Applying Snapback Repellers in Ecology

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Outline

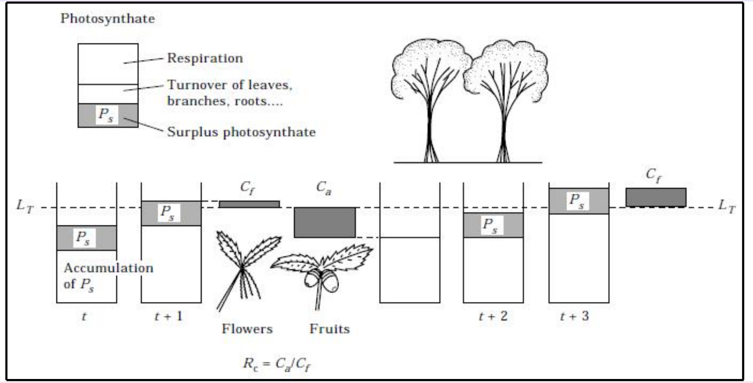
Generalized Resource Budget Model

A. Satake and Y. Iwasa, *Pollen Coupling of Forest Trees: Forming Synchronized and Periodic Reproduction out of Chaos*, **J. theor. Biol.**, Vol. 203 (2000), 63–84.

Resource Budget Model

Y. Isagi, K. Sugimura, A. Sumida and H. Ito, *How does masting happen and synchronize?*, **J. theor. Biol.**, Vol. 187 (1997), 231–239.

Isagi's Resource Budget Model



- $S^{(t)}$: Amount of energy at the beginning of year t.
- P_s: Photosynthate (constant from year to year).
- L_T: Resource threshold.
- C_f: Flowering energy.
- C_a: Fruiting energy.
- R_c: The ratio of C_a/C_f.

Satake's Generalized Resource Budget Model

Main problem

$$Y^{(t+1)} = \begin{cases} Y^{(t)} + 1, & \text{if } Y^{(t)} \leq 0, \\ -kY^{(t)} + 1, & \text{if } Y^{(t)} > 0, \end{cases}$$

where k: depletion coefficient $(k = a(R_c + 1) - 1)$,
$t = 0, 1, 2, \ldots$.

Non-dimensionalization

$S^{(t)}$: energy reserved at the t-th year

Isagi's Resource Budget Model

$$S^{(t+1)} = \begin{cases} S^{(t)} + P_s, & \text{if } S^{(t)} + P_s \leq L_T, \\ S^{(t)} + P_s - C_f - C_a, & \text{if } S^{(t)} + P_s > L_T. \end{cases}$$

Let $a > 0, \quad C_f \equiv a(S^{(t)} + P_s - L_T)$

Satake's Generalized Resource Budget Model

$$S^{(t+1)} = \begin{cases} S^{(t)} + P_s, \\ S^{(t)} + P_s - a(R_c + 1)(S^{(t)} + P_s - L_T). \end{cases}$$

Non-dimensionalization

$$
\begin{aligned}
S^{(t+1)} &= \begin{cases} S^{(t)} + P_s, \\ S^{(t)} + P_s - a(R_c+1)(S^{(t)} + P_s - L_T) \end{cases} \\
&= \begin{cases} S^{(t)} + P_s, \\ S^{(t)} + P_s {\color{red}-L_T} - a(R_c+1)(S^{(t)} + P_s - L_T) {\color{blue}+L_T} \end{cases} \\
&= \begin{cases} S^{(t)} + P_s, \\ (1 - a(R_c+1))\left(S^{(t)} + P_s - L_T\right) + L_T. \end{cases}
\end{aligned}
$$

Non-dimensionalization

$$S^{(t+1)}+P_s-L_T = \begin{cases} (S^{(t)}+P_s-L_T)+P_s, \\ (1-a(R_c+1))\,(S^{(t)}+P_s-L_T) \\ \qquad +L_T-L_T+P_s, \end{cases}$$

$$\left(S^{(t+1)}+P_s-L_T\right)/P_s = \begin{cases} (S^{(t)}+P_s-L_T)/P_s+1, \\ (1-a(R_c+1))\,(S^{(t)}+P_s-L_T)/P_s+1. \end{cases}$$

