# Dynamics of Three Vortices in Bose-Einstein Condensates[a]

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## Outline

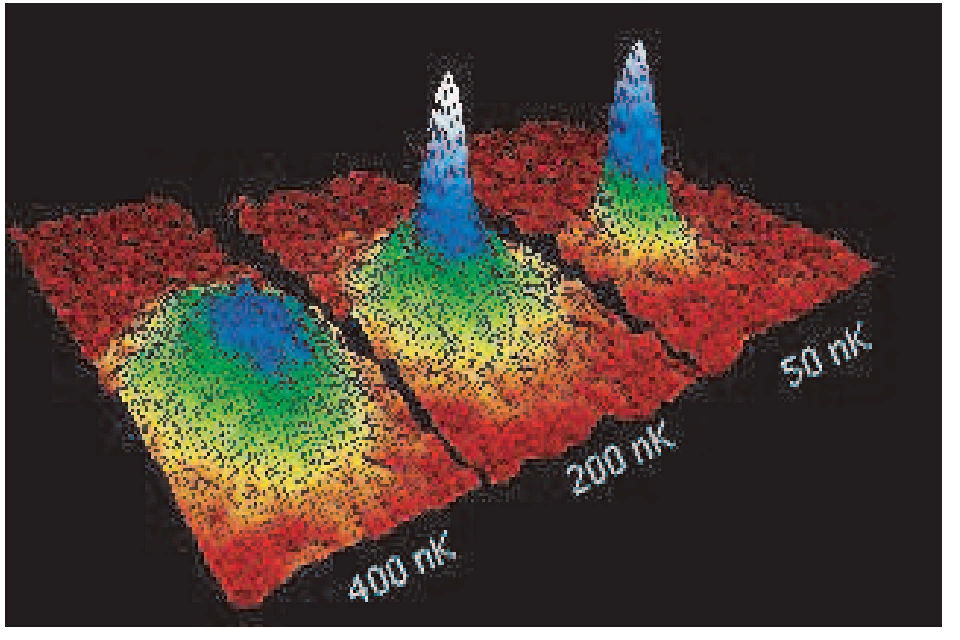
- Motivation
- Introduction of Vortices in Bose-Einstein Condensates (BEC)
- Mathematical Model
- Numerical Study of Three Vortices
- Conclusion

## Motivation

- Make a study of vortices's behavior in a two-dimensional trapped Bose-Einstein Condensates.
  - PDE: time-dependent Gross-Pitaevskii equation.
  - ODE: asymptotic motion equations of vortices.

# Introduction of Vortices in BEC

- What is BEC?

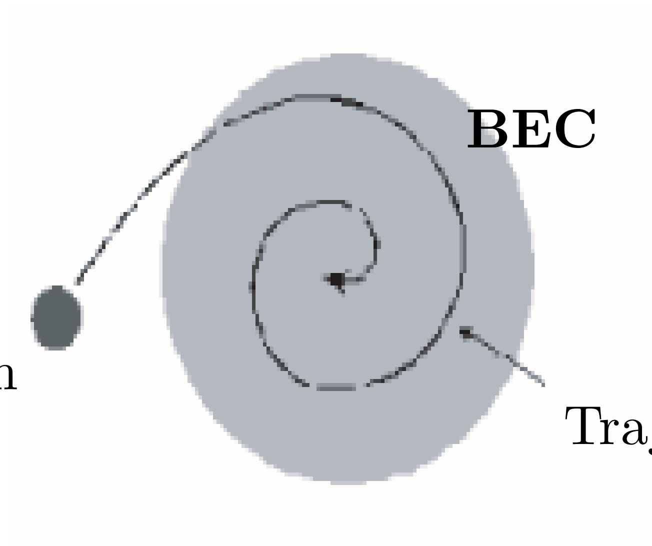



BEC at 400, 200, and 50 nK [5].

- What are vortices?

- How do vortices happen?
  - Idea 1: rotation (standard way in fluid mechanics).
  - Idea 2: laser beam moving slowly through the condensate (without rotation), by B. Jackson et al. (1998, theoretical); K. Staliunas (1999, experiment).

- Idea of K. Staliunas – stirred Bose-Einstein Condensates:
  **(1)** Create one component BEC.
  **(2)** The laser beam enters the condensate spiraling clockwise.
  **(3)** Reaching the center of the condensate it is switched off.




Moving laser beam

Trajectory of laser beam

# Mathematical Model

- Time-dependent Gross-Pitaevskii equation (PDE)

$$i\,u_t = -\Delta u + V_\epsilon(x,y)\,u + \frac{1}{\epsilon^2}(|u|^2-1)\,u, \quad t>0, \tag{1}$$

with the initial data $u|_{t=0} = u_0(x,y)$ and $(x,y) \in \mathbb{R}^2$.

$u$: a complex-valued order parameter,

$\epsilon > 0$: a small parameter,

$V_\epsilon(x,y) = \alpha_\epsilon\, x^2 + \beta_\epsilon\, y^2$: a harmonic trap potential,

$\alpha_\epsilon,\ \beta_\epsilon > 0$: depending on $\epsilon$.

This time-dependent Gross-Pitaevskii equation was introduced as a phenomenological equation for the order parameter in superfluids.

- asymptotic motion equations of vortices (ODE)

  Suppose $u_0$ has $d$ vortex centers at $q_j(0) = (q_{jx}(0), q_{jy}(0))^\top$.

  Under some specific assumptions on $u_0$, we obtain

$$\begin{cases} \dot{q}_{jx} = -\sum_{\substack{k=1 \\ k \neq j}}^{d} n_k \frac{q_{jy}-q_{ky}}{|q_j-q_k|^2} - \omega_1 q_{jy} , \\ \dot{q}_{jy} = \sum_{\substack{k=1 \\ k \neq j}}^{d} n_k \frac{q_{jx}-q_{kx}}{|q_j-q_k|^2} + \omega_2 q_{jx} , \end{cases} \tag{2}$$

  where $q_j = q_j(t) = (q_{jx}(t), q_{jy}(t))$, $n_j$: winding numbers and $\omega_1 = -\omega + 2\beta_0$, $\omega_2 = -\omega + 2\alpha_0$. For the stability of the vortex structure in $u$, we require $n_j \in \{\pm 1\}, j = 1, \ldots, d$.

# Numerical Study of Three Vortices

We consider $d = 3$, then obtain

**(1)** the bounded and collisionless trajectories of three vortices form chaotic, quasi 2- or quasi 3-periodic orbits,

**(2)** a new phenomenon of $1 : 1$-topological synchronization is observed in the chaotic trajectories of vortices with the same sign of winding numbers..

Let $d$ be the number of vortices.

- Aref 1979 [3]: The Kirchhoff equations (3) form an integrable system if $d \leq 3$. (Theoretical Proof)
- Aref 1983 [4]: The Kirchhoff equations may have chaotic motions in a bounded region if $d > 3$. (Numerical Simulation)

$$\begin{cases} \dot{q}_{jx} = -\sum_{\substack{k=1 \\ k \neq j}}^{d} n_k \frac{q_{jy} - q_{ky}}{|q_j - q_k|^2}, \\ \dot{q}_{jy} = \sum_{\substack{k=1 \\ k \neq j}}^{d} n_k \frac{q_{jx} - q_{kx}}{|q_j - q_k|^2}. \end{cases} \tag{3}$$

- Characterize the motion:
  - Lyapunov exponent,
  - Poincaré map,
  - Spectrums of waveforms.
- Indicator for ratio topologically synchronized chaotic regimes (Afraimovich et al. (1999, 2000), [1, 2]):
  - the Poincaré dimension for Poincaré recurrences.

- Lyapunov Exponents

$m_1(t), \ldots, m_6(t)$: eig.s of the transition matrix $\Phi_t(x_0)$ of (2) with $d = 3$ and $\Phi_0(x_0) = \mathrm{I}_6$. The Lyapunov exponents of $x_0$ are given by

$$\lambda_i(x_0) = \lim_{t\to\infty} \frac{1}{t} \ln |m_i(t)|, \quad i = 1, \ldots, 6. \tag{4}$$

$K$-periodic: $\lambda_1 = \lambda_2 = \cdots = \lambda_K = 0, \quad 0 > \lambda_{K+1} \geq \lambda_{K+2} \geq \lambda_d,$

Chaotic: $\lambda_1 > 0, \quad \sum \lambda_i < 0.$

- Poincaré Dimension for Poincaré Recurrences

$q^t : X \times Y \to X \times Y$: dynamical system, $A$: cpt. inv. $\subset X \times Y$, $\pi_1 A = A_1 \subset X, \pi_2 A = A_2 \subset Y$.

