Verticillate Structures in Multi-Component Bose-Einstein Condensates

Shu-Ming Chang[a]

Department of Mathematics

National Tsing Hua University

Hsinchu, Taiwan

December 18, 2004

[a]This is a joint work with Chang-Shou Lin (NCCU), Tai-Chia Lin (NTU), Wen-Wei Lin (NTHU) & Shih-Feng Shieh (NTHU).

Outline

- Introduction of Bose-Einstein Condensation (BEC)
- Coupled Nonlinear Schrödinger Equations and Coupled Gross-Pitaevskii Equations (CGPEs)
- Nonlinear Algebraic Eigenvalue Problems (NAEPs)
- Fixed Point Iteration for NAEPs
- Numerical Results
- Conclusions

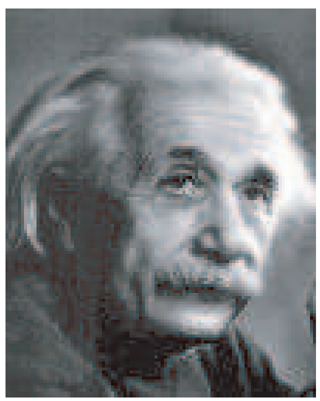
Results

- Theoretical
  - Propose a Jacobi-type fixed point iteration (J-FPI) and a Gauss-Seidel-type fixed point iteration (GS-FPI) to solve Multi-Component BEC.
  - Prove that the GS-FPI method converges locally and linearly to a fixed point if and only if the associated minimized energy functional problem has a strictly local minimum at the feasible fixed point.
- Numerical
  - Simulate multi-component BEC.
  - A New phenomenon: verticillate multipling structure.

1 Introduction of BEC

- What is BEC?

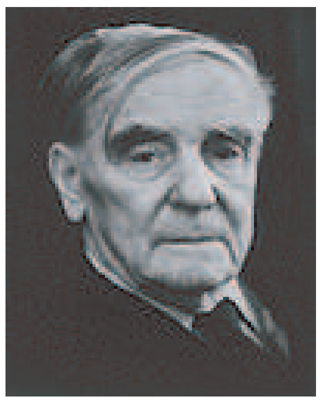
- Theoretical prediction 1924 ...
  - S. Bose: derived Planck's black body radiation law from considering the cavity radiation as an ideal photon gas and worked out Bose statistics for photons.
  - A. Einstein: generalized Bose statistics to other Bosonic particles and atoms (Bose-Einstein statistics) and predicted if the atoms were cold enough, almost all of the particles would congregate in the ground states (BEC).



A. Einstein
(1879 $\sim$ 1955)

S. Bose
(1894 $\sim$ 1974)

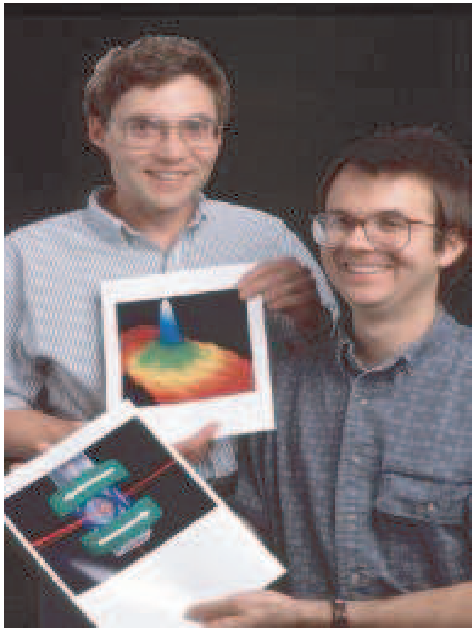
- Physical experiments
  - Superfluid He^4 1938:
    P. L. Kapitza, Allen and Misener: discovered the superfluidity of liquid helium.
    F. London: proposed that the superfluid fraction consisting of those atoms which have “condensed” to the ground state.

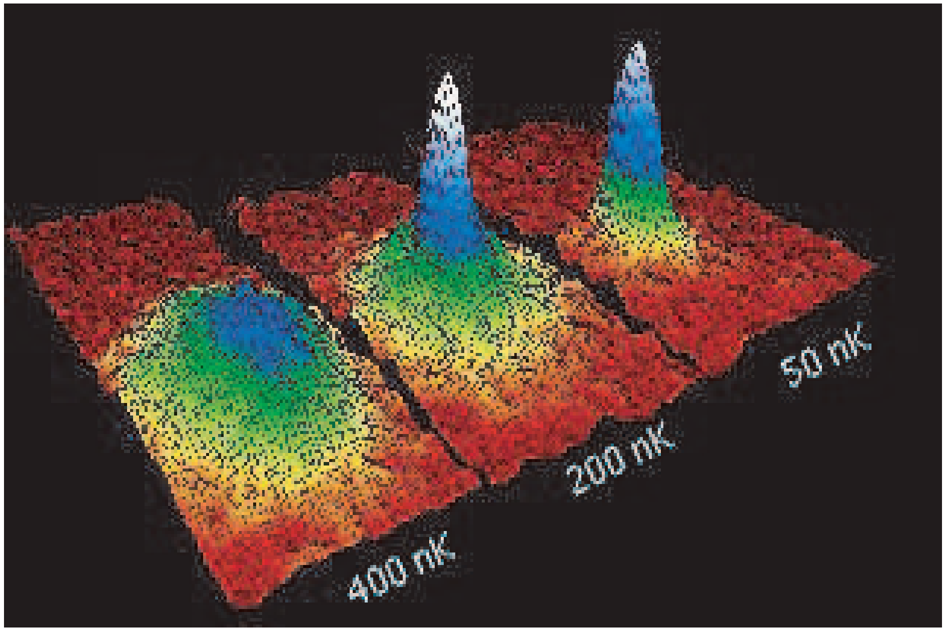


P. L.Kapitza
(1894 ∼ 1984)

F. London
(1900 ∼ 1954)

- – E. A. Cornell & C. E. Wieman (JILA, 1995):
  first observed BEC of rubidium (^{87}Rb) atoms at 20 nK, i.e. 0.000 000 02 K.