Let $Y^{(t)} = (S^{(t)}+P_s-L_T)/P_s, \quad k \equiv a(R_c+1)-1.$

Coupled Systems

$$Y_i^{(t+1)} = \begin{cases} Y_i^{(t)} + 1 & \text{if } Y_i^{(t)} \leq 0, \\ -\kappa P_i^{(t)} Y_i^{(t)} + 1 & \text{if } Y_i^{(t)} > 0, \end{cases}$$

where

$$P_i^{(t)} = \left\{ \frac{1}{N-1} \sum_{j \neq i}^{N} [Y_j^{(t)}]_+ \right\}^{\beta}.$$

- Satake and Iwasa proved by computing the positive Lyapunov exponent that if the depletion coefficient k is greater than one, then the generalized budget resource model is chaotic. However, a positive Lyapunov exponent means only sensitivity in Devaney's chaos.
- When the depletion coefficient k is a positive integer, Satake and Iwasa proved that the generalized budget resource model is periodic.

$$\downarrow$$

1. Snapback repeller method $\Rightarrow$ Devaney's chaos.
2. Investigate the difference between odd depletion coefficients and even depletion coefficients.

References

- F. R. Marotto, *Snap-Back Repellers Imply Chaos in* $\mathbb{R}^n$, **Math. Anal. Appl.**, Vol. 63 (1978), 199–223.
- F. R. Marotto, *On Redefining a Snap-Back Repeller*, **Chaos, Solitons and Fractals**, Vol. 25 (2005), 25–28.

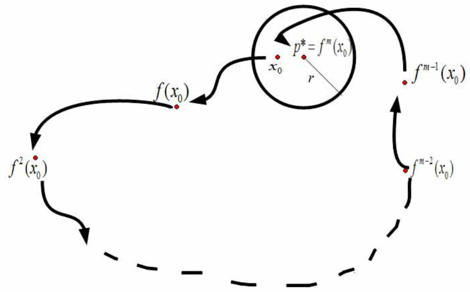
expanding fixed point

Let $p^* \in \mathbb{R}^n$, suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ be diff. in $B_r(p^*)$, if $f(p^*) = p^*$ and $|\sigma(Df(x))| > 1 \ \forall\, x \in B_r(p^*)$.

snapback repeller

Suppose p^* is an *expanding fixed point* of f in $B_r(p^*)$ for some $r > 0$ and $\exists$ const. $s > 0$ s.t. $\|f(x) - f(y)\| > s\|x - y\| \ \forall$ $x, y \in B_r(p^*)$, if $\exists\ x_0 \in B_r(p^*)$ with $x_0 \neq p^*$ and $m \in \mathbb{N}$ s.t. $f^m(x_0) = p^*$ and $\det(Df^m(x_0)) \neq 0$.

Technique



Theorem

Let snapback repeller p^*, f, m, and x_0 be the same as above. If f is C^1 in some neighborhood of x_j, $\det(Df(x_j)) \neq 0$, $0 \leq j \leq m-1$, and f has a snapback repeller p^*, then f is chaotic in the sense of Devaney.

- Y. Shi and G. Chen. Chaos of discrete dynamical systems in complete metric spaces. Chaos, Solitons and Fractals, 22:555-571, 2004.
- Y. Shi and G. Chen. Discrete chaos in Banach spaces. Science in China Ser. A Mathematics, 48(2):222-238, 2005.
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- M. C. Li, M. J. Lyu and P. Zgliczyński. Topological entropy for multidimensional perturbations of snap-back repellers and one-dimensional maps. Nonlinearity, 21:2555-2567, 2008.
- Y. Shi and P. Yu. Chaos induced by regular snap-back repellers. J. Math. Anal. Appl., 337:1480-1494, 2008.
- M. C. Li and M. J. Lyu. A simple proof for persistence of snap-back repellers. Journal of Mathematical Analysis and Applications, 352:669-671, 2009.
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Devaney's Chaos

Let X be a metric space, $f\colon X \to X$ conti.