**Definition 1** *Let $U_1 \subset X$ ($U_2 \subset Y$). The numbers*

$$\tau_x(U_1) = \inf_{x_0 \in U_1} t(x_0, U_1), \quad \tau_y(U_2) = \inf_{y_0 \in U_2} t(y_0, U_2), \tag{5}$$

*$x$-Poincaré and $y$-Poincaré recurrences for $U_1$ and $U_2$, resp..*

$$M_x(\alpha_x, \varepsilon, q) = \inf_{G_1} \sum_k \exp^{-p_x \tau_x(U_{1k})} (\mathrm{diam} U_{1k})^{\alpha_x}, \tag{6}$$

$$M_y(\alpha_y, \varepsilon, q) = \inf_{G_2} \sum_k \exp^{-p_y \tau_y(U_{2k})} (\mathrm{diam} U_{2k})^{\alpha_y}, \tag{7}$$

$$\dim_P(A_1) = p_0^{(x)}, \quad \dim_P(A_2) = p_0^{(y)}. \tag{8}$$

**Theorem 1** [1, 2] *If a dynamical system* $q^t : X \times Y \to X \times Y$ *is* $\frac{m_0}{n_0}$*-topologically synchronized, then*

$$dim_P(A_2) = \frac{m_0}{n_0} dim_P(A_1). \tag{9}$$

The asymptotic equalities of (6) and (7) show that we may expect

$$\langle \exp\left(-p_0^{(x)} \tau_x(U_{1k}^{\varepsilon})\right)\rangle \sim \varepsilon^{b_1}, \quad \langle \exp\left(-p_0^{(y)} \tau_y(U_{2k}^{\varepsilon})\right)\rangle \sim \varepsilon^{b_2}, \tag{10}$$

where $b_i = \dim_B(A_i)$ (box dimension) for $i = 1, 2$, and $\langle \cdot \rangle$ denotes the arithmetic average over $k$. We may also expect that $b = b_1 = b_2$. In this case (10) implies the asymptotic equalities

$$\langle \tau_x(U_{1k}^{\varepsilon})\rangle \sim \frac{-b}{p_0^{(x)}} \ln \varepsilon, \quad \langle \tau_y(U_{2k}^{\varepsilon})\rangle \sim \frac{-b}{p_0^{(y)}} \ln \varepsilon, \tag{11}$$

where $\mathrm{diam} U_{1k} \leq \varepsilon$ and $\mathrm{diam} U_{2k} \leq \varepsilon$.

## Case $(n_1, n_2, n_3) = (1, -1, -1)$

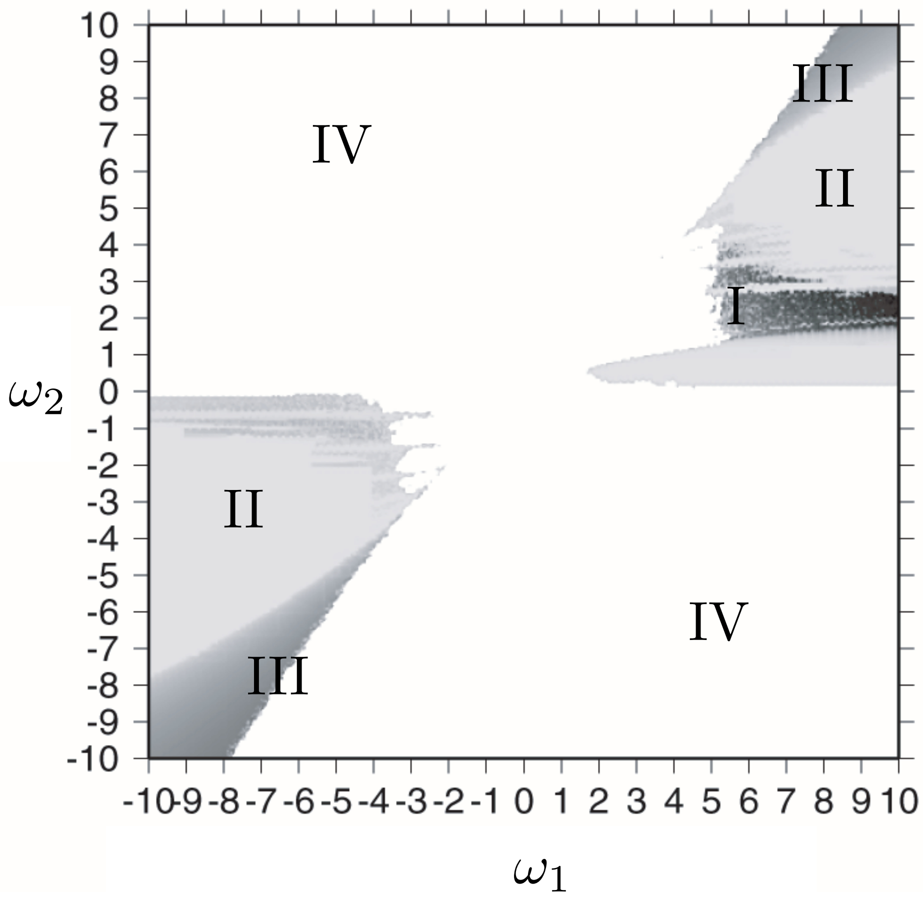



Figure 1: The first Lyapunov exponent.

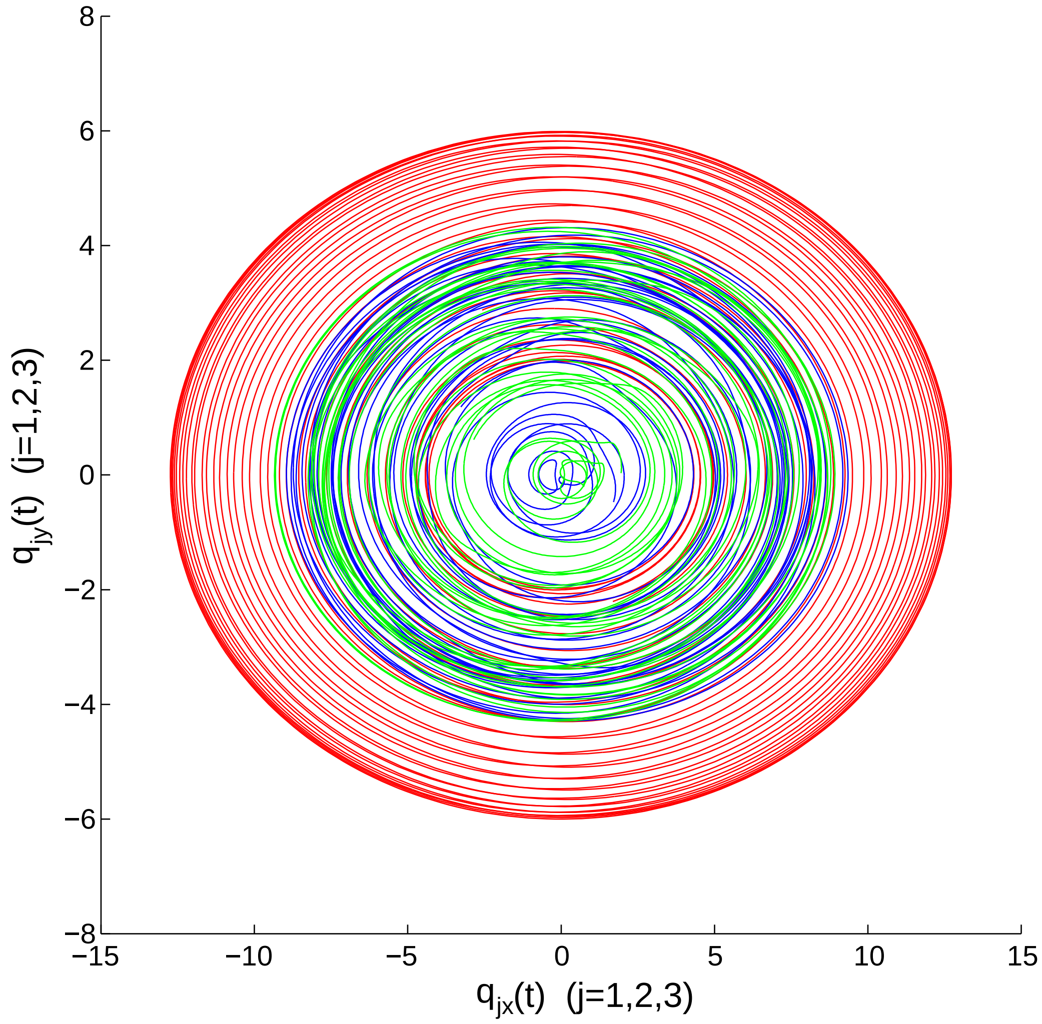



Figure 2: Chaotic trajectories: $(\omega_1, \omega_2) = (9.88, 2.24), t = 25,050 \sim 25,100$ sec.