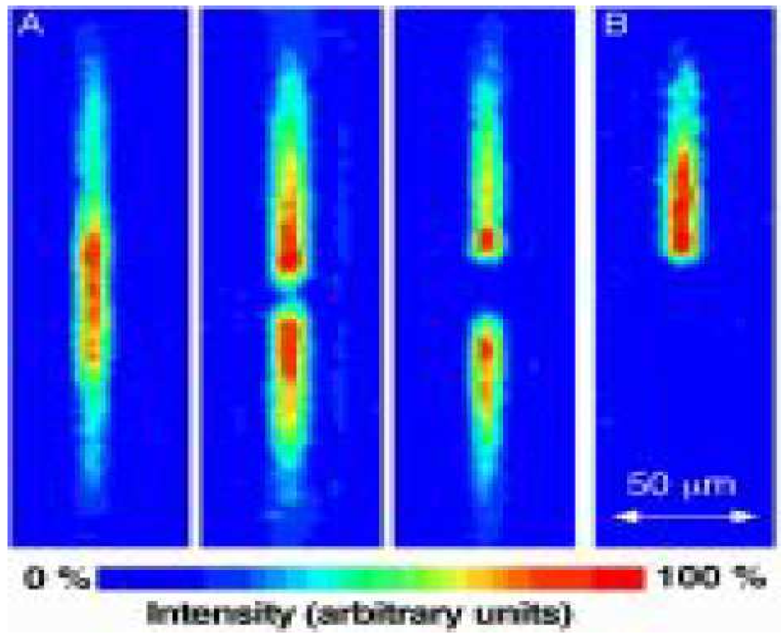




C. E. Wieman & E. A. Cornell

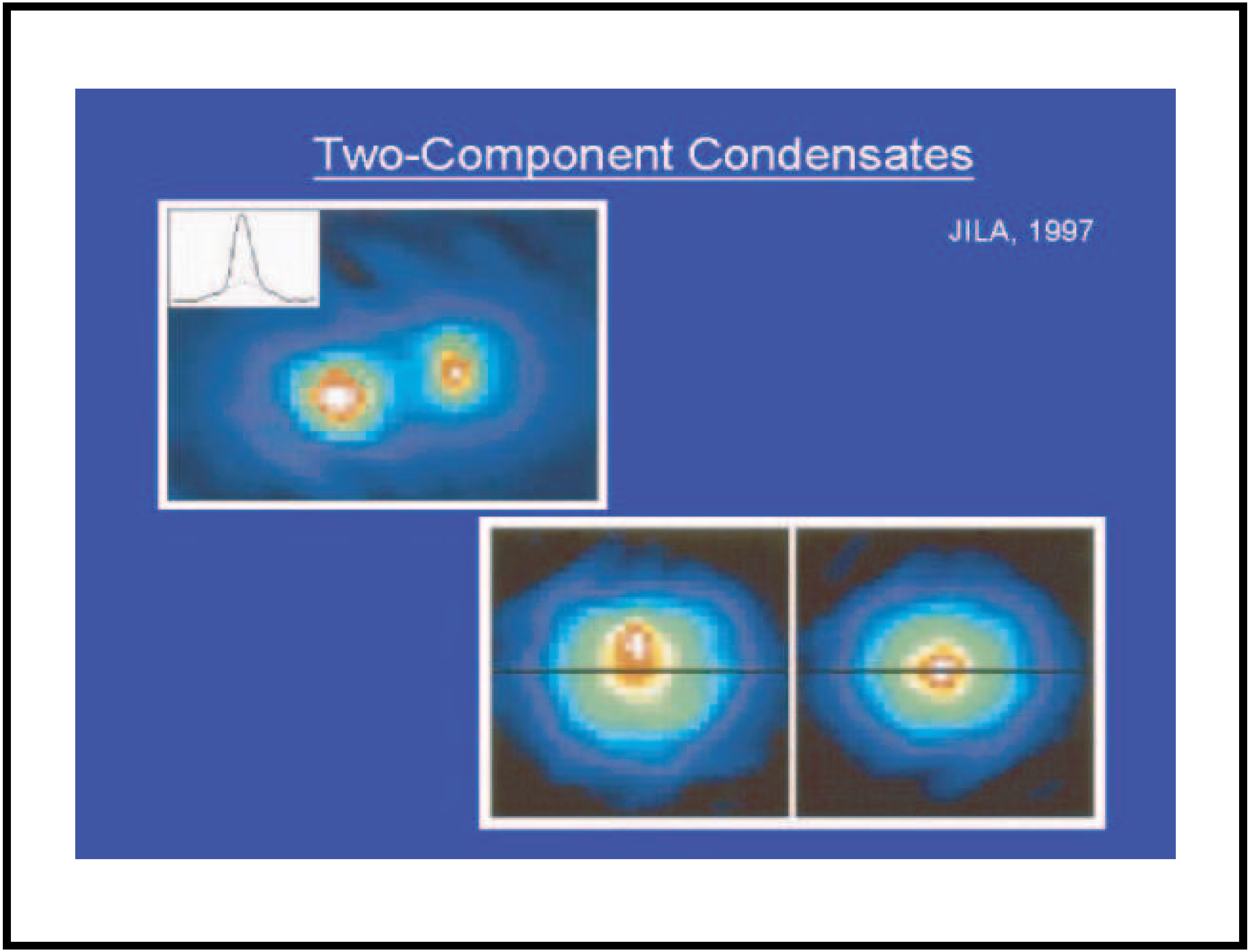
BEC at 400, 200, and 50 nK

– W. Ketterle (MIT, 1995):
observed BEC of sodium (^{23}Na) atoms.



W. Ketterle

Two-Component BEC


Two-Component Condensates
JILA, 1997

- Experimental implementation
  - The BEC named Science Magazine's "Molecule of the Year 1995"!
  - Nobel Prize in Physics (2001), E. A. Cornell, C. E. Wieman (JILA), W. Ketterle (MIT):
    for the achievement of BEC in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates.
- Applications of BEC: atom laser, quantum computer, MEMS.
- Mathematical model: nonlinear Schrödinger equations, Gross-Pitaevskii equations (GPEs), coupled nonlinear Schrödinger equations, coupled Gross-Pitaevskii equations (CGPEs).
- Numerical simulation: method, guide for experiment etc.

2 Coupled Nonlinear Schrödinger Eqs. and CGPEs

$$\iota\hbar\frac{\partial\psi_j(\boldsymbol{x},t)}{\partial t}=-\frac{\hbar^2}{2m_a}\nabla^2\psi_j+\widehat{V}_j\psi_j+\mu_{jj}|\psi_j|^2\psi_j+\sum_{j\neq i}\mu_{ij}|\psi_i|^2\psi_j,\quad j=1,\ldots,m.$$

- Coupled Gross-Pitaevskii equations (CGPEs):

$$\iota\hbar\frac{\partial\boldsymbol{\psi}(\boldsymbol{x},t)}{\partial t}=-\frac{\hbar^2}{2m_a}\nabla^2\boldsymbol{\psi}(\boldsymbol{x},t)+\widehat{\boldsymbol{V}}(\boldsymbol{x})\circ\boldsymbol{\psi}(\boldsymbol{x},t)+\widehat{\boldsymbol{B}}(\boldsymbol{\psi})\circ\boldsymbol{\psi}(\boldsymbol{x},t),\tag{2.1}$$