sensitivity $\exists\,\delta > 0$ st. $\forall\, x \in X$, any nbd(x), $\exists\, y \in \mathrm{nbd}(x)$ and $n \in \mathbb{N}$ such that $|f^{\,n}(x) - f^{\,n}(y)| > \delta$;

transitivity for any pair of nonempty open sets $U, V \subset X$, $\exists\, k > 0$ st. $f^{\,k}(U) \cap V \neq \emptyset$;

density of periodic points

Li & Yorke's Chaos

Let I be an interval, $f : I \to I$ conti., if f has an uncountable scrambled set $S \subset I$ which satisfies the following conditions:

(i) $\forall\, p, q \in S$ with $p \neq q$,

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0,$$
$$\liminf_{n\to\infty} |f^n(p) - f^n(q)| = 0;$$

(ii) $\forall\, p \in S$ and periodic point $q \in I$,

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0.$$

Topological Entropy

$f: X \to X$ conti. with metric d. $S \subset X$: (n, ϵ)-separated for f with $n \in \mathbb{Z}^+$ and $\epsilon > 0$ provided that for every pair of distinct points $x, y \in S$, $x \neq y$, there is at least one k with $0 \leq k < n$ st. $d(f^k(x), f^k(y)) > \epsilon$. The number of different orbits of length n (as measured by ϵ) is defined by

$$r(n, \epsilon, f) = \{\#(S) : S \subset X \text{ is a } (n, \epsilon)\text{-separated set for f }\},$$

where $\#(S)$ is the cardinality of elements in S. Let

$$h_{\text{top}}(\epsilon, f) = \limsup_{n \to \infty} \frac{\log(r(n, \epsilon, f))}{n},$$

and define the **topological entropy** of f as

$$h_{\text{top}}(f) = \lim_{\epsilon \to 0, \epsilon > 0} h_{\text{top}}(\epsilon, f).$$

Theorem

Assume $f\colon X \to X$ is uniformly continuous or X is compact and $n \in \mathbb{N}$. Then $h_{\mathrm{top}}(f^n) = n \cdot h_{\mathrm{top}}(f)$.

C. Robinson

Dynamical Systems: Stability, Symbolic Dynamics, and Chaos, 2nd Ed., CRC, Boca Raton, Florida, 1998.

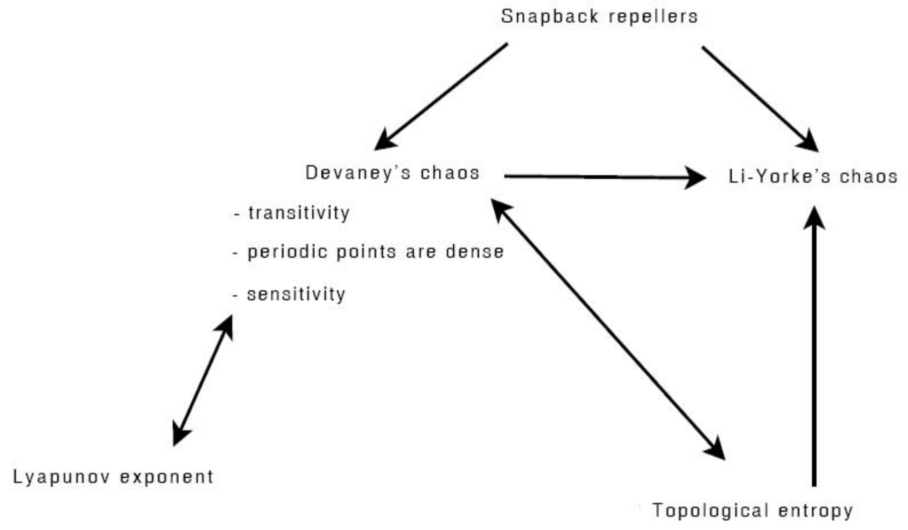
Lyapunov Exponent

Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function. For each point x_0, define the **Lyapunov exponent** of x_0, $\lambda(x_0)$, as follows:

$$\begin{aligned} \lambda(x_0) &= \limsup_{n\to\infty} \frac{1}{n} \log(|(f^n)'(x_0)|) \\ &= \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log(|f'(x_k)|), \end{aligned}$$

where $x_j = f^j(x_0)$.

