gamma_* = \frac{1}{2} - \frac{2}{p}\,, C_2 = C_2(p) > 0$, and K_2 is a positive constant independent of Λ_i. In particular, we set $R = \kappa \sqrt{\log \Lambda_i}\,, \epsilon_1 = 1$, and $p = 5/2$, where κ is a positive constant depending only on η and ϵ_0. We may determine κ later. Then we obtain

$$\operatorname{osc}_{B_R(0)} \hat{u} \leq K_2\, \kappa^{1/5}\, (\log \Lambda_i)^{-1/5}\,, \quad \text{for } R = \kappa \sqrt{\log \Lambda_i}\,. \tag{2.32}$$

Since $u_{\Lambda_i}(0) \geq 2\epsilon_0$ i.e. $\hat{u}(0) \geq 2\epsilon_0$, then by (2.32), we have

$$\hat{u}(x) \geq \epsilon_0 \quad \text{for } x \in B_R(0)\,, \tag{2.33}$$

as $i \geq N_0$, where N_0 is a large constant which depends only on $\epsilon_0\,, \kappa\,, C_0$, and the upper bound of λ_1.

Let $\check{u}(x) = \hat{u}\left(\sqrt{\log \Lambda_i}\, x\right)$ and $\check{v}(x) = \hat{v}\left(\sqrt{\log \Lambda_i}\, x\right)$ for $x \in B_\kappa(0)$. Then (2.33) implies

$$\check{u}(x) \geq \epsilon_0 \quad \text{for } x \in B_\kappa(0)\,. \tag{2.34}$$

Moreover, $\check{u}$ and $\check{v}$ satisfy

$$-\Delta\check{u}+\alpha\frac{\log^2\Lambda_i}{\Lambda_i}\,\check{u}^3+\left(\log^2\Lambda_i\right)\check{v}^2\,\check{u} = \lambda_1\frac{\log^2\Lambda_i}{\Lambda_i}\,\check{u} \quad \text{in } B_\kappa(0)\,, \tag{2.35}$$

$$-\Delta\check{v}+\beta\frac{\log^2\Lambda_i}{\Lambda_i}\,\check{v}^3+\left(\log^2\Lambda_i\right)\check{u}^2\,\check{v} = \lambda_2\frac{\log^2\Lambda_i}{\Lambda_i}\,\check{v} \quad \text{in } B_\kappa(0)\,. \tag{2.36}$$

By (2.34) and (2.36), we have

$$\Delta\check{v}\geq\frac{1}{2}\epsilon_0^2\,\left(\log^2\Lambda_i\right)\,\check{v} \quad \text{in } B_\kappa(0)\,. \tag{2.37}$$

Let w be the solution of

$$\begin{cases}\Delta w & =\frac{1}{2}\epsilon_0^2\,\left(\log^2\Lambda_i\right)\,w \quad \text{in } B_\kappa(0)\,,\\ w|_{\partial B_\kappa(0)} & =\sup\limits_{B_\kappa(0)}\,\check{v}\equiv K_{\Lambda_i}\,.\end{cases}$$

Then by the maximum principle, we obtain

$$\check{v}\leq w \quad \text{in } B_\kappa(0)\,. \tag{2.38}$$

Since the equation of w is linear, we may write $w=K_{\Lambda_i}W$, where W is the solution of

$$\begin{cases}\Delta W & =\frac{1}{2}\epsilon_0^2\,\left(\log^2\Lambda_i\right)\,W \quad \text{in } B_\kappa(0)\,,\\ W|_{\partial B_\kappa(0)} & =1\,.\end{cases}$$

Hence $W(r)=I_0(\sqrt{\frac{1}{2}}\epsilon_0\,r\log\Lambda_i)/I_0(\sqrt{\frac{1}{2}}\epsilon_0\,\kappa\log\Lambda_i)$, where I_0 is the modified Bessel function of order zero. Thus by the monotonic increasing of I_0 and the asymptotic formula $I_0(r)\sim e^r/\sqrt{2\pi r}$ as $r\to\infty$(cf. [10]), we have

$$\begin{aligned}W(r) &\leq I_0\left(\frac{1}{2}\sqrt{\frac{1}{2}}\epsilon_0\,\kappa\log\Lambda_i\right)/I_0\left(\sqrt{\frac{1}{2}}\epsilon_0\,\kappa\log\Lambda_i\right)\\ &\leq 2\Lambda_i^{-\epsilon_0\kappa/\sqrt{8}}\,,\ \forall\,0<r\leq\kappa/2\,.\end{aligned}$$

By Lemma 2.1, $K_{\Lambda_i}=\sup\limits_{B_\kappa(0)}\,\check{v}\leq C_0$. Hence by (2.38), we obtain

$$\check{v}\leq 2C_0\,\Lambda_i^{-\eta} \quad \text{in } B_{\kappa/2}(0)\,,$$

for $\eta>1$, where $\eta=\epsilon_0\kappa/\sqrt{8}$ and the constant κ is determined. Thus

$$v_{\Lambda_i}(0)=\check{v}(0)\leq\sup_{B_{\kappa/2}(0)}\,\check{v}\leq 2C_0\,\Lambda_i^{-\eta}\,,$$

for $\eta>1$. Therefore we complete the proof of Proposition 2.1.

3 Positive Ground States

In this section, we study the energy minimization problem (1.6) and prove Theorem 1.2 as follows:
To estimate the energy upper bound , we may define comparison functions by

$$U_j(x) = \begin{cases} w_j(x) & \text{for } x \in \Omega_j^0 \,, \\ 0 & \text{for } x \in \Omega \backslash \Omega_j^0 \,, j = 1, \dots, m \,, \end{cases}$$

where Ω_j^0's are disjoint smooth domains satisfying $\Omega_j^0 \subset \Omega \,, j = 1, 2 \,,$ and $\cup_{j=1}^m \overline{\Omega_j^0} = \overline{\Omega}$. In addition, each w_j is the first eigenfunction of Laplace operator in the space $H_0^1\left(\Omega_j^0\right)$. Then it is easy to check that

$$E_\Lambda(U) \leq K_0 \,, \tag{3.1}$$

where $U = (U_1, \dots, U_m)$ and K_0 is a positive constant independent of Λ.