$\boldsymbol{\psi}(\boldsymbol{x},t)=(\psi_1(\boldsymbol{x},t),\ldots,\psi_m(\boldsymbol{x},t))^\top$: wave ft.,

$\widehat{\boldsymbol{V}}(\boldsymbol{x})=(\widehat{V}_1(\boldsymbol{x}),\ldots,\widehat{V}_m(\boldsymbol{x}))^\top$: trap potential,

$\widehat{\boldsymbol{B}}(\boldsymbol{\psi})=(\widehat{B}_1(\boldsymbol{\psi}),\ldots,\widehat{B}_m(\boldsymbol{\psi}))^\top$: intra- and inter-species scattering lengths,where $\widehat{B}_j(\boldsymbol{\psi})=\mu_{j1}|\psi_1|^2+\cdots+\mu_{jm}|\psi_m|^2$.

$$\widehat{V}_j(\boldsymbol{x}) = \frac{m_a}{2}\left(\omega_{x,j}^2(x-\hat{x}_{0,j})^2 + \omega_{y,j}^2(y-\hat{y}_{0,j})^2 + \omega_{z,j}^2(z-\hat{z}_{0,j})^2\right).$$

$\mu_{j\ell} = \frac{4\pi\hbar^2 b_{j\ell}}{m_a}$, $\quad b_{j\ell}$: scattering length. (+: repulsive, −: attractive)

$$\int_{\mathbb{D}} |\psi_j(\boldsymbol{x},t)|^2 d\boldsymbol{x} = N_j^0 > 0, \quad j = 1,\ldots,m.$$

Assume that

(i) $\omega_{x,1} \le \cdots \le \omega_{x,m} \le \omega_{y,1} \le \cdots \le \omega_{y,m} \le \omega_{z,1} \le \cdots \le \omega_{z,m}$.

(ii) $b_{j\ell} = b_{\ell j}, \quad j,\ell = 1,\ldots,m.$

- Dimensionless

$$\tilde{t} = \frac{t}{t_s}, \quad \tilde{\boldsymbol{x}} = \frac{\boldsymbol{x}}{x_s}, \quad \tilde{\psi}_j(\tilde{\boldsymbol{x}}, \tilde{t}) = \frac{x_s^{3/2}}{\sqrt{N_j^0}} \psi_j(\boldsymbol{x}, t).$$

$$t_s = \frac{1}{\omega_{x,1}} \; : \text{ dimensionless "time".}$$

$$x_s = \sqrt{\frac{\hbar}{m_a \omega_{x,1}}} \; : \text{ dimensionless "length".}$$

- Dimensionless CGPEs

$$\boldsymbol{\iota}\frac{\partial\boldsymbol{\psi}(\boldsymbol{x},t)}{\partial t}=-\frac{1}{2}\nabla^2\boldsymbol{\psi}(\boldsymbol{x},t)+\boldsymbol{V}(\boldsymbol{x})\circ\boldsymbol{\psi}(\boldsymbol{x},t)+\boldsymbol{B}(\boldsymbol{\psi})\circ\boldsymbol{\psi}(\boldsymbol{x},t), \quad (2.2)$$

$$\boldsymbol{\psi}(\boldsymbol{x},t)=0,\quad \boldsymbol{x}\in\mathbb{D}, \quad (2.3)$$

$\mathbb{D}$: A smooth bounded domain in $\mathbb{R}^d$, $d=2,3$,

$$\mathrm{n}(\psi_j):=\int_{\mathbb{D}}|\psi_j(\boldsymbol{x},t)|^2d\boldsymbol{x}=1,\quad j=1,\ldots,m,$$

$$\boldsymbol{B}(\boldsymbol{\psi})=(B_1(\boldsymbol{\psi}),\ldots,B_m(\boldsymbol{\psi}))^\top, B_j(\boldsymbol{\psi})=\beta_{j1}|\psi_1|^2+\cdots+\beta_{jm}|\psi_m|^2.$$

$$
\begin{aligned}
&\boldsymbol{\psi}(\boldsymbol{x},t) = (\psi_1(\boldsymbol{x},t),\ldots,\psi_m(\boldsymbol{x},t))^\top,\\
&\boldsymbol{V}(\boldsymbol{x}) = (V_1(\boldsymbol{x}),\ldots,V_m(\boldsymbol{x}))^\top,\\
&V_j(\boldsymbol{x}) = \frac{1}{2}(\gamma_{x,j}^2(x-x_{0,j})^2 + \gamma_{y,j}^2(y-y_{0,j})^2 + \gamma_{z,j}^2(z-z_{0,j})^2),\\
&\gamma_{x,j} = \frac{\omega_{x,j}}{\omega_{x,1}}, \quad \gamma_{y,j} = \frac{\omega_{y,j}}{\omega_{x,1}}, \quad \gamma_{z,j} = \frac{\omega_{z,j}}{\omega_{x,1}},\\
&x_{0,j} = \frac{\hat{x}_{0,j}}{b_0}, \quad y_{0,j} = \frac{\hat{y}_{0,j}}{b_0}, \quad z_{0,j} = \frac{\hat{z}_{0,j}}{b_0},\\
&\beta_{j,\ell} = \frac{\mu_{j\ell}N_\ell^0}{b_0^3\hbar\omega_{x,1}} = \frac{4\pi\hbar^2 b_{j\ell}N_\ell^0}{m_a b_0^3\hbar\omega_{x,1}} = \frac{4\pi b_{j\ell}N_\ell^0}{b_0}.
\end{aligned}
$$

Energy

$$E(\boldsymbol{\psi}) = \sum_{j=1}^{m} \frac{N_j^0}{N_0} E_j(\boldsymbol{\psi}),$$

where $N_j^0 > 0$ is the number of particles with $\sum_{j=1}^{m} N_j^0 = N^0$ and

$$E_j(\boldsymbol{\psi}) = \int_{\mathbb{D}} \left[\frac{1}{2} |\nabla \psi_j|^2 + V_j(\boldsymbol{x})|\psi_j|^2 + \frac{1}{2} \sum_{k=1}^{m} \beta_{j,k} |\psi_j|^2 |\psi_k|^2 \right] d\boldsymbol{x},$$

for $j = 1, \ldots, m$.

Let $\boldsymbol{\psi}(\boldsymbol{x},t) = e^{-\boldsymbol{\iota}\boldsymbol{\lambda}^{(c)}t} \circ \boldsymbol{\phi}(\boldsymbol{x})$, where $\boldsymbol{\lambda}^{(c)} = (\lambda_1^{(c)}, \ldots, \lambda_m^{(c)})^\top$, $\boldsymbol{\phi}(\boldsymbol{x}) = (\phi(\boldsymbol{x}), \ldots, \phi_m(\boldsymbol{x}))^\top$.