Important relations



$$Y^{(t+1)} = \begin{cases} Y^{(t)} + 1, & \text{if } Y^{(t)} \leq 0, \\ -kY^{(t)} + 1, & \text{if } Y^{(t)} > 0. \end{cases}$$

$k \leq -1$

$Y^{(t)}$ tends to infinity.

$-1 < k < 1$

$Y^{(t)}$ converges to the stable equilibrium $\dfrac{1}{k+1}$.

$k = 1$

$Y^{(t)}$ converges to a period 2.

Main results

(I) $k > \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \equiv k_1$

the system is chaotic in Devaney's sense.

(II) $1 < k \leq k_1 \approx 1.3247$

the system is chaotic in Devaney's sense, too.

$$Y^n(x) = (Y \circ \cdots \circ Y)(x)$$

$p \in \mathbb{N} \cup \{0\}$

$$Y^{2^p}(x) = \begin{cases} L_{2^p}(x), & x \in \left[C_{p-3}\left(\frac{1}{k}\right), C_{p-2}\left(\frac{1}{k}\right)\right], \\ R_{2^p}(x), & x \in \left[C_{p-2}\left(\frac{1}{k}\right), 1\right], \end{cases}$$

$$L_1(x) = x + 1, \; R_1(x) = -kx + 1,$$

$$R_{2^p}(x) = (L_{2^{p-1}} \circ R_{2^{p-1}})(x),$$

$$L_{2^p}(x) = \begin{cases} -kR_{2^p}(x) + k + 1, & p \in \text{odd}, \\ \dfrac{-R_{2^p}(x) + k + 1}{k}, & p \in \text{even}. \end{cases}$$

$j \in \mathbb{N}$,

$$C_j = \begin{cases} C_{j-1} \circ A \circ A \circ C_{j-1}, & j \in \text{odd}, \\ C_{j-1} \circ B \circ C_{j-1}, & j \in \text{even} \end{cases}$$

with

$$C_0(x) = B(x),$$

$$C_{-1}(x) = x, C_{-2} = 0, C_{-3} = -k + 1,$$

where

$$A(x) = \frac{1}{k}(1 - x), \quad B(x) = \frac{1}{k}(2 - x).$$

$p = 0, 1$

$$k_0 = \frac{1+\sqrt{5}}{2} \approx 1.6180$$

$$Y(x) = \begin{cases} x+1, & x \in [-k+1,\ 0], \\ -kx+1, & x \in [0,\ 1]; \end{cases}$$

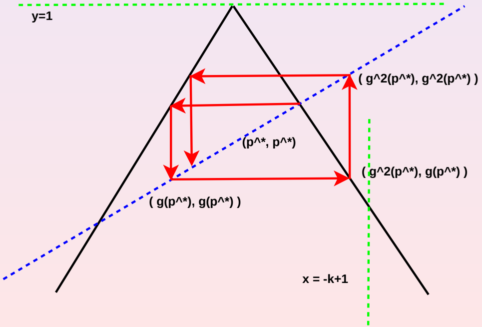
$$k_1 \approx 1.3247$$

$$Y^2(x) = \begin{cases} k^2 x - k + 1, & x \in \left[0,\ \frac{1}{k}\right], \\ -kx+2, & x \in \left[\frac{1}{k},\ 1\right]. \end{cases}$$

Proof:

Part I: $k > \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \approx 1.3247.$

Let $p^* = \frac{1}{1+k}$, $g = Y^{-1}$



Main Result (I)

Since $|g'(p^*)| < 1$, there exists $r > 0$ with $U = (p^* - r, p^* + r)$, $U \subset (0,1)$ such that $\lim_{m\to\infty} g^m(x) = p^*$ if $x \in U$. Choose

$$g(p^*) = \frac{-k}{1+k} < 0 \quad \text{and} \quad g^2(p^*) = \frac{2k+1}{k^2+k} > 0.$$