By (3.1) and the standard Direct method, there exists an energy minimizer $u_\Lambda = (u_{1,\Lambda}, \dots, u_{m,\Lambda})$ of (1.6) such that each $u_{j,\Lambda}$ is nonnegative,

$$u_{j,\Lambda} \rightharpoonup u_{j,\infty} \,, \quad \text{in } H_0^1(\Omega; \mathbb{R}) \quad (\text{up to a subsequence}) \,, \tag{3.2}$$

and

$$u_{i,\infty} \, u_{j,\infty} = 0 \quad \text{almost everywhere in} \, \Omega \,, \forall i \neq j \,. \tag{3.3}$$

Here we have used the standard inequality

$$\int_\Omega |\nabla |u||^2 \leq \int_\Omega |\nabla u|^2 \,, \quad \forall u \in H^1(\Omega; \mathbb{R}) \,,$$

to obtain the nonnegative ground state solution u_Λ. The solutions u_Λ satisfies the system (1.2), (1.3) and (1.4). We may multiply both sides of the j-th component of (1.2) by $u_{j,\Lambda}$ and integrate it over Ω. Then by (1.3) and (1.4), we have

$$\int_\Omega |\nabla u_{j,\Lambda}|^2 + \mu_j \, u_{j,\Lambda}^4 + \Lambda \sum_{i \neq j} \tilde{\beta}_{ij} u_{i,\Lambda}^2 u_{j,\Lambda}^2 = \lambda_j \,, j = 1, \dots, m \,. \tag{3.4}$$

Hence by (3.4), λ_j's are positive constants which may depend on Λ. Since each $u_{j,\Lambda}$ is nonnegative, and λ_j's are positive constants, then by (1.2), we have

$$\Delta u_{j,\Lambda} - \left(\mu_j u_{j,\Lambda}^2 + \Lambda \sum_{i \neq j} \tilde{\beta}_{ij} u_{i,\Lambda}^2 \right) u_{j,\Lambda} \leq 0 \,, \quad \text{in } \Omega \,. \tag{3.5}$$

Thus by (3.5) and the strong maximum principle, each $u_{j,\Lambda}$ must be positive in Ω. It is easy to check that the multipliers λ_j's satisfy

$$\begin{aligned}\sum_{j=1}^m \lambda_j &= \sum_{j=1}^m \int_\Omega |\nabla u_{j,\Lambda}|^2 + \mu_j u_{j,\Lambda}^4 + \Lambda \sum_{i\neq j} \tilde\beta_{ij} u_{i,\Lambda}^2 u_{j,\Lambda}^2 \\ &\leq 4E_\Lambda(u_\Lambda) \leq 4E_\Lambda(U)\,.\end{aligned}$$

From the energy upper bound (3.1), λ_j's must be bounded quantities as $\Lambda \to \infty$.

Since u_Λ is the energy minimizer, then by (3.3), we have

$$E_\Lambda(u_\Lambda) \leq E_\Lambda(u_\infty) = \sum_{j=1}^m \int_\Omega \frac{1}{2}|\nabla u_{j,\infty}|^2 + \frac{\mu_j}{4}\, u_{j,\infty}^4\,, \tag{3.6}$$

where $u_\infty = (u_{1,\infty}, \ldots, u_{m,\infty})$. Hence by (3.2), (3.6) and Fatou's Lemma, we obtain

$$\Lambda \int_\Omega \sum_{i\neq j} \tilde\beta_{ij} u_{i,\Lambda}^2\, u_{j,\Lambda}^2 \to 0\,, \quad j = 1, \ldots, m\,, \tag{3.7}$$

and

$$\int_\Omega |\nabla u_{j,\Lambda}|^2 \to \int_\Omega |\nabla u_{j,\infty}|^2\,, \quad j = 1, \ldots, m\,. \tag{3.8}$$

Thus by (3.2) and (3.8), we have the strong convergence as follows:

$$u_{j,\Lambda} \to u_{j,\infty} \quad \text{in } H_0^1(\Omega;\mathbb{R}) \quad (\text{up to a subsequence})\,. \tag{3.9}$$

Now we want to prove the nodal line $\Gamma = \{x \in \Omega : u_{j,\infty}(x) = 0\,, j = 1, \ldots, m\}$ having no interior point by contradiction. Suppose the nodal line Γ having some interior points. Let Ω_1' be the interior of $\Omega \setminus \cup_{j=2}^m \Omega_j$. Then $\Omega_1' \supset \Omega_1$ and $|\Omega_1'| > |\Omega_1|$. Now we define the comparison functions by $\tilde U = \left(\tilde U_1, \ldots, \tilde U_m\right)$,

$$\tilde U_1(x) = \begin{cases} \varphi(x) & \text{for } x \in \Omega_1'\,, \\ 0 & \text{for } x \in \Omega\setminus\Omega_1'\,, \end{cases}$$

and

$$\tilde U_j(x) = \begin{cases} 0 & \text{for } x \in \Omega\setminus\Omega_j\,, \\ u_{j,\infty}(x) & \text{for } x \in \Omega_j\,, j = 2, \ldots, m\,, \end{cases}$$

where φ is the energy minimizer of the following minimization problem:

$$\text{Minimize} \int_{\Omega_1'} \frac{1}{2}|\nabla\psi|^2 + \frac{\mu_1}{4}\,\psi^4 \quad \text{for } \psi \in H_0^1(\Omega_1';\mathbb{R})\,, \int_{\Omega_1'} \psi^2 = 1\,.$$

Then

$$E_\Lambda(\tilde{U}) = \int_{\Omega_1'} \frac{1}{2}|\nabla\varphi|^2 + \frac{\mu_1}{4}\,\varphi^4 + \sum_{j=2}^{m} \int_{\Omega_j} \frac{1}{2}|\nabla u_{j,\infty}|^2 + \frac{\mu_j}{4}\,u_{j,\infty}^4\,. \tag{3.10}$$

Since $E_\Lambda(\tilde{U}) \geq E_\Lambda(u_\Lambda)$, then by (3.2), (3.10) and Fatou's Lemma, we have $\int_{\Omega_1'} \frac{1}{2}|\nabla\varphi|^2 + \frac{\mu_1}{4}\,\varphi^4 \geq \int_{\Omega_1} \frac{1}{2}|\nabla u_{1,\infty}|^2 + \frac{\mu_1}{4}\,u_{1,\infty}^4$. This may contradict with $\Omega_1' \supset \Omega_1$ and the definition of φ. Therefore we may complete the proof of the nodal line $\{x \in \Omega : u_{j,\infty}(x) = 0\,, j = 1, \ldots, m\}$ having no interior point.