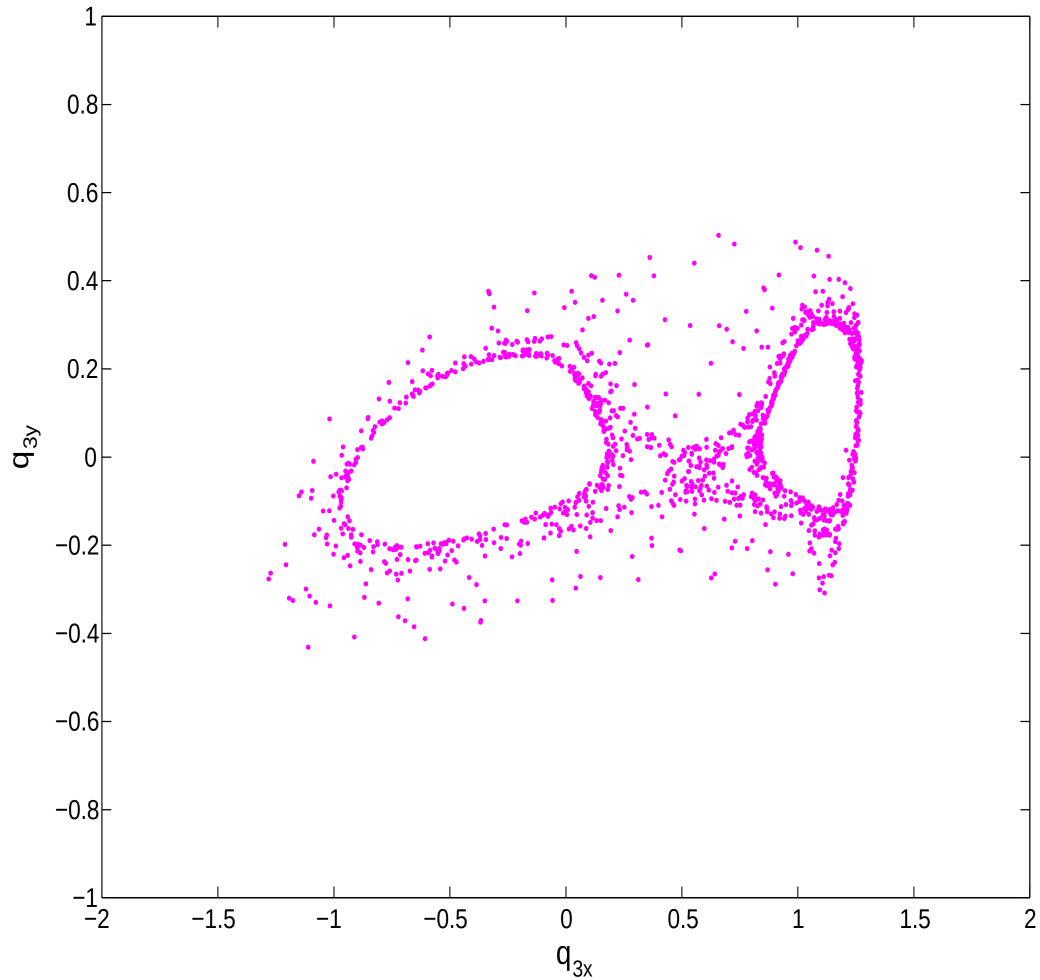



Figure 3: Chaotic second-Poincaré maps (4 dim.), $t = 1,393 \sim 5,000,000$ sec.

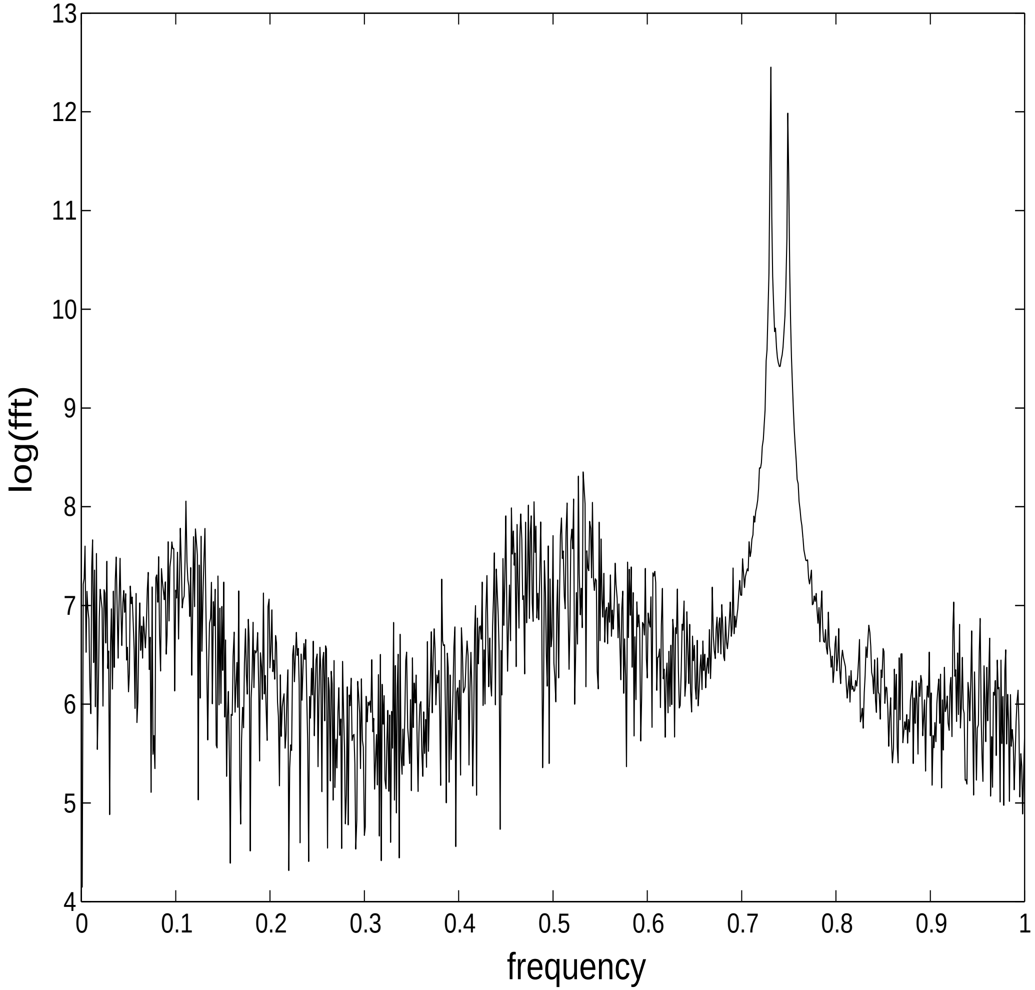



Figure 4: Chaotic spectrum of waveforms, $t = 1,050 \sim 25,500$ sec.

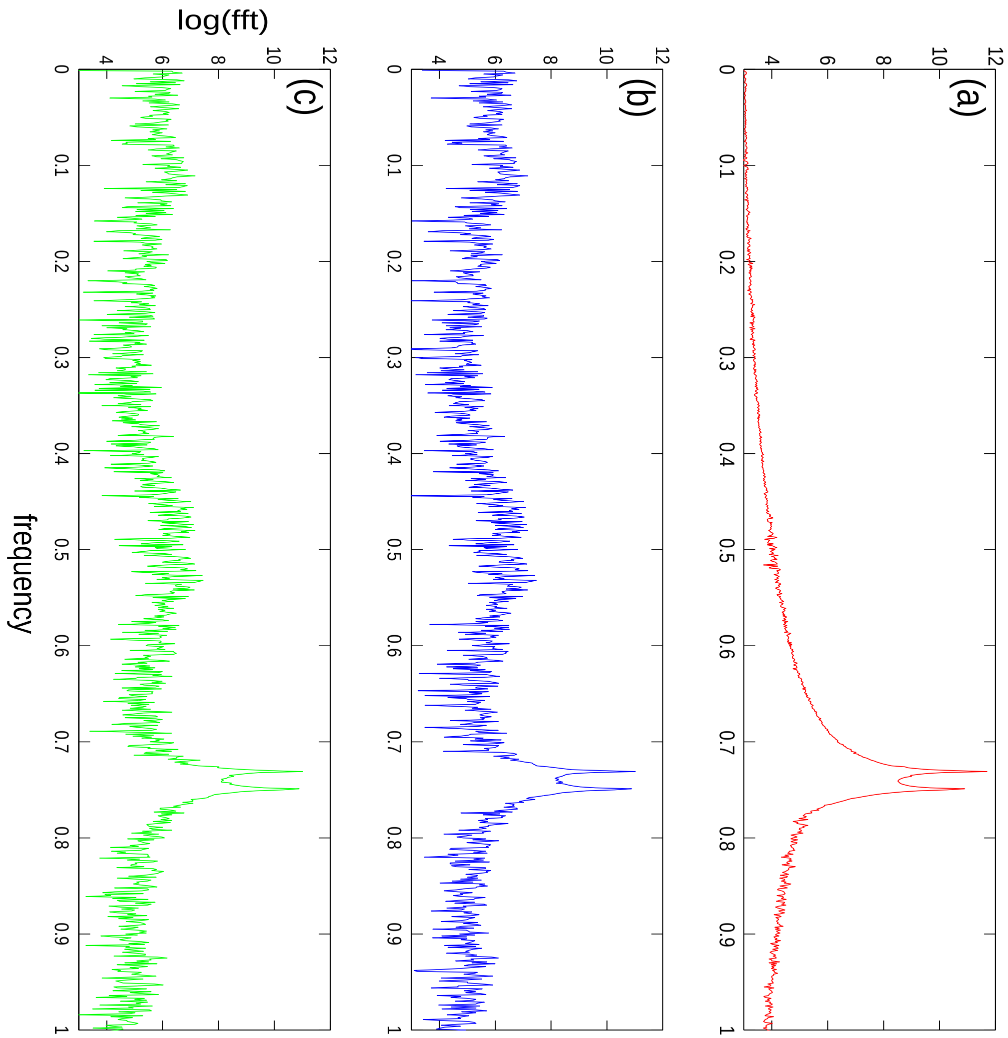



Figure 5: Chaotic individual spectrum, $t = 1,000 \sim 25,500$ sec.