Substituting $\boldsymbol{\psi}(\boldsymbol{x},t)$ into CGPEs gives the NEP for $(\boldsymbol{\lambda}, \boldsymbol{\phi})$:

$$\boldsymbol{\lambda}^{(c)} \circ \boldsymbol{\phi}(\boldsymbol{x}) = -\frac{1}{2}\nabla^2\boldsymbol{\phi}(\boldsymbol{x}) + \boldsymbol{V}(\boldsymbol{x}) \circ \boldsymbol{\phi}(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{\phi}) \circ \boldsymbol{\phi}(\boldsymbol{x}), \qquad (2.4)$$

with $\int_{\mathbb{D}} |\phi_j(\boldsymbol{x})|^2 d\boldsymbol{x} = 1,\ j = 1, \ldots, m$, where

$$\boldsymbol{B}(\boldsymbol{\phi}) = (B_1(\boldsymbol{\phi}), \ldots, B_m(\boldsymbol{\phi}))^\top, \quad B_j(\boldsymbol{\phi}) = \sum_{k=1}^{m} \beta_{jk}|\phi_k|^2.$$

Multiplying the j-th eq. in NEP (2.4) by $\phi_j(\boldsymbol{x})$, the eigenvalue $\lambda_j^{(c)}$ and the corresp. eigenfunction ϕ_j for (2.4) satisfy

$$\lambda_j^{(c)} = \int_{\mathbb{D}} \left[\frac{1}{2}|\nabla\phi_j|^2 + V_j(\boldsymbol{x})|\phi_j|^2 + \sum_{k=1}^{m} \beta_{jk}|\phi_j|^2|\phi_k|^2 \right] d\boldsymbol{x}$$

$$= E_j(\boldsymbol{\phi}) + \frac{1}{2}\int_{\mathbb{D}} \sum_{k=1}^{m} \beta_{jk}|\phi_j|^2|\phi_k|^2 d\boldsymbol{x}.$$

The ground state $\boldsymbol{\phi}_g(\boldsymbol{x})$ of multi-comp. BEC can be found by minimizing $E(\boldsymbol{\phi})$:

$$\begin{aligned} &\underset{\boldsymbol{\phi}=(\phi_1,\ldots,\phi_m)^\top}{\text{Minimize}} E(\boldsymbol{\phi}) \\ &\text{subject to } \int_{\mathbb{D}} |\phi_j(\boldsymbol{x})|^2 d\boldsymbol{x} = 1, \quad j = 1,\ldots,m, \end{aligned} \tag{2.5}$$

where $E(\boldsymbol{\phi}) = \sum_{j=1}^{m} \frac{N_j^0}{N^0} E_j(\boldsymbol{\phi})$ with

$$E_j(\boldsymbol{\phi}) = \int_{\mathbb{D}} \left[\frac{1}{2}|\nabla\phi_j|^2 + V_j(\boldsymbol{x})|\phi_j|^2 + \frac{1}{2}\sum_{k=1}^{m} \beta_{jk}|\phi_j|^2|\phi_k|^2 \right] d\boldsymbol{x}.$$

The NEP (2.4) can be regarded as the Euler-Lagrange eq. of the opt. problem (2.5).

3 Nonlinear Algebraic Eigenvalue Problems (NAEPs)

For computational purpose, we derive the discretization of NEP and the associated opt. problem. We consider $\mathbb{D} \subseteq \mathbb{R}^2$ a bounded domain.

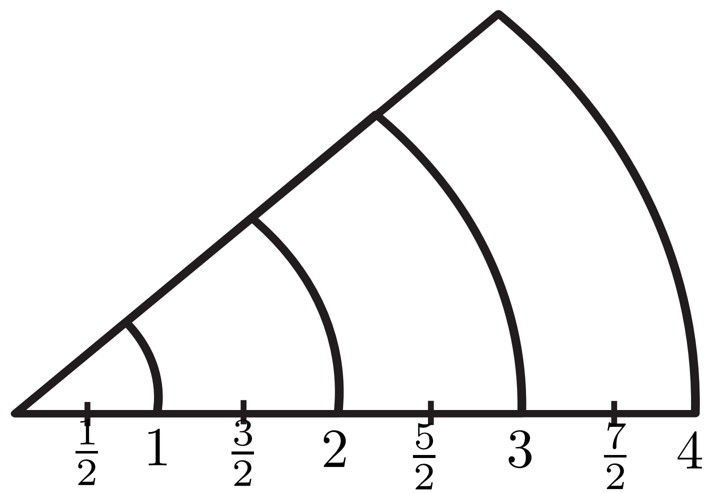
The central finite difference discretizes $-\nabla^2 \phi_j(\boldsymbol{x})$ into

$$\boldsymbol{A}\boldsymbol{u}_j = \boldsymbol{A}[u_{j1}, \ldots, u_{jl}, \ldots, u_{jN}]^\top, \quad \boldsymbol{A} \in \mathbb{R}^{N\times N},$$

where $\boldsymbol{u}_j$ is an approx. of the j-th wave ft. $\phi_j(\boldsymbol{x})$.

$\widehat{\boldsymbol{A}}$ and $\widehat{\boldsymbol{A}}^\top$ are irreducible and diag.-dominant with positive diag. and nonpositive off-diag. elements. $\widehat{\boldsymbol{A}}$ is symmetrizable to a s.p.d. $\boldsymbol{A}$ by a $\boldsymbol{D} > 0$, i.e.,

$$\widehat{\boldsymbol{A}} = \boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}, \quad \boldsymbol{A}^\top = \boldsymbol{A} \succ 0.$$



Note that $\boldsymbol{D} = \mathrm{diag}(d_{l_1,l_2})$ with $d^2_{l_1,l_2} = (l_1 - \frac{1}{2})\delta r^2 \delta\theta$ is equal to the area of the $(l_1 + \nu(l_2 - 1))$-th sector.

Applying $-\nabla^2 \approx \boldsymbol{A}$ to NEP and rewriting $u_j \equiv hu_j, \beta_{jk} = \frac{\beta_{jk}}{h^2}$, the discretization of NEP, referred as a NAEPs, can be formulated by

$$\frac{1}{2}\boldsymbol{A}\boldsymbol{u}_j + \boldsymbol{V}_j \circ \boldsymbol{u}_j + \sum_{k=1}^{m} \beta_{jk} \boldsymbol{u}_k^{②} \circ \boldsymbol{u}_j = \lambda_j^{(c)} \boldsymbol{u}_j, \tag{3.1}$$

$$\boldsymbol{u}_j^\top \boldsymbol{u}_j = 1, \quad j = 1, \ldots, m, \tag{3.2}$$

where $\boldsymbol{V}_j = [V_j, \ldots, V_j], j = 1, \ldots, m$. h is the grid size.