Now we claim that Ω_j's are m disjoint domains for $\mu_j \geq 0\,, j = 1, \ldots, m$. As $\mu_j = 0\,, \forall j$, it is obvious that Ω_j's are m disjoint domains i.e. each set Ω_j is connected. For general μ_j's, we need a crucial assumption that $u_{j,\infty}$ depend on μ_j continuously for $j = 1, \ldots, m$. Now we prove the claim by contradiction. Suppose Ω_j is not a domain for some $\mu_j > 0$. Then by the continuity of $u_{j,\infty}$ to μ_j, we may assume that for some $\mu_j > 0$, Ω_j can be divided into two subdomains Ω_j^+ and Ω_j^-, where the measure of Ω_j^- is sufficiently small such that $\lambda\left(\Omega_j^-\right) \geq K_*$, and $K_* > 0$ is a large constant determined later. Hereafter, $\lambda\left(\Omega_j^-\right)$ is the first eigenvalue of $-\Delta$ on the space $H_0^1\left(\Omega_j^-\right)$. Furthermore, we may assume

$$\int_{\Omega_j^+} u_{j,\infty}^2 = 1 - \epsilon\,, \quad \int_{\Omega_j^-} u_{j,\infty}^2 = \epsilon\,, \quad 0 < \epsilon < \frac{1}{2}\,.$$

Let $v_j^+ = u_{j,\infty}/\sqrt{1-\epsilon}$ in Ω_j^+, and $v_j^- = u_{j,\infty}/\sqrt{\epsilon}$ in Ω_j^-. Then

$$\int_{\Omega_j^+} \left(v_j^+\right)^2 = 1\,, \quad \int_{\Omega_j^-} \left(v_j^-\right)^2 = 1\,. \tag{3.11}$$

By (3.1) and (3.2), we obtain

$$\int_{\Omega_j^+} \frac{1}{2}\left|\nabla v_j^+\right|^2 + \frac{\mu_j}{4}\left(v_j^+\right)^4 \leq K_j\,, \tag{3.12}$$

where K_j is a positive constant depending only on the upper bound K_0 in (3.1). Hence we have

$$\begin{aligned}
&\int_{\Omega_j} \frac{1}{2}|\nabla u_{j,\infty}|^2 + \frac{\mu_j}{4}u_{j,\infty}^4 \\
= \;& (1-\epsilon)\int_{\Omega_j^+} \left[\frac{1}{2}|\nabla v_j^+|^2 + (1-\epsilon)\frac{\mu_j}{4}\left(v_j^+\right)^4\right] + \epsilon\int_{\Omega_j^-} \left[\frac{1}{2}|\nabla v_j^-|^2 + \epsilon\frac{\mu_j}{4}\left(v_j^-\right)^4\right] \\
\geq \;& \int_{\Omega_j^+} \left[\frac{1}{2}|\nabla v_j^+|^2 + \frac{\mu_j}{4}\left(v_j^+\right)^4\right] - 4\epsilon K_j + \frac{1}{2}\epsilon\lambda\left(\Omega_j^-\right) \;\text{(by }\; (3.11), (3.12)) \\
\geq \;& \nu_j + \epsilon\left(\frac{1}{2}K_* - 4K_j\right) ,
\end{aligned}$$

i.e.

$$\int_{\Omega_j} \frac{1}{2}|\nabla u_{j,\infty}|^2 + \frac{\mu_j}{4}u_{j,\infty}^4 \geq \nu_j + \epsilon\left(\frac{1}{2}K_* - 4K_j\right) , \tag{3.13}$$

where $\nu_j = \xi_j\left(\Omega_j^+\right)$, and ξ_j is defined in (1.11). On the other hand, since $u_{j,\infty}$ is the limit function of the energy minimizers $u_{j,\Lambda}$'s, then it is easy to check that

$$\int_{\Omega_j} \frac{1}{2}|\nabla u_{j,\infty}|^2 + \frac{\mu_j}{4}u_{j,\infty}^4 \leq \nu_j \, . \tag{3.14}$$

By (3.13) and (3.14), we may get contradiction and complete the proof of Theorem 1.2 if we set the constant K_* satisfying $K_* > 8K_j$.

4 Verticillate Structures of m Nodal Domains

In this section we study the numerical behavior of phase separation of general m-mixture of BECs for sufficiently large scattering length Λ. Because of phase separation, as the number of multispecies m becomes larger and larger, more and more segregated domains may occur. As in Section 1, a natural question raised here is how these segregated domains distribute when Λ is sufficiently large. It will be shown later in this section by numerical computation that multiple verticillate structures of m $(2 \leq m \leq 33)$ nodal domains occur for m-component ground states.

Recently, a generalization of the normalized gradient flow (NGF) method [4] and the time-splitting spectral method [5] have been developed in [3] for computing the ground state solutions of (1.2) of a multi-component BEC.

Instead, based on the fixed point iteration method [8] we propose a Gauss-Seidel-type iteration method (GSI), which is inspired by the eigenvalue approach for computing the ground states and the other bound states of the multi-component BEC.

Hereafter, we use the bold face letters or symbols to denote a matrix or a vector. For $\mathbf{u} = (u_1, \ldots, u_N)^\top$, $\mathbf{v} = (v_1, \ldots, v_N)^\top \in \mathbb{R}^N$, $\mathbf{u} \circ \mathbf{v} = (u_1 v_1, \ldots, u_N v_N)^\top$ denotes the Hadamard product $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u}^{ⓡ} = \mathbf{u} \circ \cdots \circ \mathbf{u}$ denotes the r-time Hadamard product of $\mathbf{u}$, $[\![\mathbf{u}]\!] := \mathrm{diag}(\mathbf{u})$ the diagonal matrix of $\mathbf{u}$. For $\mathbf{A} \in \mathbb{R}^{M \times N}$, $\mathbf{A} > 0$ (≥ 0) denotes a positive (nonnegative) matrix with positive (nonnegative) entries, $\mathbf{A} \succ 0$ (with $\mathbf{A}^\top = \mathbf{A}$) denotes a symmetric positive definitive matrix.

We now discretize the VGPEs of (1.2) into a nonlinear algebraic eigenvalue problem and derive the discretized version of the associated minimized energy functional problem. We consider the equation (1.2) on a 2-dimensional unit disk $\Omega = \mathbb{D}$ and rewrite the Laplacian operator $-\nabla^2$ on $u_j(\mathbf{x})$ in the polar coordinate system. Based on the recently proposed discretization scheme [20] the standard central finite difference method discretizes $-\Delta \mathbf{u}_j(\mathbf{x})$ into

$$\widehat{\mathbf{A}}\mathbf{u}_j = \widehat{\mathbf{A}}[u_{j1}, \ldots, u_{jl}, \ldots, u_{jN}]^\top, \ \widehat{\mathbf{A}} \in \mathbb{R}^{N \times N}, \tag{4.1}$$

where $\mathbf{u}_j$ is an approximation of the j-th wave function $u_j(\mathbf{x})$ for $j = 1, \ldots, m$. The matrix $\widehat{\mathbf{A}}$ is irreducible and diagonally-dominant with positive diagonal and nonpositive off-diagonal entries. Moreover, $\widehat{\mathbf{A}}$ is symmetrizable to a symmetric positive definitive matrix $\mathbf{A}$ by a positive diagonal matrix $\mathbf{D} > 0$, i.e.,

$$\widehat{\mathbf{A}} = \mathbf{D}^{-1}\mathbf{A}\mathbf{D}, \quad \mathbf{A}^\top = \mathbf{A} \succ 0. \tag{4.2}$$