Solve $g(p^*) > -k+1$ and $g^2(p^*) < 1$, choosing $k > \frac{1+\sqrt{5}}{2}$ allows j to be found such that

$$g^j(p^*) > 0 \quad \text{for all } j \geq 3$$

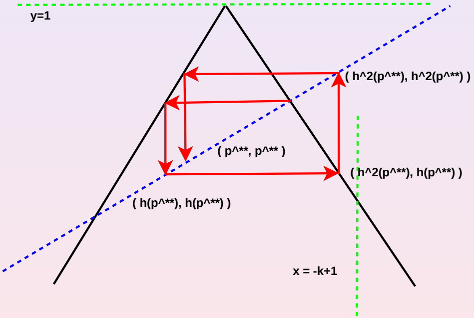
by the definition of Y. Computing $|g^j(p^*) - p^*|$, yield $|g^j(p^*) - p^*| = \frac{1}{k^{j-1}} \to 0$ as $j \to \infty$. That is, for this r, there exists a natural number $J > 0$ such that

$$g^j(p^*) \in U \quad \text{as } j \geq J.$$

Fix J and let $x_0 = g^J(p^*)$, then $x_0 \in U$ and $Y^J(x_0) = p^*$. Since $|Y'(p)| = k > 1$ for all $p \in U$, and $(Y^J)'(x_0) \neq 0$, p^* is a snapback repeller of Y.

Main Result (I)

$$\text{Let } p^{**} = \frac{2}{1+k},\ h = (Y^2)^{-1}$$



Main Result (I)

Since $|h'(p^{**})| < 1$, there exists $r > 0$ with $V = (p^{**} - r, p^{**} + r)$, $V \subset (\frac{1}{k}, 1)$ such that $\lim_{m\to\infty} h^m(x) = p^{**}$ if $x \in V$. Choose

$$h(p^{**}) < \frac{1}{k} \quad \text{and} \quad h^2(p^{**}) > \frac{1}{k}.$$

Solve $h(p^{**}) > -k + 2$ and $h^2(p^{**}) < 1$, choosing $k > \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}$ allows j to be found such that

$$h^j(p^{**}) > \frac{1}{k} \quad \text{for all } j \geq 3$$

by the definition of Y^2.

Main Result (I)

Computing $|g^j(p^*) - p^*|$, yield $|h^j(p^{**}) - p^{**}| = \frac{k-1}{k^{j+1}} \to 0$ as $j \to \infty$. That is, for this r, there exists a natural number $J' > 0$ such that

$$h^j(p^{**}) \in V \quad \text{as } j \geq J'.$$

Fix this J', let $y_0 = h^{J'}(p^{**})$, then $y_0 \in V$ and $(Y^2)^{J'}(y_0) = p^{**}$. Since $|(Y^2)'(p)| = \ k \ > \ 1$ for all $p \in V$, and $\left[(Y^2)^{J'}\right]'(y_0) \neq 0$, p^{**} is a snapback repeller of Y^2.

- Y^2 is chaotic in the Devaney sense $\Leftrightarrow h_{\text{top}}(Y^2) > 0$.
- $h_{\text{top}}(Y^2) = 2 \cdot h_{\text{top}}(Y) > 0 \Leftrightarrow h_{\text{top}}(Y) > 0$.
- $h_{\text{top}}(Y) > 0 \Leftrightarrow Y$ is chaotic in Devaney's sense as

$$k > \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}.$$

$p = 2$

$$Y^{2^2}(x) = \begin{cases} k^2x - 2k + 2, & x \in \left[\frac{1}{k}, B\left(\frac{1}{k}\right)\right], \\ -k^3x + 2k^2 - k + 1, & x \in \left[B\left(\frac{1}{k}\right), 1\right]. \end{cases}$$

$p = 3$

$$
Y^{2^3}(x) = \begin{cases} k^6 x - 2k^5 + k^4 - k^3 + 2k^2 - k + 1, \\ \qquad x \in \left[B\left(\frac{1}{k}\right), \underbrace{BAAB}_{C_1}\left(\frac{1}{k}\right) \right], \\ -k^5 x + 2k^4 - k^3 + k^2 - 2k + 2, \\ \qquad x \in \left[C_1\left(\frac{1}{k}\right), 1 \right]. \end{cases}
$$