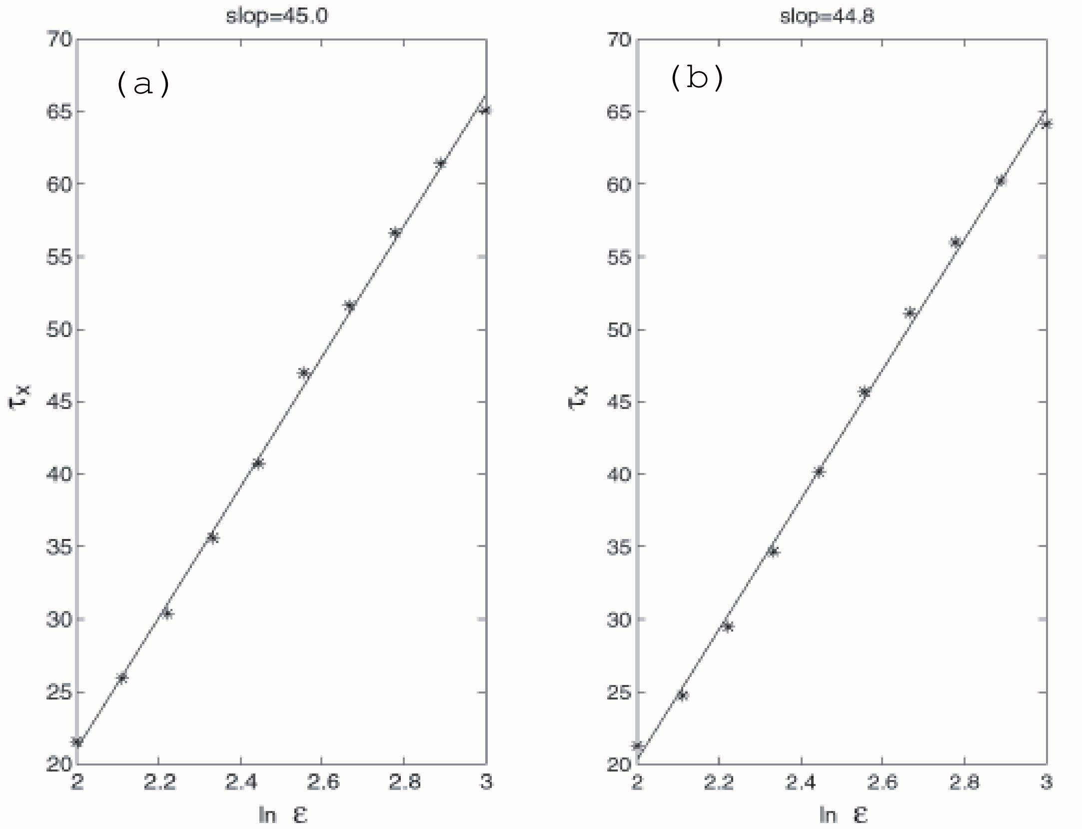



Figure 6: The ratio of slopes $= 45.0/44.8 \approx 1.006$

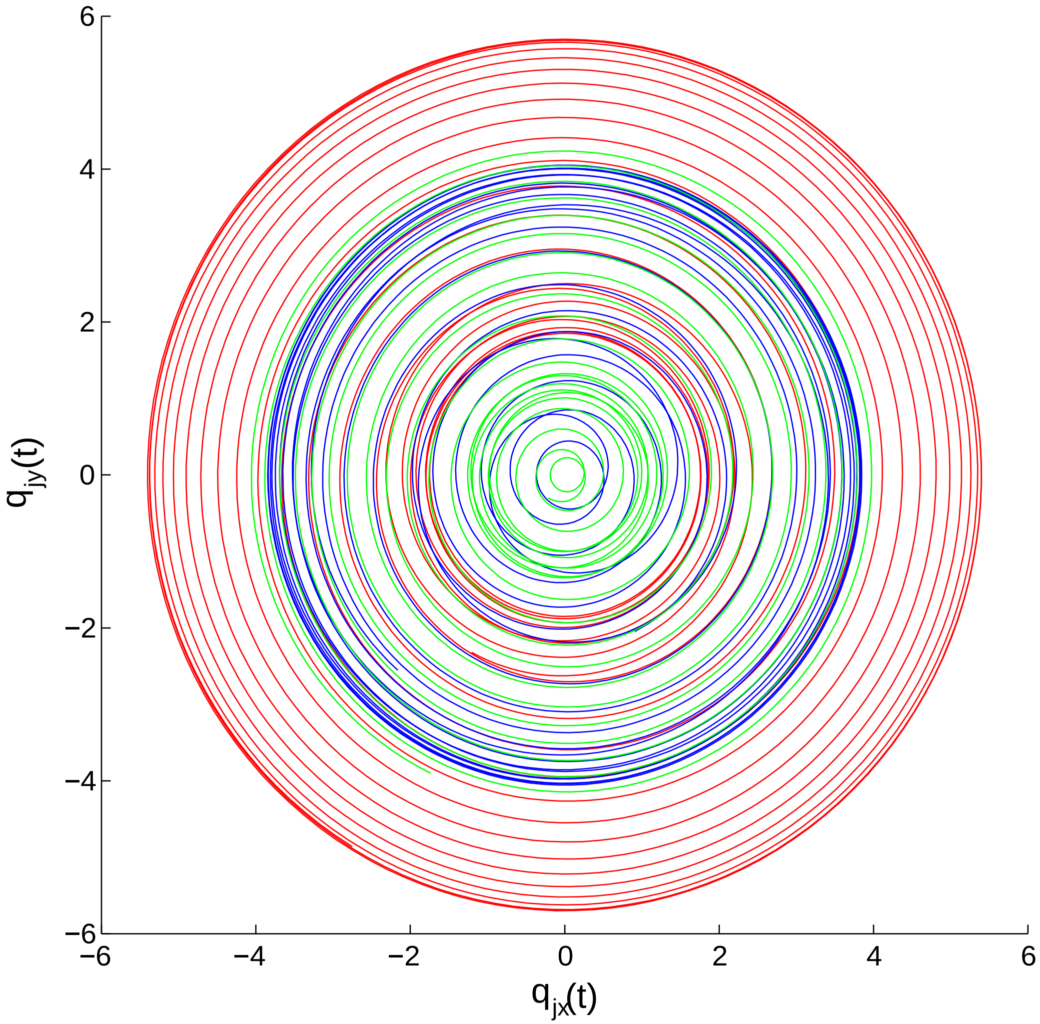



Figure 7: Quasi 3-periodic trajectories: $(\omega_1, \omega_2) = (9, 10), t = 25,080 \sim 25,095$ sec.