It can be shown [8] that the square of the l-th diagonal element of $\mathbf{D}$ is equal to the area of the l-th sector corresponding to an integrated partition for $\mathbb{D}$. Applying (4.1) to (1.2) and normalizing each $\mathbf{u}_j$ with respect to $\mathbf{D}^2$, the discretization of VGPEs in (1.2), referred as a nonlinear algebraic eigenvalue problem (NAEP), can be formulated as

$$\mathbf{A}_j(\mathbf{D}\mathbf{u}_j) + \Lambda \sum_{i \neq j} \tilde{\beta}_{ij} \mathbf{u}_i^{②} \circ (\mathbf{D}\mathbf{u}_i) = \lambda_j(\mathbf{D}\mathbf{u}_j), \tag{4.3a}$$

where

$$\mathbf{u}_j^\top \mathbf{D}^2 \mathbf{u}_j = 1, \quad \mathbf{A}_j := \mathbf{A} + 2[\![\mathbf{V}_j + \mu_j \mathbf{u}_j^{②}]\!], \tag{4.3b}$$

for $j = 1, \ldots, m$. Furthermore, the associated optimization problem of (4.3) becomes

$$\begin{aligned} &\underset{\mathbf{u}=(\mathbf{u}_1,\ldots,\mathbf{u}_m)}{\text{Minimize}}\ E(\mathbf{u}) \\ &\text{subject to } \mathbf{u}_j^\top \mathbf{D}^2 \mathbf{u}_j = 1,\ j = 1, \ldots, m. \end{aligned} \tag{4.4a}$$

where

$$E(\mathbf{u}) \equiv \sum_{j=1}^{m} \left(\frac{1}{2} \mathbf{u}_j^\top \mathbf{DADu}_j + (\mathbf{V}_j + \mu_j \mathbf{u}_j^{②})^\top (\mathbf{Du}_j)^{②} \right) + \frac{1}{2} \Lambda \sum_{1 \le j \le i \le m} \tilde{\beta}_{ij} \mathbf{u}_i^{②T} (\mathbf{Du}_j)^{②}. \tag{4.4b}$$

The derivation of (4.3) and (4.4) can be found in [8].

Define the set

$$\mathcal{M} = \{\mathbf{v} \in \mathbb{R}^N | \mathbf{v}^\top \mathbf{D}^2 \mathbf{v} = 1,\ \mathbf{v} \ge 0\}, \quad \mathring{\mathcal{M}} = \text{ interior of } \mathcal{M}. \tag{4.5}$$

For convenience, we now suppose that

$$\tilde{\beta}_{ji} = \tilde{\beta}_{ij} > 0\ (j \ne i), \quad j, i = 1, \ldots, m. \tag{4.6}$$

For any given $\mathbf{V}_j \ge 0$ and $(\mathbf{u}_1, \ldots, \mathbf{u}_m) \in \overset{m}{\underset{j=1}{\times}} \mathcal{M}$, the matrix

$$\bar{\mathbf{A}}_j \equiv \mathbf{A}_j + 2[\![\mathbf{V}_j]\!] + \Lambda \sum_{i \ne j} [\![\tilde{\beta}_{ij} \mathbf{u}_i^{②}]\!] \tag{4.7}$$

is an irreducible M-matrix. By the Perron-Frobenius Theorem (see e.g., [6]) there is a unique positive eigenvector $\mathbf{D}\bar{\mathbf{u}}_j > 0$ with $\bar{\mathbf{u}}_j^\top \mathbf{D}^2 \bar{\mathbf{u}}_j = 1$ corresponding to the maximal eigenvalue $\omega_j^{\max}$ of $\bar{\mathbf{A}}_j^{-1}$ which satisfies

$$\bar{\mathbf{A}}_j (\mathbf{D}\bar{\mathbf{u}}_j) = \left(\mathbf{A}_j + \Lambda \sum_{i \ne j} [\![\tilde{\beta}_{ij} \mathbf{u}_i^{②}]\!] \right) (\mathbf{D}\bar{\mathbf{u}}_j) = \lambda_j^{\min} (\mathbf{D}\bar{\mathbf{u}}_j), \tag{4.8}$$

where $\lambda_j^{\min} = 1/\omega^{\max}$, $j = 1, \ldots, m$.

Define a function $\mathbf{f} : \overset{m}{\underset{j=1}{\times}} \mathcal{M} \to \overset{m}{\underset{j=1}{\times}} \mathcal{M}$ by

$$\begin{aligned} \mathbf{f}(\mathbf{u}_1, \ldots, \mathbf{u}_m) &\equiv (\mathbf{f}_1(\mathbf{u}_1, \ldots, \mathbf{u}_m), \ldots, \mathbf{f}_m(\mathbf{u}_1, \ldots, \mathbf{u}_m)) \\ &= (\bar{\mathbf{u}}_1, \ldots, \bar{\mathbf{u}}_m), \end{aligned} \tag{4.9}$$

where $\bar{\mathbf{u}}_j > 0$ is well-defined by (4.8) for $j = 1, \ldots, m$. We now construct a Gauss-Seidel-type mapping $\mathbf{g} : \bigtimes_{j=1}^{m} \mathcal{M} \to \bigtimes_{j=1}^{m} \mathcal{M}$ by

$$\mathbf{g}(\mathbf{u}_1, \ldots, \mathbf{u}_m) = (\bar{\mathbf{u}}_1, \ldots, \bar{\mathbf{u}}_m), \tag{4.10a}$$

where

$$\begin{aligned}
\bar{\mathbf{u}}_1 &= \mathbf{g}_1(\mathbf{u}_1, \ldots, \mathbf{u}_m) = \mathbf{f}_1(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m), \\
\bar{\mathbf{u}}_2 &= \mathbf{g}_2(\mathbf{u}_1, \ldots, \mathbf{u}_m) = \mathbf{f}_2(\bar{\mathbf{u}}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m), \\
&\ \ \vdots \qquad\qquad\qquad\qquad \vdots \\
\bar{\mathbf{u}}_m &= \mathbf{g}_m(\mathbf{u}_1, \ldots, \mathbf{u}_m) = \mathbf{f}_m(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_{m-1}, \mathbf{u}_m),
\end{aligned} \tag{4.10b}$$

with $\{\mathbf{f}_j\}_{j=1}^m$ as given by (4.9). The mapping $\mathbf{g}$ in (4.10) can be used to naturally define a Gauss-Seidel-type iteration (GSI). The following Theorem from [8] gives a necessary and sufficient condition for the convergence of the above GSI.