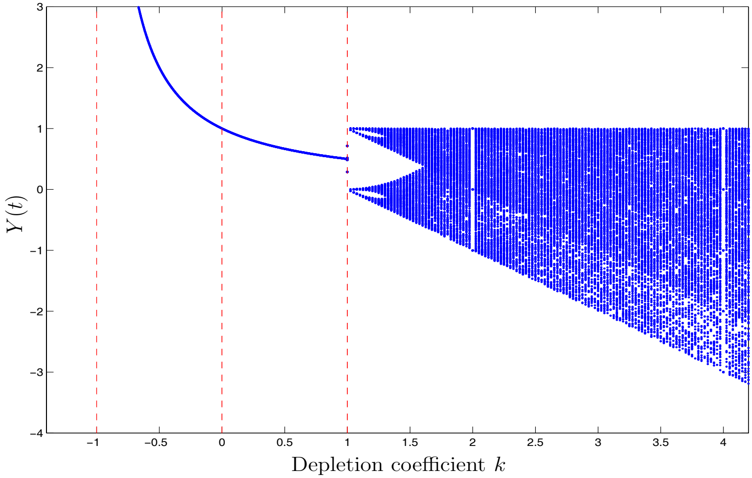
MatLab

p	k_p
0	**1.6180**33988749895
1	**1.3247**17957244745
2	**1.1347**24138401520
3	**1.0682**97188920740
4	**1.0327**70966453956
5	**1.0164**43864419055
6	**1.0081**40050503278
7	**1.0041**60992268882
8	**1.0036**64292317828
9	**1.0037**95792338565

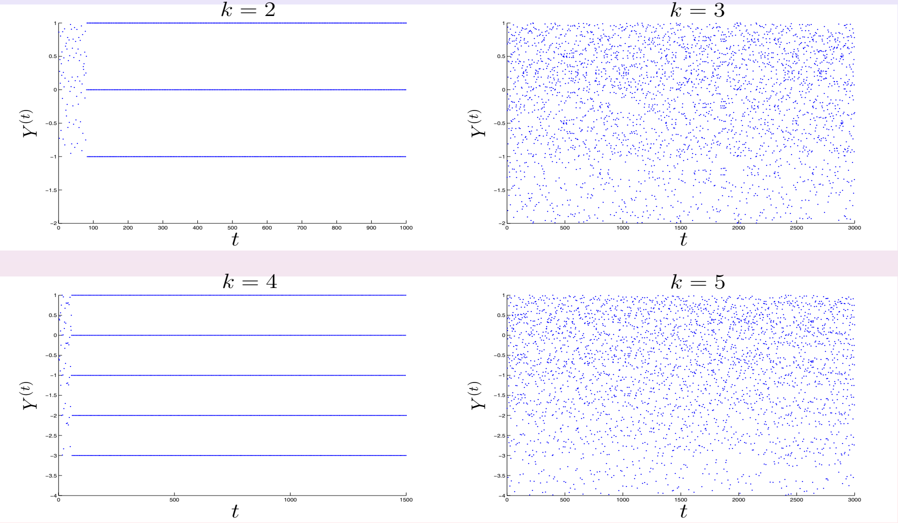
Maple

p	k_p
0	**1.6180**339887498948482
1	**1.3247**179572447460259
2	**1.1347**241384015194926
3	**1.0682**971889208412763
4	**1.0327**709664410429093
5	**1.0164**438640594170720
6	**1.0081**400320211663423
7	**1.0040**736663886927402
8	**1.0020**317763334169970
9	**1.0010**161163502399878
10	**1.0005**077430745001149
11	**1.0002**538857993064976

$Y^{(t+1)} = -kY^{(t)} + 1, \; Y^{(t)} > 0$



$Y^{(t+1)} = -kY^{(t)} + 1,\ Y^{(t)} > 0$



For any initial value $x \in \mathbb{Q}$

$k \in \mathbb{N}$

$Y^{(t)}(x)$ is periodic **eventually**.

Binary representation with finite digits

$k \in \mathbb{N} \setminus \{1\}$

$k \in$ even

$Y^{(t)}$ **always** converges to the periodic cycle
$S \equiv \{-k+1, -k+2, \ldots, 0, 1\}$
with period $k+1$ for any initial value.

$k \in$ odd

$Y^{(t)}$ **can not** converge to the periodic cycle
$S \equiv \{-k+1, -k+2, \ldots, 0, 1\}$
as the initial value $x \notin S$.

Thank you for your attention!