Let $\boldsymbol{u} = (\boldsymbol{u}_1^\top, \dots, \boldsymbol{u}_m^\top)^\top$. Since the j-th kinetic energy

$$\int_{\mathbb{D}} \frac{1}{2} |\nabla \phi_j|^2 d\boldsymbol{x} = -\int_{\mathbb{D}} \phi_j (\nabla^2 \phi_j) d\boldsymbol{x},$$

we approximate it by $\frac{1}{2}\boldsymbol{u}_j^\top \boldsymbol{A} \boldsymbol{u}_j$. Then the discretized eq. of the j-th energy $E_j(\boldsymbol{\phi})$ becomes

$$E_j(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{u}_j^\top \boldsymbol{A} \boldsymbol{u}_j + \boldsymbol{V}_j^\top \boldsymbol{u}_j^{②} + \frac{1}{2} \sum_{k=1}^{m} \beta_{jk} \boldsymbol{u}_k^{② \top} \boldsymbol{u}_j^{②}.$$

Multiplying NAEPs (3.1) by $\boldsymbol{u}_j^\top$, the eigenvalues $\boldsymbol{\lambda}^{(c)} = (\lambda_1^{(c)}, \ldots, \lambda_m^{(c)})^\top$ and the assoc. EVs $\boldsymbol{u} = (\boldsymbol{u}_1^\top, \ldots, \boldsymbol{u}_m^\top)^\top$ satisfy

$$\begin{aligned} \lambda_j^{(c)} &= \frac{1}{2}\boldsymbol{u}_j^\top \boldsymbol{A}\boldsymbol{u}_j + \boldsymbol{V}_j^\top \boldsymbol{u}_j^{②} + \sum_{k=1}^{m} \beta_{jk} \boldsymbol{u}_k^{②\top} \boldsymbol{u}_j^{②} \\ &= E_j(\boldsymbol{u}) + \frac{1}{2}\sum_{k=1}^{m} \beta_{jk} \boldsymbol{u}_k^{②\top} \boldsymbol{u}_j^{②}, \quad j = 1, \ldots, m. \end{aligned}$$

Furthermore,

$$E(\boldsymbol{u}) = \sum_{j=1}^{m} \frac{N_j^0}{N^0} E_j(\boldsymbol{u}) = \sum_{j=1}^{m} \frac{N_j^0}{N^0} \left(\lambda_j^{(c)} - \frac{1}{2}\sum_{k=1}^{m} \beta_{jk} \boldsymbol{u}_k^{②\top} \boldsymbol{u}_j^{②} \right).$$

For convenience, we let $N_j^0/N^0 = 1/m$. Then

$$E(\boldsymbol{u}) = \frac{1}{m}\sum_{j=1}^{m} E_j(\boldsymbol{u}) = \frac{1}{2m}\sum_{j=1}^{m} \boldsymbol{u}_j^\top \boldsymbol{A}\boldsymbol{u}_j + \frac{1}{m}\sum_{j=1}^{m} \boldsymbol{V}_j^\top \boldsymbol{u}_j^{②}$$
$$+ \frac{1}{2m}\sum_{j=1}^{m} \beta_{jj}\boldsymbol{u}_j^{②\top}\boldsymbol{u}_j^{②} + \frac{1}{m}\sum_{1\le j<k\le m} \beta_{jk}\boldsymbol{u}_k^{②\top}\boldsymbol{u}_j^{②}.$$

The discretization of the opt. problem (2.5) becomes

$$\begin{aligned} &\underset{\boldsymbol{u}=(\boldsymbol{u}_1^\top,\ldots,\boldsymbol{u}_m^\top)^\top}{\text{Minimize}} E(\boldsymbol{u}) \\ &\text{subject to } \boldsymbol{u}_j^\top \boldsymbol{u}_j = 1, \quad j = 1,\ldots,m. \end{aligned} \tag{3.3}$$

Applying the optimality condition to the opt. problem (3.3), a local minimum $((\lambda_1^{(L)}, \ldots, \lambda_m^{(L)}), (\boldsymbol{u}_1^\top, \ldots, \boldsymbol{u}_m^\top)^\top)$ satisfies the Karash-Kuhn-Tucker (KKT) eq.

$$\frac{1}{m}\left(\boldsymbol{A} + [\![\boldsymbol{V}_j]\!] + 2\beta_{jj}[\![\boldsymbol{u}_j^{②}]\!]\right)\boldsymbol{u}_j + \frac{2}{m}\sum_{k\neq j}\beta_{jk}\boldsymbol{u}_k^{②} \circ \boldsymbol{u}_j = 2\lambda_j^{(L)}\boldsymbol{u}_j, \tag{3.4}$$

where $\{\lambda_j^{(L)}\}_{j=1}^m$ are Lagrange multipliers. Multiplying (3.4) by $m/2$ gives

$$\frac{1}{2}\boldsymbol{A}\boldsymbol{u}_j + \boldsymbol{V}_j \circ \boldsymbol{u}_j + \sum_{k=1}^m \beta_{jk}\boldsymbol{u}_k^{②} \circ \boldsymbol{u}_j = m\lambda_j^{(L)}\boldsymbol{u}_j.$$

We now define

$$\boldsymbol{A}_j := \boldsymbol{A} + 2[\![\boldsymbol{V}_j]\!] + 2\beta_{jj}[\![\boldsymbol{u}_j^{②}]\!],$$

$$\lambda_j := 2m\lambda_j^{(L)}, \quad \beta_{jk} := 2\beta_{jk}, \quad j \neq k,$$

for $j, k = 1, \ldots, m$. Then the NAEPs becomes

$$\boldsymbol{A}_j \boldsymbol{u}_j + \sum_{k \neq j} \beta_{jk} \boldsymbol{u}_k^{②} \circ \boldsymbol{u}_j = \lambda_j \boldsymbol{u}_j, \quad j = 1, \ldots, m$$

and the associated opt. problem (3.3) becomes

$$\underset{\boldsymbol{u}=(\boldsymbol{u}_1^\top, \ldots, \boldsymbol{u}_m^\top)^\top}{\text{Minimize}} \; E(\boldsymbol{u})$$

$$\text{subject to } \boldsymbol{u}_j^\top \boldsymbol{u}_j = 1, \quad j = 1, \ldots, m,$$

where

$$E(\boldsymbol{u}) \equiv \sum_{j=1}^{m} \left(\frac{1}{2} \boldsymbol{u}_j^\top \boldsymbol{A} \boldsymbol{u}_j + \boldsymbol{V}_j^\top \boldsymbol{u}_j^{②} \right) + \frac{1}{2} \sum_{1 \leq j \leq k \leq m} \beta_{jk} \boldsymbol{u}_k^{②\top} \boldsymbol{u}_j^{②}.$$