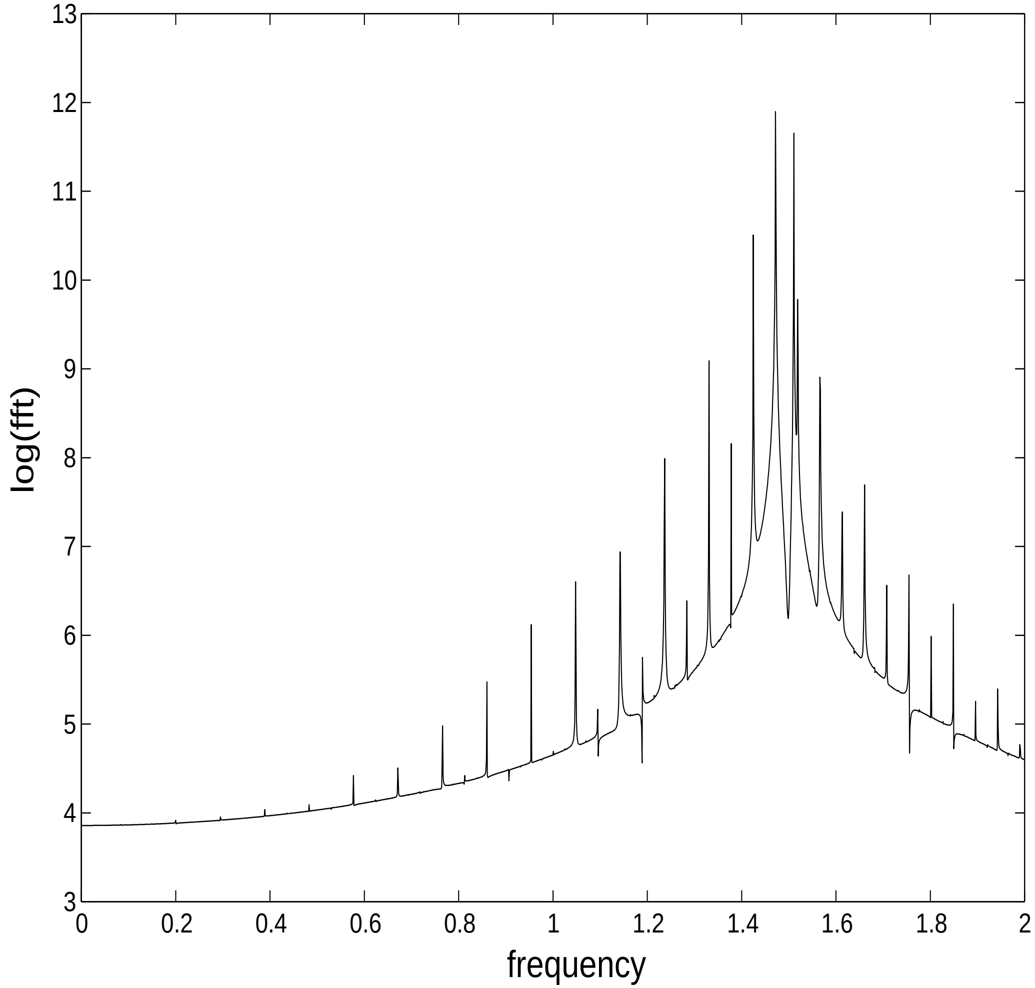



Figure 8: Quasi 3-periodic spectrum, $t = 1,000 \sim 25,500$ sec.

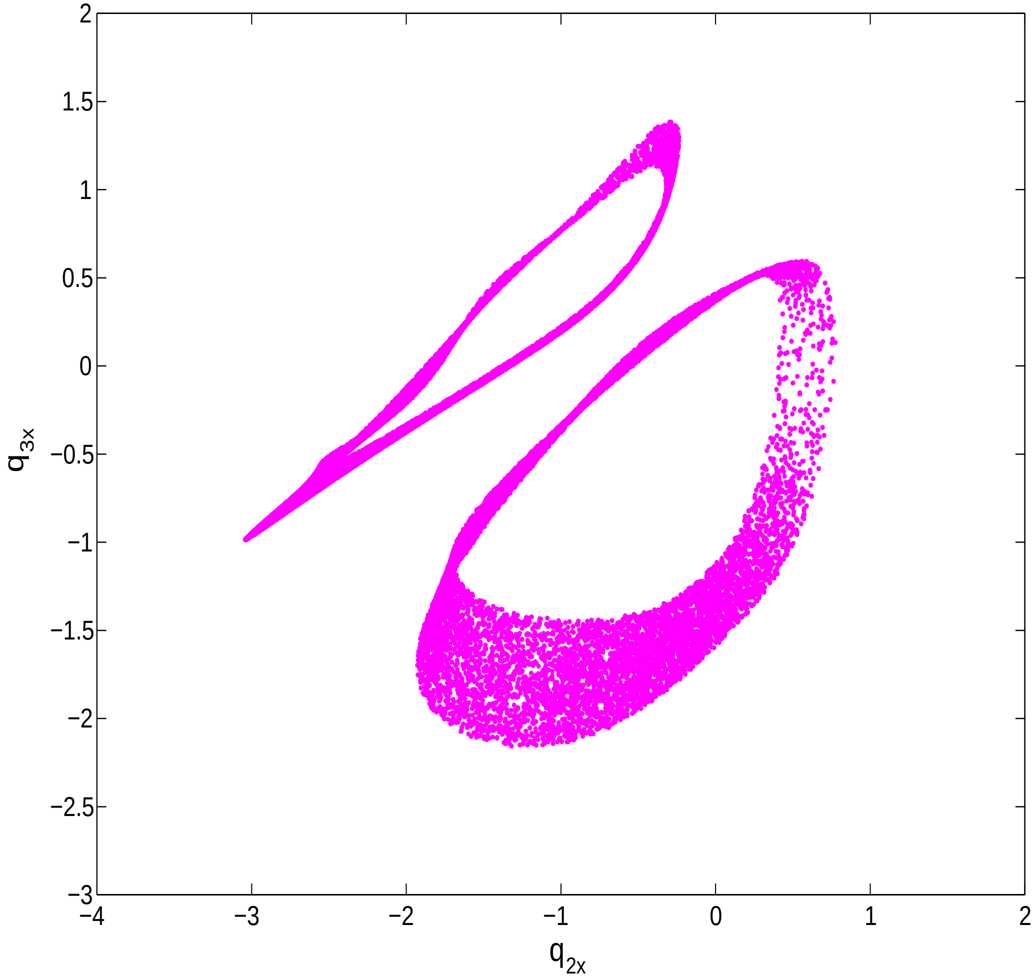



Figure 9: Quasi 3-periodic second-order Poincaré maps (4 dim.), $t = 41,179 \sim 4,000,000$ sec.

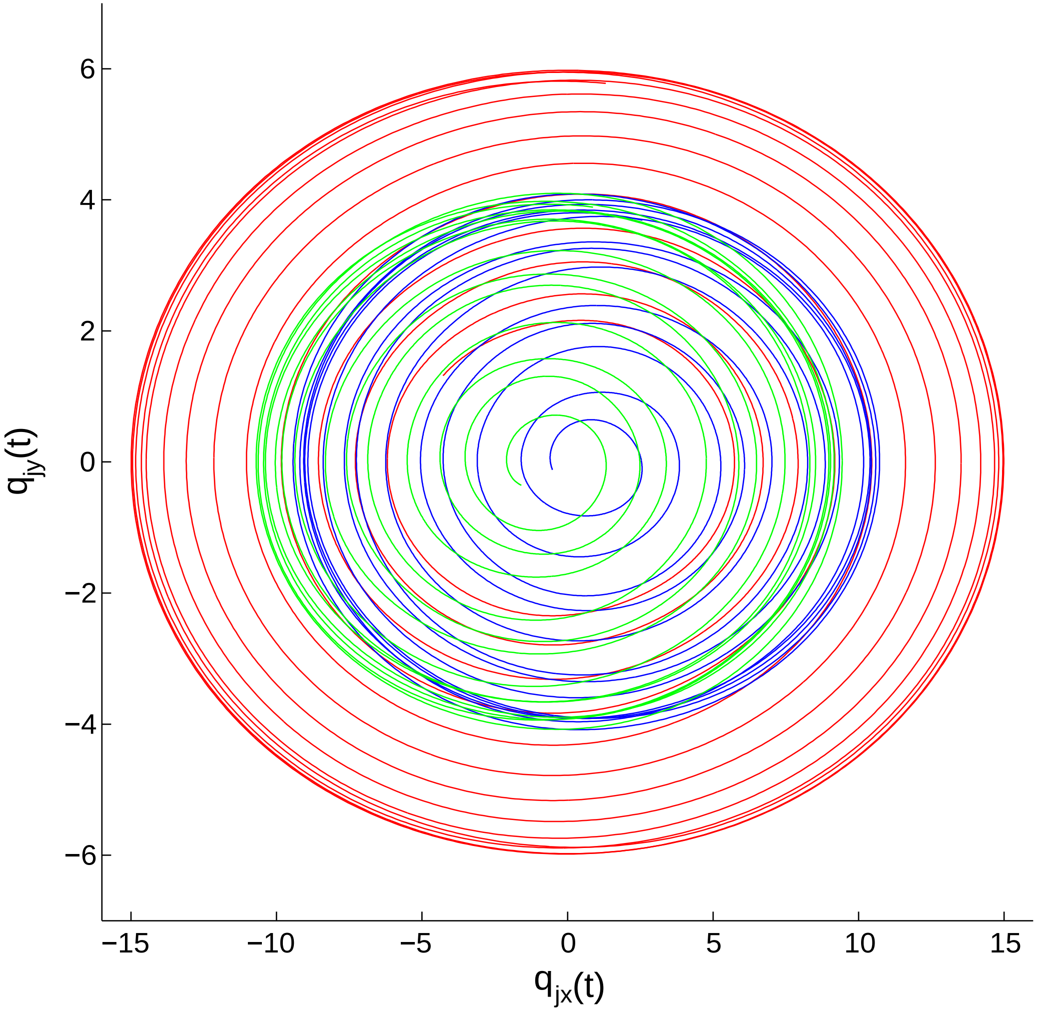



Figure 10: Quasi 2-periodic trajectories: $(\omega_1, \omega_2) = (6, 1), t = 25,155 \sim 25,190$ sec.