Theorem 4.1 ([8]) *Suppose that* $\mu_j (j = 1, \ldots, m)$ *in* (4.3) *are sufficiently small positive numbers. Let* $(\boldsymbol{\lambda}^*, \mathbf{u}^*) = ((\boldsymbol{\lambda}_1^*, \ldots, \boldsymbol{\lambda}_m^*), (\mathbf{u}_1^*, \ldots, \mathbf{u}_m^*))$ *be a fixed point of* (4.3) *satisfying* (4.6)*. The GSI method defined by* (4.10) *converges to* $(\boldsymbol{\lambda}^*, \mathbf{u}^*)$ *locally and linearly if and only if* $\mathbf{u}^* = (\mathbf{u}_1^*, \ldots, \mathbf{u}_m^*)$ *is a strictly local minimum of* (4.4)*.*

We simulate the multi-component BECs from $m = 2$ to $m = 33$ by using GSI method in (4.10). By Theorem 4.1 the GSI method can converge to a bound state or a ground state solution of (4.3) which depends on whether the associated energy is the smallest one.

It is well-known that when the scattering length $\Lambda = 0$ in (4.3) the NAEP of (4.3) is decoupled and have m identical ground state solutions. On the other hand, by Theorem 1.1 when $\Lambda \to \infty$ the VGPEs have m disjoint ground state solutions. We now compute the energy state solutions of (4.3) by GSI method, taking Λ as a parameter varying from 0 to 10^6.

In the numerical simulation, we first show that for a fixed m there is a $\Lambda_1(m) > 0$ (dependent on m) such that the NAEP (4.3) have only identical ground state solutions for $0 \le \Lambda < \Lambda_1(m)$, and a bifurcation occurs at $\Lambda = \Lambda_1(m)$, so that some ground state solutions begin to separate and some ground state solutions are still identical, for $\Lambda > \Lambda_1(m)$. Since $\Lambda > 0$ is a repulsive scattering length, it is expected that the ground state solutions of (4.3) should be mutually separated when Λ is continually increased. We continue this process and observe that there is a second bifurcation point $\Lambda_2(m)$ ($\Lambda_2(m) > \Lambda_1(m)$) so that more ground state solutions separate. We finally reach a bifurcation point $\Lambda^*(m)$ so that the ground state solutions of (4.3) have a phase separation, for $\Lambda > \Lambda^*(m)$. As we continue increasing Λ beyond $\Lambda^*(m)$, the structure of the phase separation will stay unchanged and reach a stage of totally disjoint phases, when Λ approaches to 10^6 (a value common to all m). In the above general bifurcation process, it is helpful to point out that the final stage is reached via a sequence of transition intervals such as $[0, \Lambda_1(m)], [\Lambda_1(m), \Lambda_k(m)], [\Lambda_k(m), \Lambda^*(m)]$ and so on. The number k may take on one, two and so on.

We observe that at the final stage the disjoint phases have a verticillate or multiple verticillate structure which depends on m, the number of components in BECs. We now elaborate on the verticillate structures using the following parameters. For a given positive integer $n_1 > 0$ with $n_1 \leq m$, we use the index (n_1) to denote the verticillate structure of the unit disk that is partitioned uniformly by the n_1 supports of the ground state solutions. In general, for a given sequence of positive integers $0 < n_1 < n_2 < \cdots < n_r$ with $\sum_{i=1}^r n_i \leq m$ and a sequence of concentric disks $D_1 \subset D_2 \subset \cdots \subset D_r := \mathbb{D}$, we define the index $(n_1, \ldots, n_r)$ to describe the multiple verticillate structure of the unit disk in which the n_i supports of the bound state solutions uniformly partition the ring $D_i \backslash D_{i-1}, 1 \leq i \leq r$ with D_0 being the empty set. In short,

$$\begin{aligned} (n_1) &:= \text{an } n_1\text{-verticillate structure of the phase separation,} \\ (n_1, \ldots, n_r) &:= \text{an } (n_1, \ldots, n_r)\text{–multiple verticillate structure of the} \\ &\quad\ \text{phase separation.} \end{aligned}$$

In Figure 4.1(a)-(e) we plot the energy of ground states or bound states versus the number m of components in BECs. Here the energies are computed by (4.4b). We denote by "$*$" the minimal energy and by "$\diamond$" the excited energy. A proper index for the verticillate or multiple verticillate structure of the phase separations is indicated near a "$*$" or "$\diamond$". For $m = 2, \ldots, 5$, we observe that the ground states have (m)-verticillate structures and m equal nodal domain Ω_j's, where the tops of $\mathbf{u}_j, j = 1, \ldots, m$ form the vertices of a m-polygon. Furthermore, two bound states have (1,3)-and (1,4)-verticillate structures, respectively, for $m = 4$ and 5. As $m = 6, 7, 8$, a new structure for ground states emerges where one nodal domain Ω_{j_0} occupies the center of Ω and the rest $m-1$ nodal domains equally distribute around the outside of Ω_{j_0}. For $m=6$ (7 or 8), we observe that a double verticillate structure (1,5) ((1,6) or (1,7)) for ground states and the single verticillate structure (6) ((7 or (8))) become bound state solutions. As $m=9,10,11$, two nodal domains Ω_{j_1} and Ω_{j_2} locate near the center of Ω and the rest $m-2$ nodal domains equally distribute around the outside of Ω_{j_1} and Ω_{j_2}. As m increases from 12 to 16, three, four, and five nodal domains may occur near the center of Ω and the rest nodal domains equally distribute the rest of domain Ω. Basically, for these cases, the tops of $\mathbf{u}_j$ in the nodal domains are located at vertices of two eccentric polygons. We term this change of verticillate structures as a *verticillate doubling*. It is naturally expected that we should have verticillate tripling or quadrupling for ground states where m increases. More precisely, as $m=17,18,\ldots,21$, one nodal domain Ω_{j_0} begin to occupy the center of Ω, respectively, 6, 5, 5, 7, 7 nodal domains $\Omega_{j_1},\ldots,\Omega_{j_r}$ (say!) equally distribute around the outside of Ω_{j_0} and, respectively, the rest 10, 12, 13, 12, 13 nodal domains equally distribute around the outside of $\Omega_{j_1},\ldots,\Omega_{j_r}$. Similarly, in Figure 4.1(d) and (e) we observe the triple and quadruple verticillate structures of nodal domains for $22\leq m\leq 33$. Especially, in Figure 4.1(c) and (e), respectively, we observe that there is a *verticillate tripling* at $m=17$ and a *verticillate quadrupling* at $m=32$.