4 Fixed Point Iteration for NAEPs

Define

$$\mathcal{M} = \{\boldsymbol{v} \in \mathbb{R}^N | \boldsymbol{v}^\top \boldsymbol{v} = 1,\ \boldsymbol{v} \geq 0\}, \quad \overset{\circ}{\mathcal{M}} = \text{ interior of } \mathcal{M}.$$

We suppose

$$\beta_{jj} > 0 \text{ small}, \quad \beta_{jk} = \beta_{kj} > 0\ (j \neq k), \quad j, k = 1, \ldots, m.$$

$\boldsymbol{A}$ in (3.1) is diagonal dominant and $\boldsymbol{Ae} \gneqq 0$, where $\boldsymbol{e} = (1, \ldots, 1)^\top$. For $\boldsymbol{V}_j \geq 0$ and $(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) \in \mathop{\times}\limits_{j=1}^{m} \mathcal{M}$, the matrix

$$\bar{\boldsymbol{A}}_j \equiv \boldsymbol{A}_j + \sum_{k=1}^{m} [\![\beta_{jk} \boldsymbol{u}_k^{②}]\!],$$

with $\boldsymbol{A}_j = \boldsymbol{A} + 2[\![\boldsymbol{V}_j]\!]$ is an irreducible M-matrix. Then $\bar{\boldsymbol{A}}_j^{-1} \geq 0$ is an irreducible and nonnegative matrix.

By Perron-Frobenious Theorem there is a unique positive eigenvector $\bar{\boldsymbol{u}}_j > 0$ with $\bar{\boldsymbol{u}}_j^\top \bar{\boldsymbol{u}}_j = 1$ corresp. to the maximal eigenvalue $\mu_j^{\max}$ of $\bar{\boldsymbol{A}}_j^{-1}$. i.e., $\bar{\boldsymbol{u}}_j > 0$ is uniquely determined by $(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m)$ and satisfies

$$\bar{A}_j \bar{\boldsymbol{u}}_j \equiv \left(\boldsymbol{A}_j + \sum_{k=1}^{m} [\![\beta_{jk} \boldsymbol{u}_k^{②}]\!] \right) \bar{\boldsymbol{u}}_j = \lambda_j^{\min} \bar{\boldsymbol{u}}_j,$$

where $\lambda_j^{\min} = 1/\mu_j^{\max}$ and $\bar{\boldsymbol{u}}_j^\top \bar{\boldsymbol{u}}_j = 1$, for $j = 1, \ldots, m$.

We now define a function $\boldsymbol{f} : \mathop{\times}\limits_{j=1}^{m} \mathcal{M} \to \mathop{\times}\limits_{j=1}^{m} \mathcal{M}$ by

$$\boldsymbol{f}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = (\bar{\boldsymbol{u}}_1, \ldots, \bar{\boldsymbol{u}}_m),$$

where $\bar{\boldsymbol{u}}_j > 0$ is well-defined, $j = 1, \ldots, m$.

Theorem 4.1 *The function $\boldsymbol{f}$ given has a fixed point in $\mathop{\times}\limits_{j=1}^{m} \mathring{\mathcal{M}}$. In other words, there is a point $(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_m^*) \in \mathop{\times}\limits_{j=1}^{m} \mathring{\mathcal{M}}$ and $\boldsymbol{\lambda} = (\lambda_1^*, \ldots, \lambda_m^*)$ which solve the NAEPs, that is,*

$$\boldsymbol{A}_j \boldsymbol{u}_j^* + \sum_{k=1}^{m} \beta_{jk} \boldsymbol{u}_k^{*②} \circ \boldsymbol{u}_j^* = \lambda_j^* \boldsymbol{u}_j^*, \quad j = 1, \ldots, m.$$

We define the restricted Lagragian function of the opt. problem by

$$L(\boldsymbol{u}) = E(\boldsymbol{u}) - \frac{1}{2}\sum_{j=1}^{m} \lambda_j (\boldsymbol{u}_j^\top \boldsymbol{u}_j - 1),$$

where

$$E(\boldsymbol{u}) \equiv \frac{1}{2}\sum_{j=1}^{m} \boldsymbol{u}_j^\top \boldsymbol{A}_j \boldsymbol{u}_j + \frac{1}{2}\sum_{1\le j<k\le m} \beta_{jk} \boldsymbol{u}_k^{②\top} \boldsymbol{u}_j^{②}.$$

Theorem 4.2 *Let $\boldsymbol{u}^* = (\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_m^*)$ be a KKT point of the opt. problem assoc. with the Lagrangian multipliers $(\lambda_1^*, \ldots, \lambda_m^*)$. Denote the Hessian of $L(\boldsymbol{u})$ at $\boldsymbol{u}^*$ by $\nabla^2 L(\boldsymbol{u}^*) = \left[\nabla^2 L(\boldsymbol{u}^*)_{ij}\right]_{i,j=1}^{m}$, where*

$$\nabla^2 L(\boldsymbol{u}^*)_{jj} = \left(\boldsymbol{A}_j + \sum_{k=1}^{m} [\![\beta_{jk} \boldsymbol{u}_k^{*②}]\!] - \lambda_j^* \boldsymbol{I}_N\right)$$

and

$$\nabla^2 L(\boldsymbol{u}^*)_{ij} = \nabla^2 L(\boldsymbol{u}^*)_{ji} = 2[\![\beta_{ji} \boldsymbol{u}_i^* \circ \boldsymbol{u}_j^*]\!], \quad j \neq i,$$

Let $\boldsymbol{d} = (\boldsymbol{d}_1^\top, \ldots, \boldsymbol{d}_m^\top)^\top \in \mathbb{R}^{Nm}$. The positivity condition

$$\boldsymbol{d}^\top (\nabla^2 L(\boldsymbol{u}^*)) \boldsymbol{d} > 0$$

holds, for all $\boldsymbol{d}$ with ${\boldsymbol{u}_j^*}^\top \boldsymbol{d}_j = 0$, $j = 1, \ldots, m$, if and only if $\boldsymbol{u}^*$ is a strictly local minimum of the opt. problem.

Jacobi Iteration (JI)

Define $\boldsymbol{f} : \bigtimes_{j=1}^{m} \mathcal{M} \to \bigtimes_{j=1}^{m} \mathcal{M}$ by

$$\boldsymbol{f}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = (\bar{\boldsymbol{u}}_1, \ldots, \bar{\boldsymbol{u}}_m),$$

where $\bar{\boldsymbol{u}}_j > 0$ is well-defined, $j = 1, \ldots, m$.