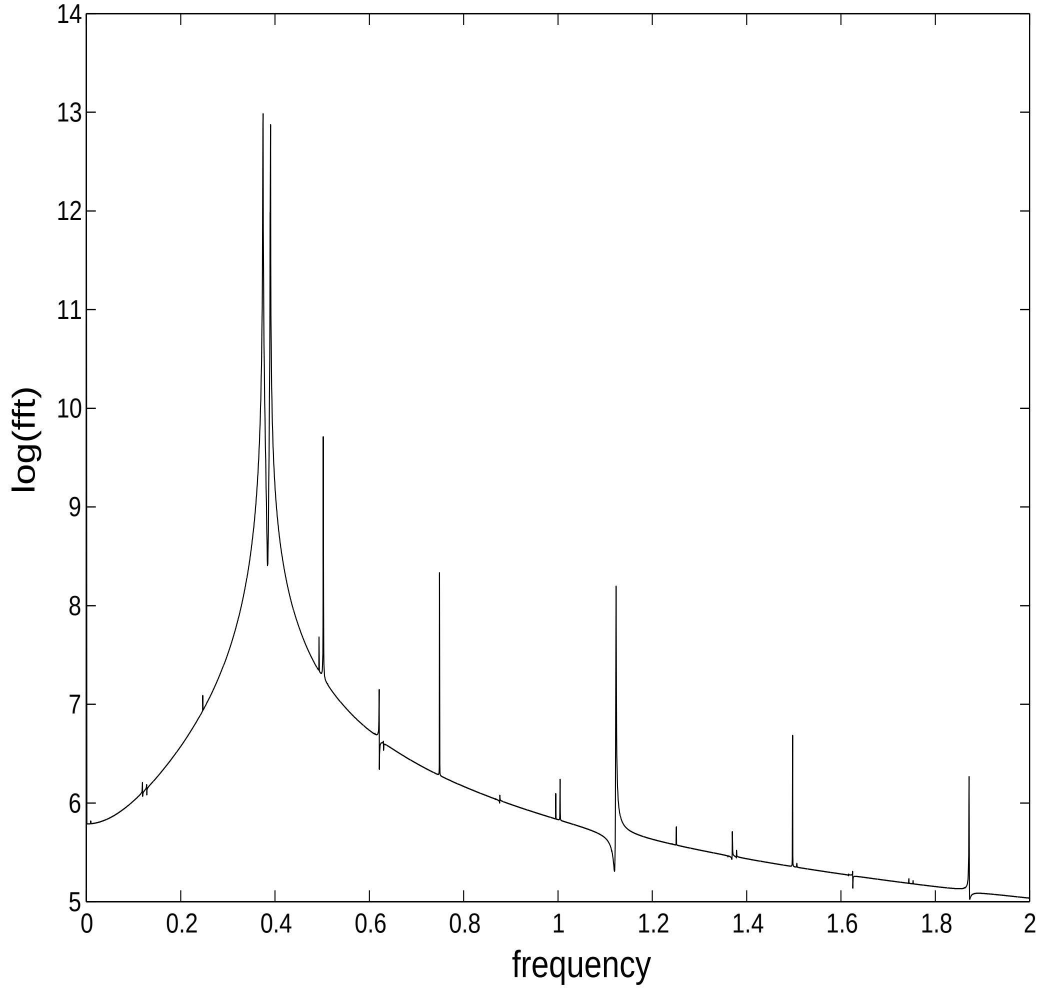



Figure 11: Quasi 2-periodic spectrum, $t = 2,000 \sim 25,500$ sec.

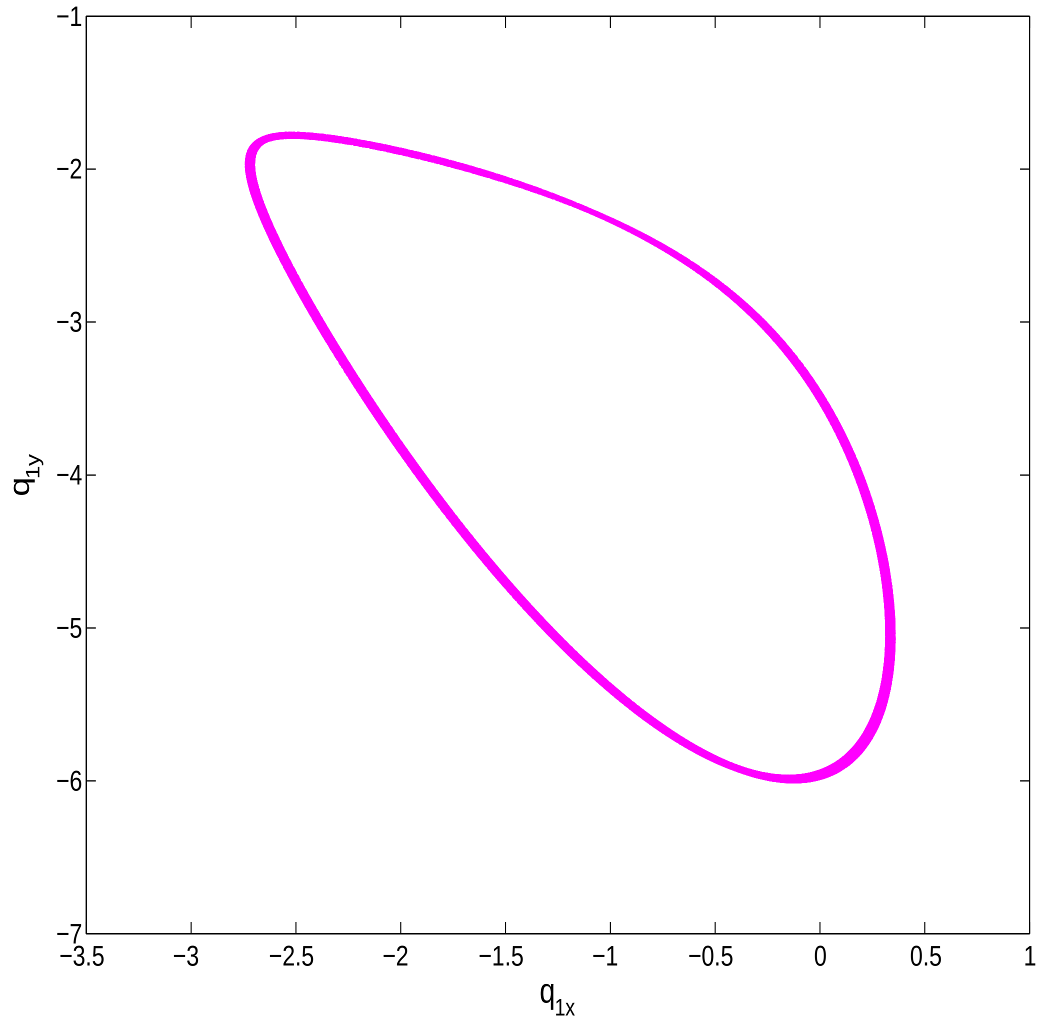



Figure 12: Quasi 2-periodic first-order Poincaré maps (5 dim.), $t = 37,193 \sim 1,000,000$ sec.

## Case $(n_1, n_2, n_3) = (1, 1, 1)$

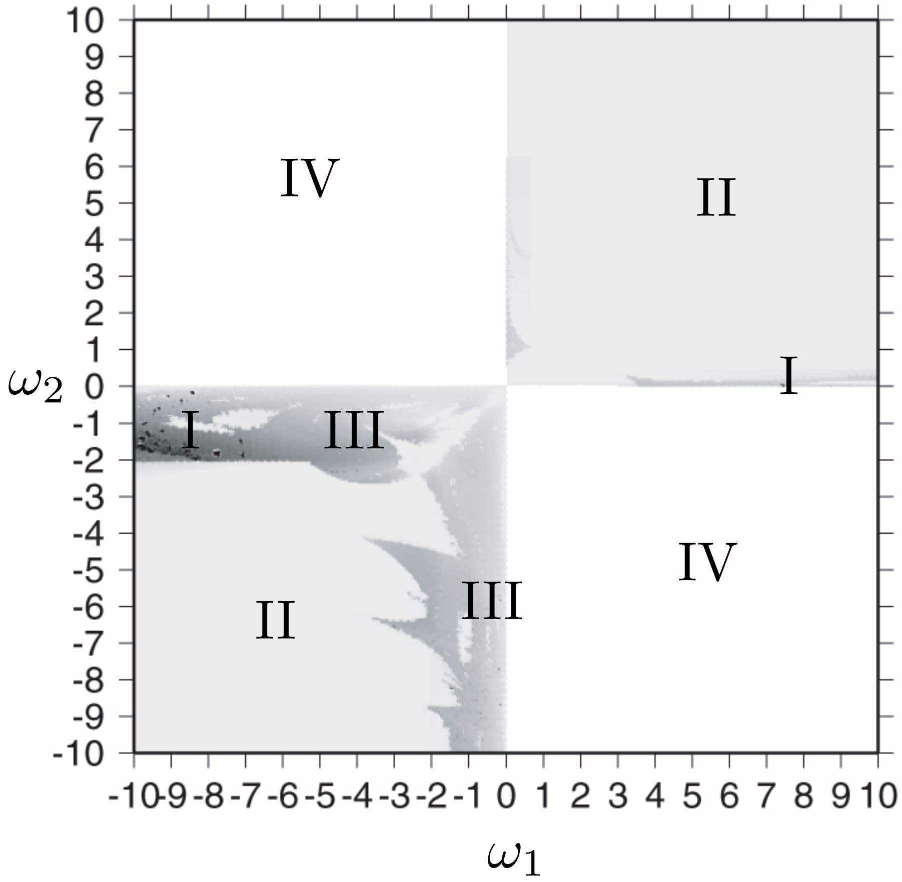



Figure 13: The first Lyapunov exponent.