Furthermore, Theorem 4.1 shows that GSI method can converge to different local minima of the optimization problem (4.4). In Figure 4.1 we see that there is only one local minimum, i.e., one unique global minimum of (4.4) for $m=2$ or 3, but there exist other local minimums of (4.4) for $m\geq 4$ which are denoted by "$\diamondsuit$". In Figure 4.1(b), we even find that there exist the other two local minimums of (4.4) for $m=9,\ 10$ and 12.

In order to understand the different patterns of multiple verticillate structures for the ground state and the bound state solutions, in Figure 4.2 and Figure 4.3 we plot the nodal domains for the ground state and bound state solutions with associated energies, for $m=5$ and $m=6$, respectively. We observe that for $m=5$, the ground state has a (5)-verticillate structure with energy $=15.81$ and a bound state has a (1,4)-verticillate structure with energy $=16.22$; however, for $m=6$, the ground state has a (1,5)-verticillate structure with energy $=18.06$ and a bound state has a (6)-verticillate structure with energy $=19.15$. A *verticillate doubling* occurs firstly here at $m=6$. In addition, in our simulation we notice that the number n_1 for the first verticillate structure on D_1 cannot be larger than five. We conclude from Figure 4.1 that one more verticillate multiplying for the ground state solutions will occur when a $n_1 > 5$ is experienced.

For the sake of comprehension of the distribution of multiple verticillate structures of all nodal domains for ground states, in Figure 4.4, we plot the nodal domains for $m = 2, \dots, 33$ with sufficiently large and positive repulsive scattering length $\Lambda \approx 10^6$. The Figure here shows that m segregated nodal domains of m nodal domains of m-mixture of BECs are clearly separated by $\Lambda \approx 10^6$. We see that as the number m becomes larger and larger, the distribution of the nodal domains is arranged in whorls more and more, and then the ringlike levels are getting increasing. In the Figure 4.4, we observe that a *verticillate doubling*, *tripling* and *quadrupling* occurs at $m = 6, 17$ and 32, respectively.

To study the numerical behavior of the energy versus the repulsive scattering length Λ we consider the case of nine-component BECs (m=9) and plot its bifurcation diagram in Figure 4.5. In this case that $m = 9$, we find that there are four different kinds of verticillate structures for bound states with various Λ that is enough to illustrate the verticillate structures of a general m. In our numerical result, we observe that the VGPEs have only identical ground state solutions, i.e., (1)-verticillate structure for $\Lambda < \Lambda_1(9)$, and bifurcate into the $(1, 7)$-verticillate structures, for $\Lambda_1(9) \le \Lambda$, where there are two identical components on D_1 and seven component solutions uniformly partition the ring $D_2 \setminus D_1$. Note that here $D_1 \subset D_2 := \mathbb{D}$ are two concentric disks. The $(1, 7)$-verticillate ground state solutions of VGPEs again bifurcate at $\Lambda = \Lambda_2(9)$ into the $(2, 7)$-verticillate structure for bound states and the $(1, 8)$-verticillate structure for ground states, for $\Lambda \ge \Lambda_2(9)$. In fact, both of these two bound state solutions are the local minimums of the optimization problem (4.4a). The associated nodal domains of these four kinds of verticillate structures are attached near the energy curve in Figure 4.5. Notice that the dash line in Figure 4.5 means that the (9)-verticillate structures are computed by the GSI method with some artificial constraints [8]. Without these constraints the GSI method always converges to either the $(1, 8)$- or the $(2, 7)$-verticillate structure locally and linearly.

We now consider VGPE of BEC coupled only with equal neighboring repulsive scattering lengths. The corresponding NAEP as in (4.3) can be simplified by

$$\mathbf{A}_j(\mathbf{D}\mathbf{u}_j) + \Lambda[\![\mathbf{u}_{j+1}]\!]^{②}(\mathbf{D}\mathbf{u}_j) + \Lambda[\![\mathbf{u}_{j-1}]\!]^{②}(\mathbf{D}\mathbf{u}_j) = \lambda_j \mathbf{D}\mathbf{u}_j, \tag{4.11a}$$

where

$$\mathbf{u}_j^\top \mathbf{D}^2 \mathbf{u}_j = 1, \quad \mathbf{A}_j = \mathbf{A} + 2[\![\mathbf{V}_j]\!], \tag{4.11b}$$

for $j = 1, \ldots, m$.

Since the local coupled VGPEs are simpler than the globally coupled VGPEs (4.3), no transition stage occurs by computation. Numerical result shows that there is a $\Lambda_1(m) > 0$ such that the NAEP (4.11) have only identical ground state solutions when $0 \leq \Lambda < \Lambda_1(m)$ and have a phase separation of the ground state solution when $\Lambda \geq \Lambda_1(m)$. Furthermore, if m is odd, then we have an (m)-verticillate structure of the ground state solutions; if m is even, then we have a (2)-verticillate structure of the ground state solutions, i.e., m ground state solutions separate disjointedly into two groups of $m/2$ identical solutions when Λ approached to 10^6. In this case, the structure changes only once from identical solutions to phase separations and the convergence of GSI is relatively fast.

5 Conclusions

In this paper, we have studied the distribution of m segregated nodal domains of the m-mixture of BECs under positive and large repulsive scattering lengths. We showed rigorously that the components of positive bound states may repel each other and form segregated nodal domains as the repulsive scattering lengths go to infinity. By numerical computations, we observed a new phenomenon: verticillate multiplying, i.e., the generation of multiple verticillate structures, when the number of the first verticillate structure is larger than five. In addition, we have created new techniques that are quite different from the existing methods [3], and our proposed Gauss-Seidel-type iteration method is very effective in that it converges always linearly in just 10 to 20 steps.

References

[1] N. Akhmediev and A. Ankiewicz, Partially coherent solitons on a finite background, *Phys. Rev. Lett.* **82**(1999) 2661(1-4).

[2] P. Ao and S. T. Chui, Binary Bose-Einstein condensate mixtures in weakly and strongly segregated phases, *Phys. Rev. A* **58** (1998), 4836-4840.

[3] W. Z. Bao, Ground states and dynamics of multi-component Bose-Einstein condensates, preprint.

[4] W. Z. Bao and Q. Du, Computing the ground state solution of Bose-Einstein condensates by a normalized gradient flow, preprint.

[5] W. Z. Bao and D. Jaksch, An explicit unconditionally stable numerical methods for solving damped nonlinear Schrodinger equations with a focusing nonlinearity, *SIAM J. Numer. Anal*, to appear.

[6] A. Berman and R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Academic Press, New York, 1979.

[7] D. Bucur, G. Buttazzo and A. Henrot, Existence results for some optimal partition problems, *Advances in Math. Sciences Appl.*, **8** (1998) no. 2, 571-579.