Theorem 4.3 *Let* $(\boldsymbol{\lambda}^*, \boldsymbol{u}^*) = ((\lambda_1^*, \ldots, \lambda_m^*), (\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_m^*))$ *be a fixed point of NAEPs. Suppose* $\beta_{jj} > 0$ *suff. small,* $j = 1, \ldots, m$*. If the JI converges to* $(\boldsymbol{\lambda}^*, \boldsymbol{u}^*)$ *locally and linearly with an initial in* $\bigtimes_{j=1}^{m} \overset{\circ}{\mathcal{M}}$*, then* $\boldsymbol{u}^* = (\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_m^*)$ *is a strictly local min. of the opt. problem.*

Gauss-Seidel Iteration (GSI)

Define $\boldsymbol{g} : \mathop{\times}\limits_{j=1}^{m} \mathcal{M} \to \mathop{\times}\limits_{j=1}^{m} \mathcal{M}$ by

$$\boldsymbol{g}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = (\bar{\boldsymbol{u}}_1, \ldots, \bar{\boldsymbol{u}}_m),$$

where

$$\begin{aligned}
\bar{\boldsymbol{u}}_1 &= \boldsymbol{g}_1(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = \boldsymbol{f}_1(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m), \\
\bar{\boldsymbol{u}}_2 &= \boldsymbol{g}_2(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = \boldsymbol{f}_2(\bar{\boldsymbol{u}}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \ldots, \boldsymbol{u}_m), \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
\bar{\boldsymbol{u}}_m &= \boldsymbol{g}_m(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = \boldsymbol{f}_m(\bar{\boldsymbol{u}}_1, \bar{\boldsymbol{u}}_2, \ldots, \bar{\boldsymbol{u}}_{m-1}, \boldsymbol{u}_m),
\end{aligned}$$

in which $\{\boldsymbol{f}_j\}_{j=1}^m$ are given in JI. The ft. $\boldsymbol{g}$ defines a Gauss-Seidel type iteration (GSI).

Theorem 4.4 *Let* $(\boldsymbol{\lambda}^*, \boldsymbol{u}^*) = ((\boldsymbol{\lambda}_1^*, \ldots, \boldsymbol{\lambda}_m^*), (\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_m^*))$ *be a fixed point of the NAEPs. Suppose the matrix* $\boldsymbol{\mathcal{Z}}^\top \nabla^2 L(\boldsymbol{u}^*) \boldsymbol{\mathcal{Z}}$ *is nonsingular. Suppose* $\beta_{jj} > 0$ *suff. small,* $j = 1, \ldots, m$*. The GSI converges to* $(\boldsymbol{\lambda}^*, \boldsymbol{u}^*)$ *locally and linearly with an initial in* $\overset{m}{\underset{j=1}{\times}} \overset{\circ}{\mathcal{M}}$ *iff* $\boldsymbol{u}^* = (\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_m^*)$ *is a strictly local min. of the opt. problem.*

$$\boldsymbol{J}_{\boldsymbol{s}}^{*} = \boldsymbol{\Omega}^{*\frac{1}{2}} \boldsymbol{J}_{\boldsymbol{f}}^{*} \boldsymbol{\Omega}^{*-\frac{1}{2}} = -2 \left[\ * \ \right],$$

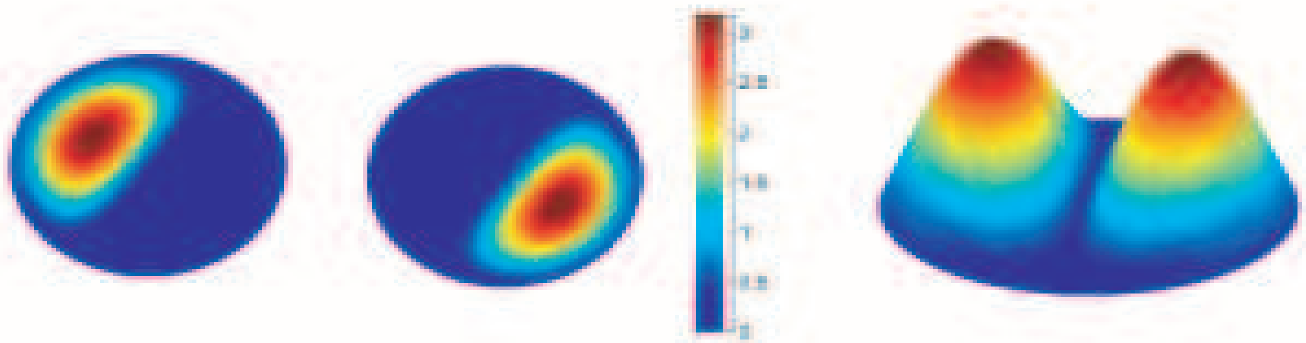
$$\begin{bmatrix} \mathbf{0} & \beta_{12} \boldsymbol{\Omega}_1^{*-\frac{1}{2}} \boldsymbol{Z}_1^{*T} [\![\boldsymbol{u}_1^* \circ \boldsymbol{u}_2^*]\!] \boldsymbol{Z}_2^* \boldsymbol{\Omega}_2^{*-\frac{1}{2}} & \cdots & \cdots & \beta_{1m} \boldsymbol{\Omega}_1^{*-\frac{1}{2}} \boldsymbol{Z}_1^{*T} [\![\boldsymbol{u}_1^* \circ \boldsymbol{u}_m^*]\!] \boldsymbol{Z}_m^* \boldsymbol{\Omega}_m^{*-\frac{1}{2}} \\ & \mathbf{0} & & & \beta_{2m} \boldsymbol{\Omega}_2^{*-\frac{1}{2}} \boldsymbol{Z}_2^{*T} [\![\boldsymbol{u}_2^* \circ \boldsymbol{u}_m^*]\!] \boldsymbol{Z}_m^* \boldsymbol{\Omega}_m^{*-\frac{1}{2}} \\ & & \ddots & & \vdots \\ & \text{Symm.} & & \ddots & \beta_{m-1,m} \boldsymbol{\Omega}_{m-1}^{*-\frac{1}{2}} \boldsymbol{Z}_{m-1}^{*T} [\![\boldsymbol{u}_1^* \circ \boldsymbol{u}_2^*]\!] \boldsymbol{Z}_m^* \boldsymbol{\Omega}_m^{*-\frac{1}{2}} \\ & & & & \mathbf{0} \end{bmatrix}$$