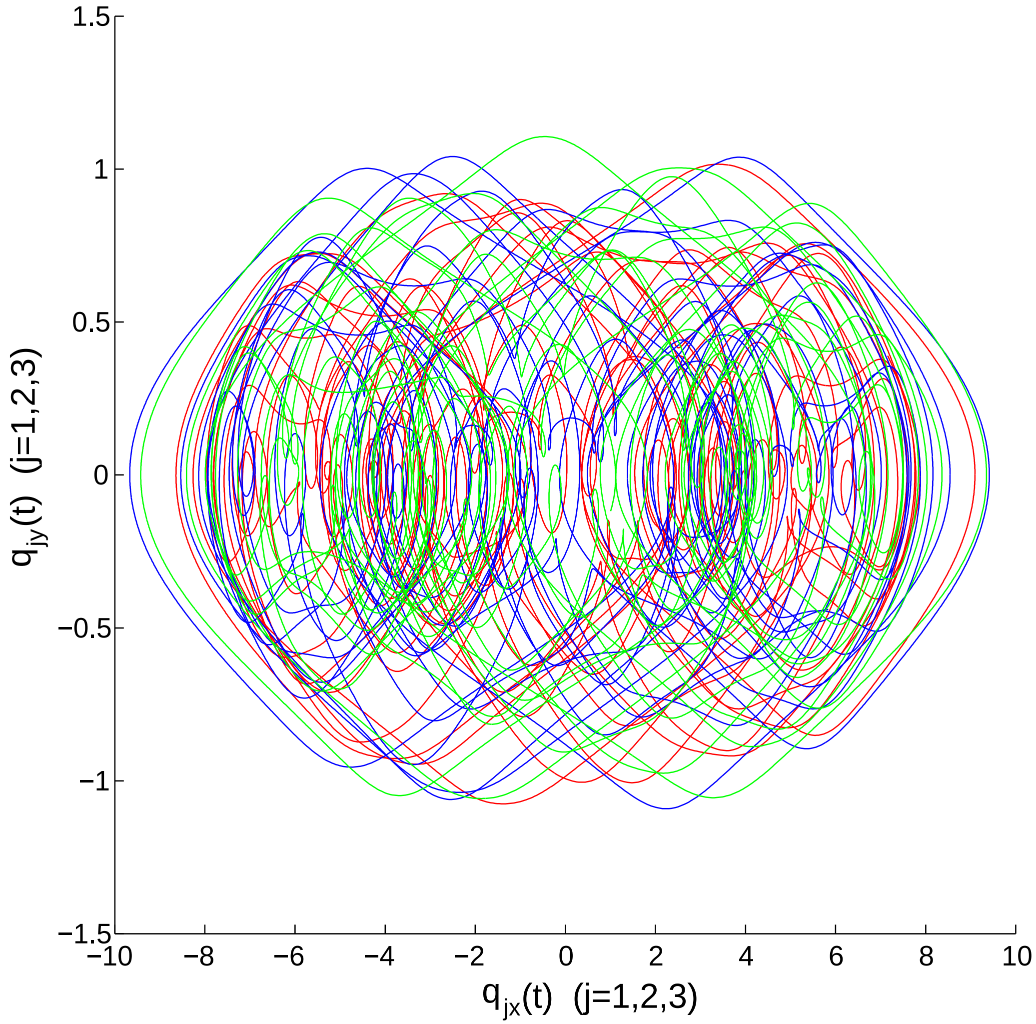



Figure 14: Chaotic trajectories: $(\omega_1, \omega_2) = (7.4, 0.025), t = 25,050 \sim 25,070$ sec.

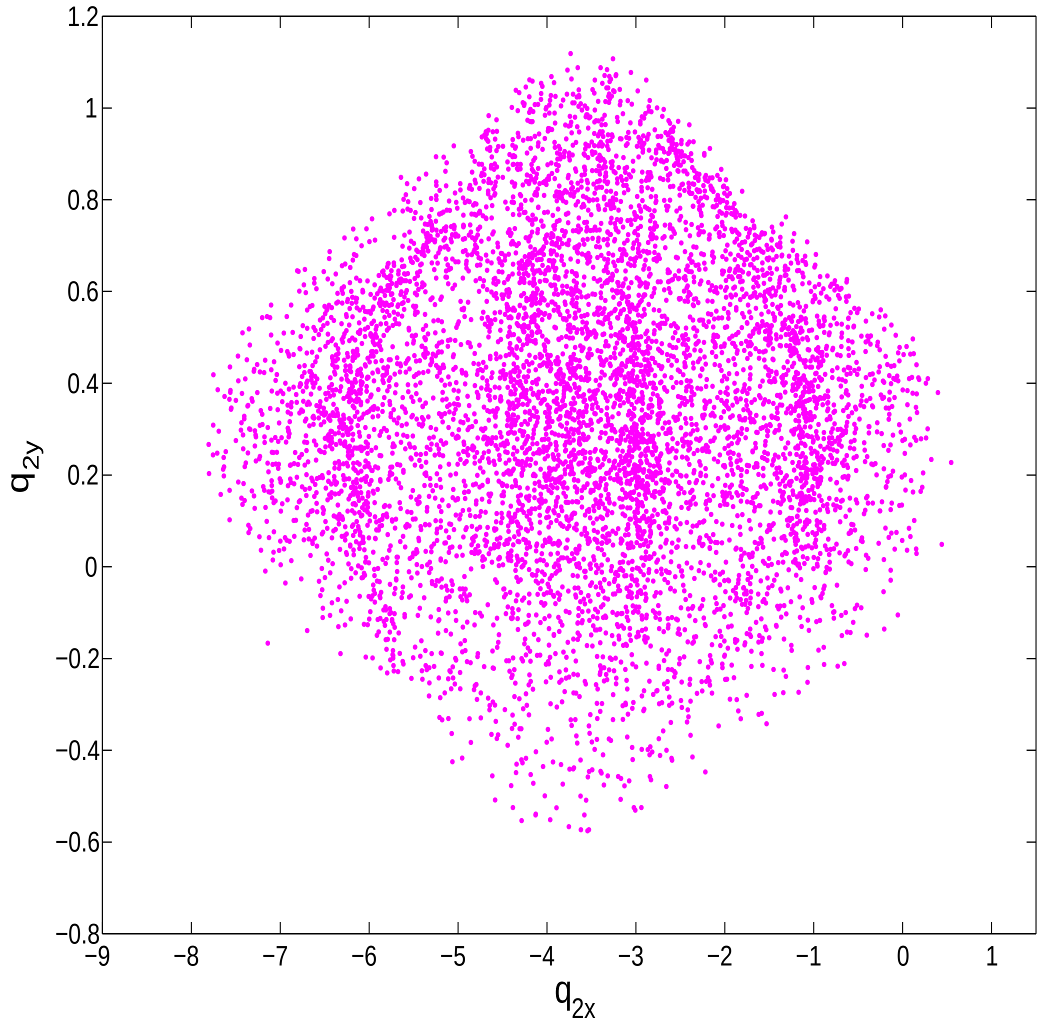



Figure 15: Chaotic first-order Poincaré maps (5 dim.), $t = 1,000 \sim 100,000$ sec.