[8] S. M. Chang, W. W. Lin and S. F. Shieh, Fixed point methods for energy states of multi-component bose-einstein condensates, submitted.

[9] K. W. Chow, Periodic solutions for a system of four coupled nonlinear Schrodinger equations, *Phys. Lett. A* **285** (2001) 319-V326.

[10] A. Erdélyi, Higher transcendental functions II, McGraw-Hill, New York, 1953.

[11] B. D. Esry, C. H. Greene, J. P. Burke Jr. and J. L. Bohn, Hartree-Fock theory for double condensates, *Phys. Rev. Lett.* **78** (1997), 3594-3597.

[12] B. D. Esry and C. H. Greene, Spontaneous spatial symmetry breaking in two-component Bose-Einstein condensates, *Phys. Rev. A* **59** (1999), 1457-1460.

[13] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order.

[14] S. Gupta, Z. Hadzibabic, M. W. Zwierlein, C. A. Stan, K. Dieckmann, C. H. Schunck, E. G. M. van Kempen, B. J. Verhaar and W. Ketterle, Radio-Frequency Spectroscopy of Ultracold Fermions, *Science* **300** (2003), 1723-1726.

[15] D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman and E. A. Cornell, Dynamics of component separation in a binary mixture of Bose-Einstein condensates, *Phys. Rev. Lett.* **81** (1998), 1539-1542.

[16] F. T. Hioe, Solitary Waves for N Coupled Nonlinear Schrödinger Equations, *Phys. Rev. Lett.* **82**(1999) 1152-1155.

[17] F. T. Hioe, T. S. Salter, Special set and solutions of coupled nonlinear schrödinger equations, *J. Phys. A: Math. Gen.* **35** (2002) 8913-8928.

[18] T. Kanna and M. Lakshmanan, Exact soliton solutions, shape changing collisions, and partially coherent solitons in coupled nonlinear Schrödinger equations, *Phys. Rev. Lett.* **86** (2001) 5043(1-4).

[19] B. Kawohl, Remarks on some old and current eigenvalue problems, in *Partial Differential Equations of Elliptic type*, pp. 165-183, A. Alvino, E. Fabes, G. Talenti eds, Symposia Mathematica, vol. XXXV, Cambridge University Press, 1994.

[20] M.-C. Lai, A note on finite difference discretizations for poisson equation on a disk, *Numerical Methods for Partial Differential Equations* **17** (2001) no. 3, 199–203.

[21] C. S. Lin, W. M. Ni and I. Takagi, Large amplitude stationary solutions to a chemotaxis system, *J. Diff. Eqns.* **72** (1988), 1-27.

[22] C. J. Myatt, E. A. Burt, R. W. Ghrist, E. A. Cornell and C. E. Wieman, Production of two overlapping Bose-Einstein condensates by sympathetic cooling, *Phys. Rev. Lett.* **78** (1997), 586-589.

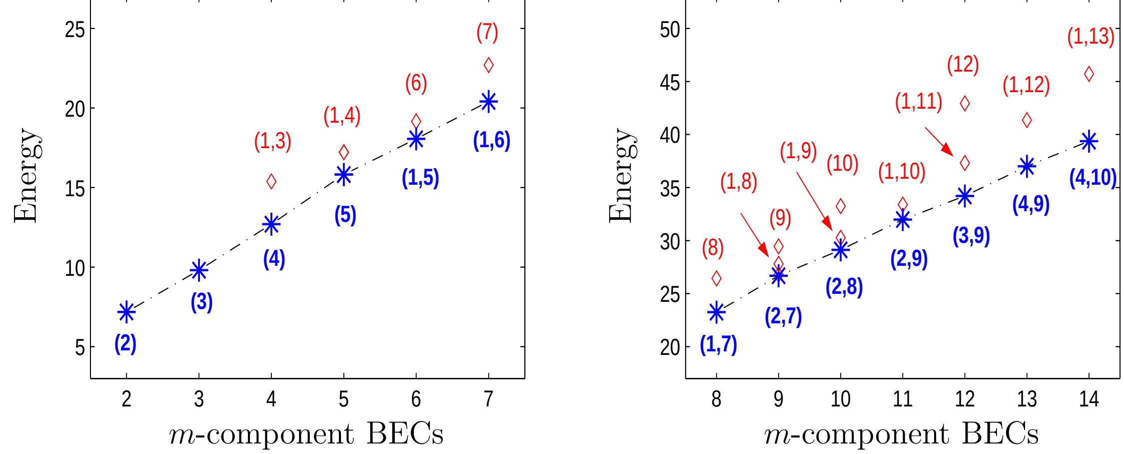
[23] A. S. Parkins and D. F. Walls, The Physics of trapped dilute-gas Bose-Einstein condensates, *Phys. Reports* **303** (1998), 1-80.

[24] Ch. Rüegg, N. Cavadini, A. Furrer, H.-U. Güdel, K. Krämer, H. Mutka, A. Wildes, K. Habicht and P. Vorderwisch, BoseVEinstein condensation of the triplet states in the magnetic insulator TlCuCl3, *Nature* **423**(2003), 62-65.

[25] E. Timmermans, Phase separation of Bose-Einstein condensates, *Phys. Rev. Lett.* **81** (1998), 5718-5721.

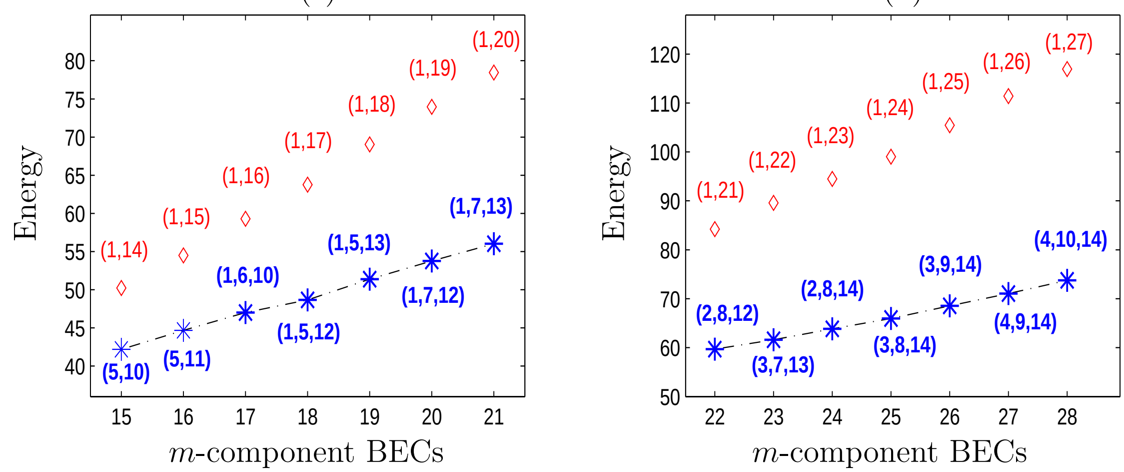
[26] M. Trippenbach, K. Góral, K. Rzążewski, B. Malomed, Y B Band, Structure of binary BoseVEinstein condensates, *J. Phys. B: At. Mol. Opt. Phys.* **33** (2000) 4017V4031.