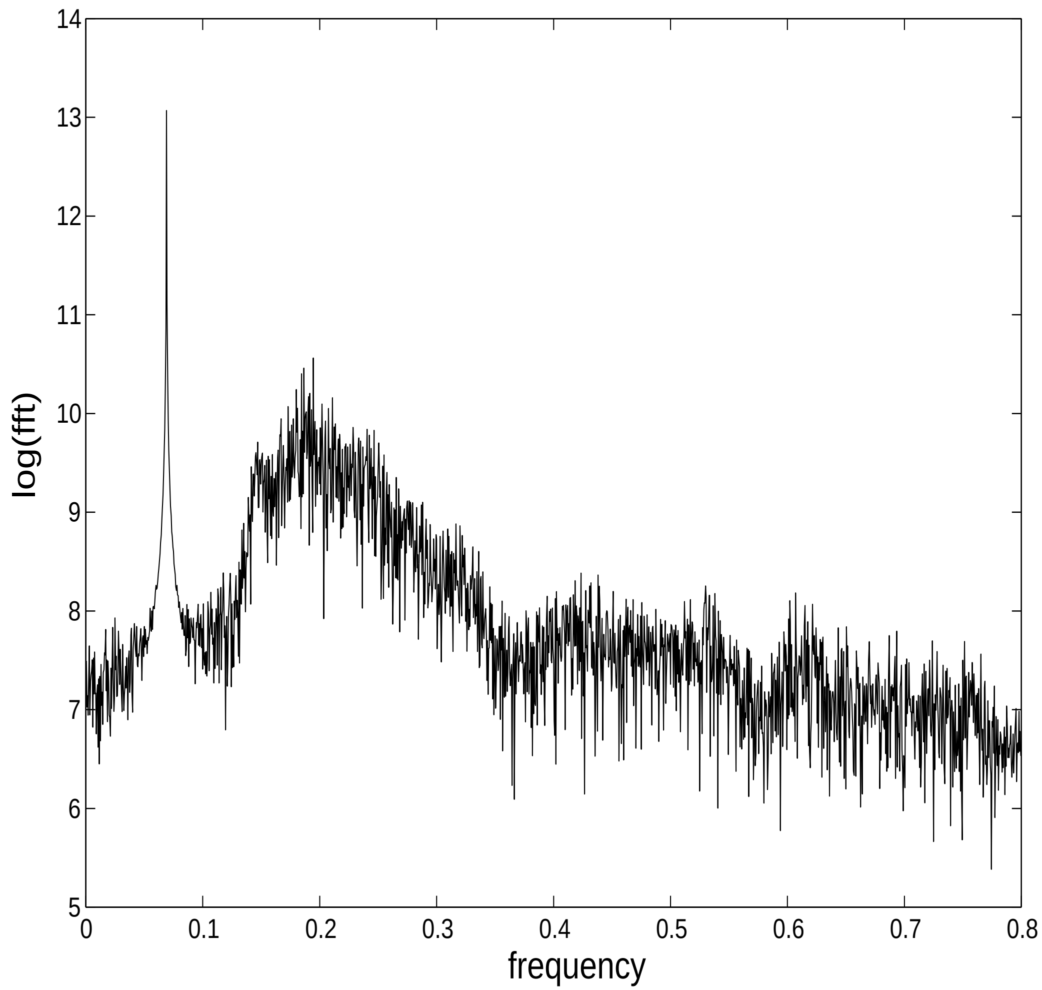



Figure 16: Chaotic spectrum, $t = 2,000 \sim 25,500$ sec.

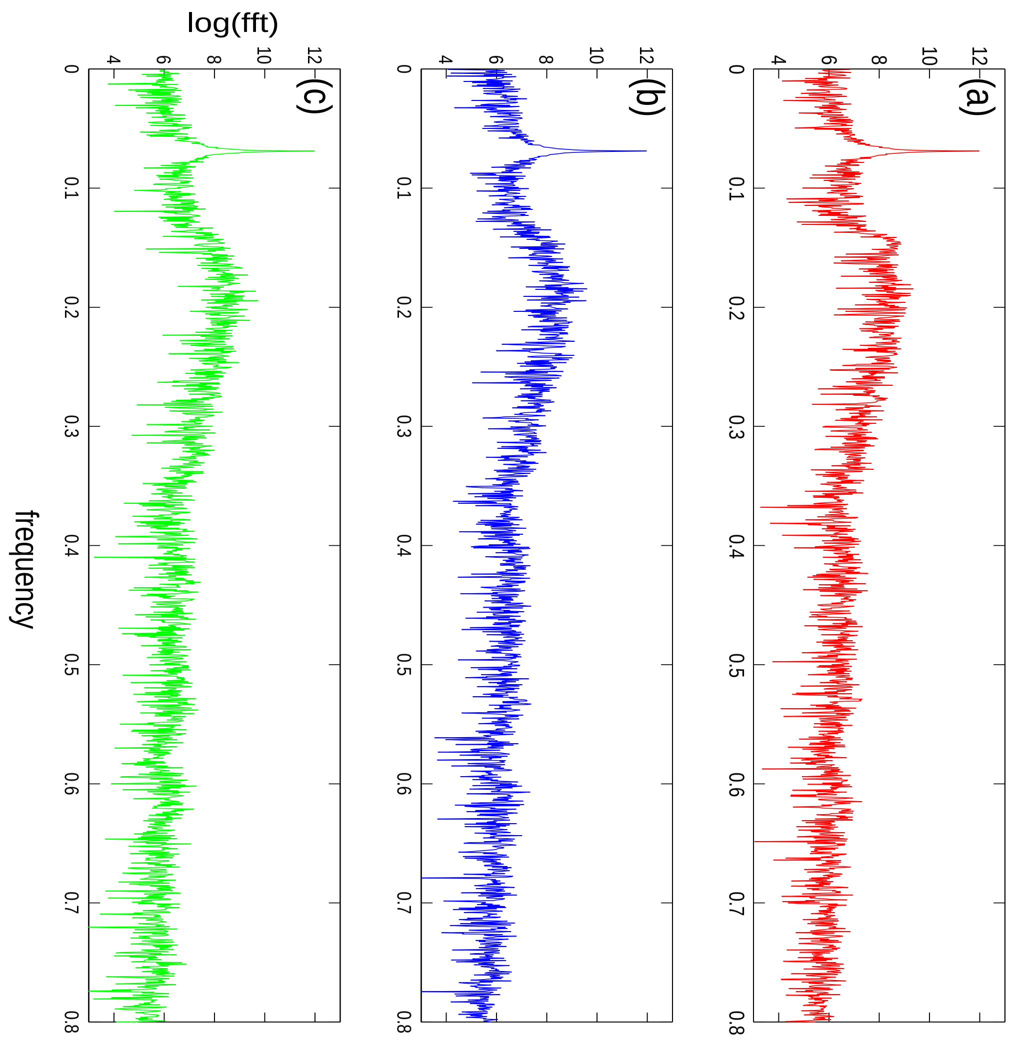



Figure 17: Chaotic individual spectrum, $t = 2,000 \sim 25,500$ sec.

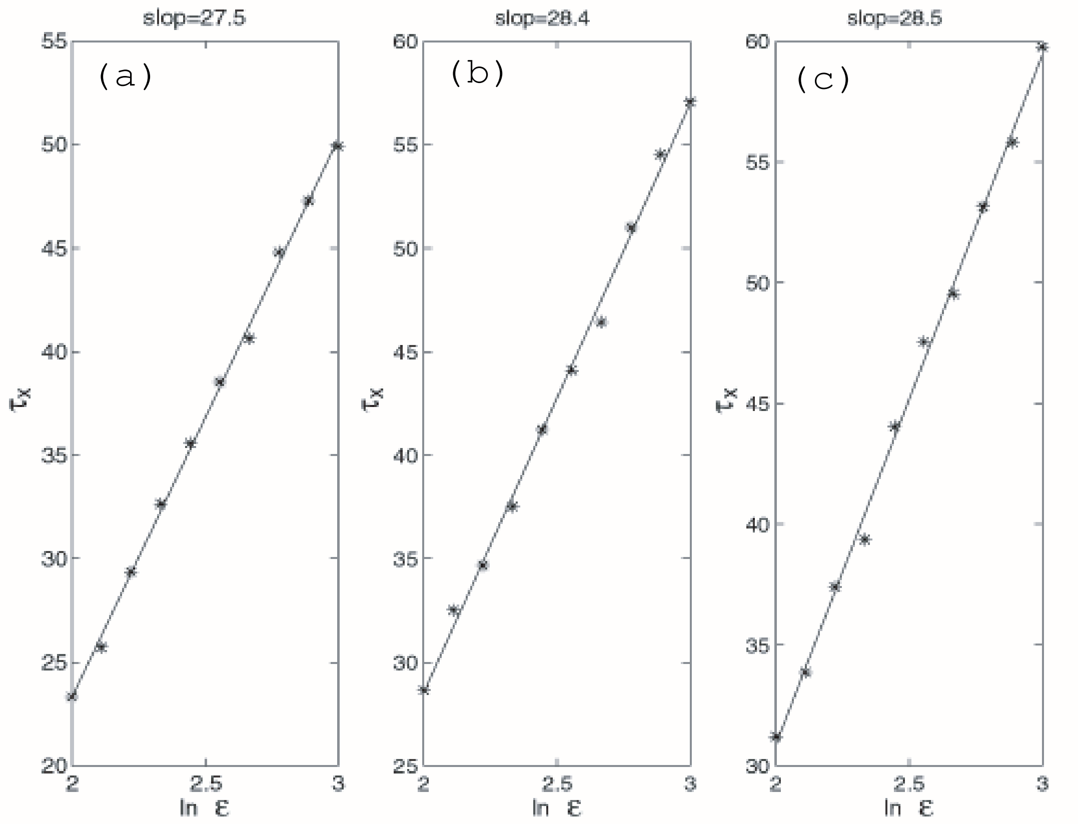



Figure 18: The ratio of slopes = 27.5 : 28.4 : 28.5 ≈ 0.97 : 0.996 : 1.

# 1 Conclusion

- Numerical
  - Harmonic trap potentials may have different effect on the motion of vortices.
  - This may become the first step to understand the effect of trap potentials on vortex dynamics.
- Future works
  - New dynamics of vortices for two-well trap potentials? periodic trap potentials?

## References


[1] V. S. Afraimovich, Poincaré recurrences of coupled subsystems in synchronized regimes, *Taiwanese Journal of Mathematics*, 3(2), 139–161 (1999).

[2] V. S. Afraimovich, W. W. Lin, and N. F. Rulkov, Fractral dimension for Poincaré recurrence as an indicator of synchronized chaotic regimes, *Int. J. Bifurcation and Chaos*, 10(10), 2323–2337 (2000).

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[4] H. Aref, Integrable, chaotic, and turbulent vortex motion in two-dimensional flows, *Ann. Rev. Fluid Mech.*, 15, 345–389 (1983).

[5] http://jilawww.colorado.edu/bec/