[27] C. Yeh, L. Bergman, Enhanced pulse compression in a nonlinear fiber by a wavelength division multiplexed optical pulse, *Phys. Rev. E* **57** (1998), 2398-2404.

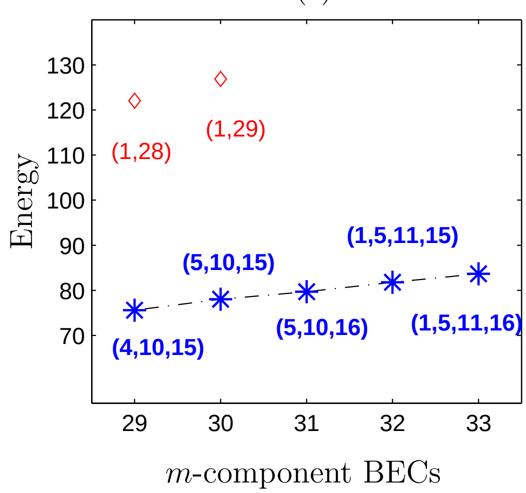


(c)

(d)



(e)



(i) Single verticillate (2) occurring at $m = 2$,

(ii) Double verticillate (1,5) occurring at $m = 6$,

(iii) Triple verticillate (1,6,10) occurring at $m = 17$,

(iv) Quadruple verticillate (1,5,11,15) occurring at $m = 32$.

Figure 4.1: Energy vs the number of components.

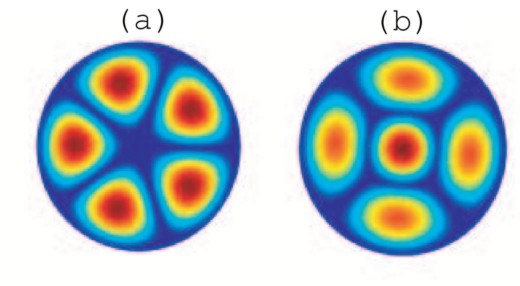


Figure 4.2: $m = 5$: (a) Ground state solutions with energy = 15.81, (b) bound state solutions with energy = 16.22.

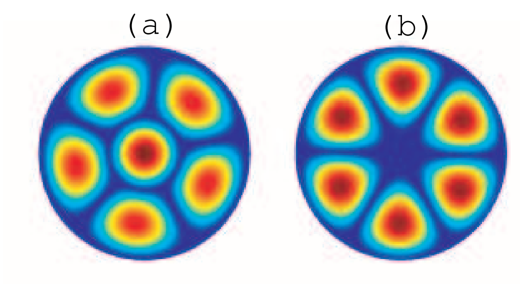


Figure 4.3: $m = 6$: (a) Ground state solutions with energy = 18.06, (b) bound state solutions with energy = 19.15.

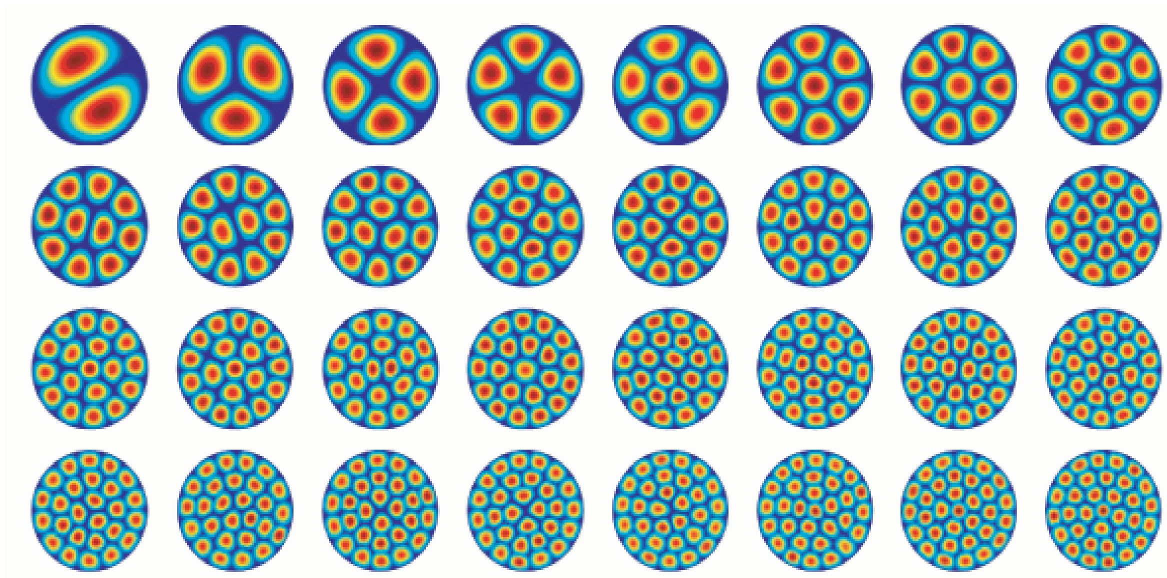


Figure 4.4: Nodal domains for $m = 2, \ldots, 33$ with $\Lambda \approx 10^6$.

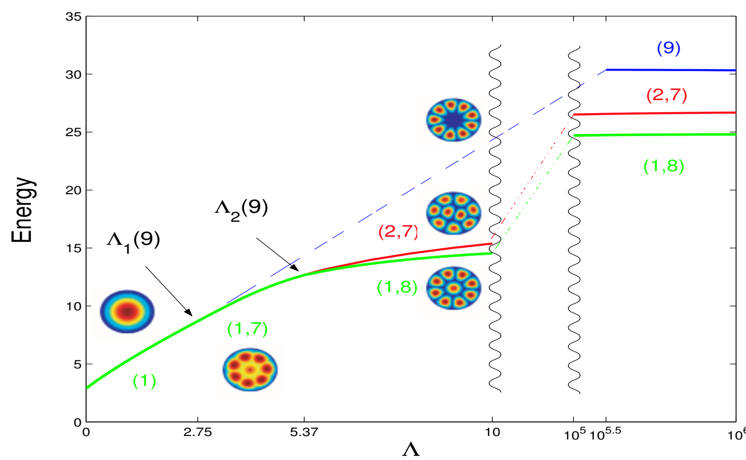


Figure 4.5: $m = 9$: Energy curves vs Λ.