where

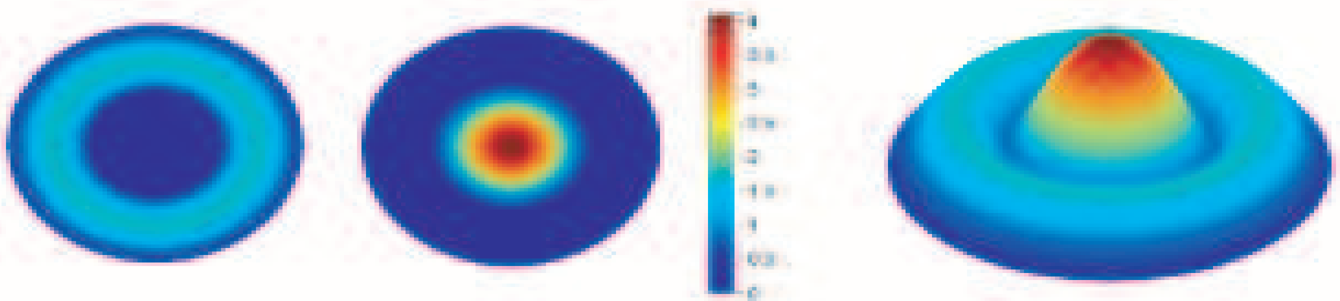
$$\boldsymbol{\Omega}^{*\frac{1}{2}} = \mathrm{diag}\{\boldsymbol{\Omega}_1^{*\frac{1}{2}}, \ldots, \boldsymbol{\Omega}_m^{*\frac{1}{2}}\}, \quad \boldsymbol{\Omega}_j^* = \mathrm{diag}\left\{ \frac{1}{\zeta_{j2}^* - \lambda_j^*}, \ldots, \frac{1}{\zeta_{jN}^* - \lambda_j^*} \right\}.$$

5 Numerical Results

Two-component BEC



(a) green: $\beta^* = 1000$, $\lambda_1^* = \lambda_2^* = 7.07$, $E(\boldsymbol{u}^*) = 7.02$



(b) red: $\beta^* = 1000$, $\lambda_1^* = 10.34$, $\lambda_2^* = 14.54$, $E(\boldsymbol{u}^*) = 12.43$

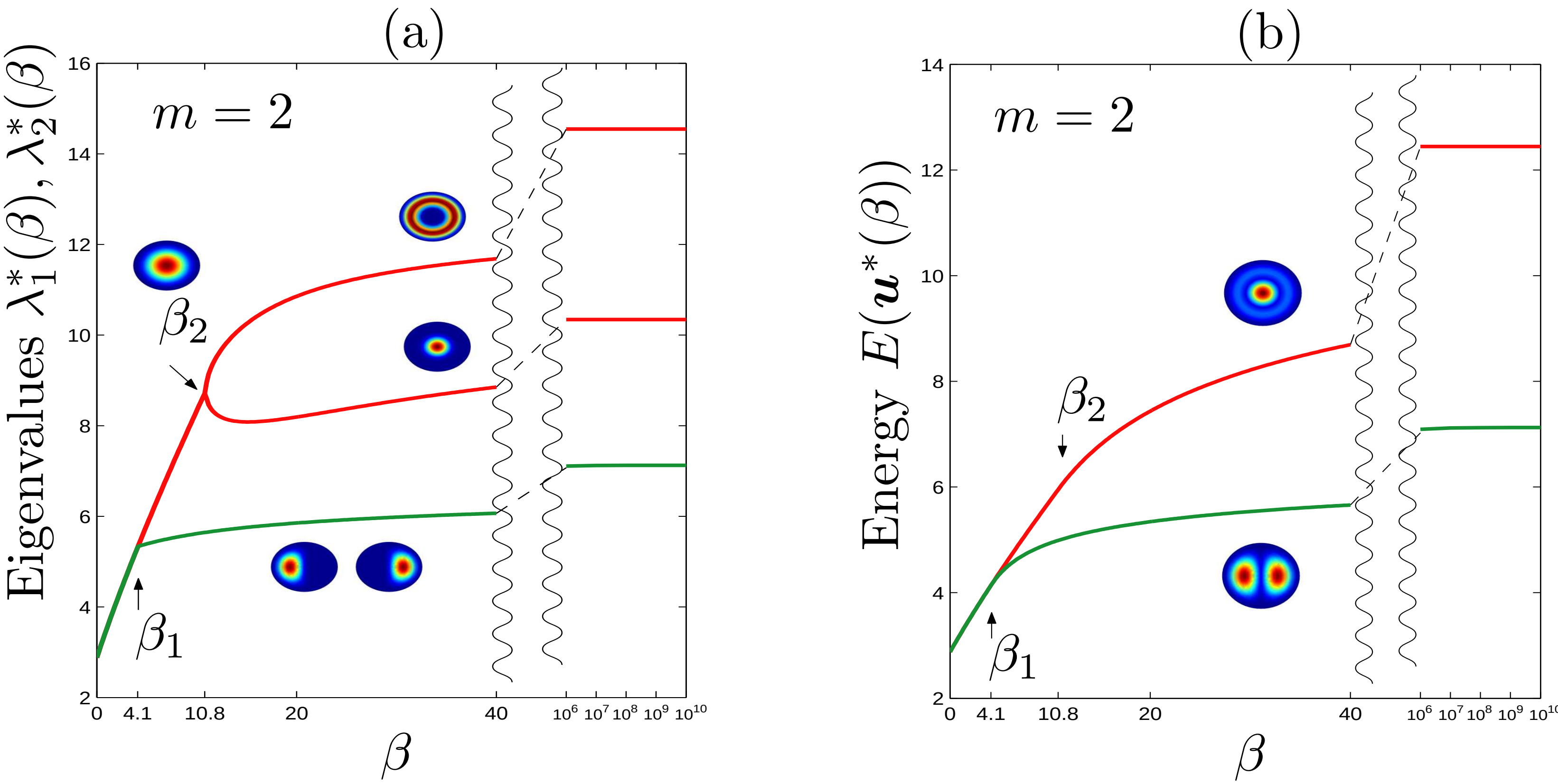
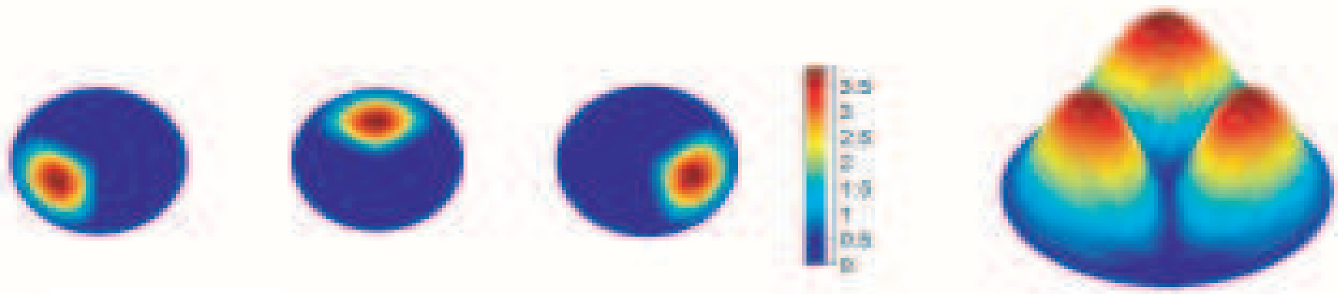
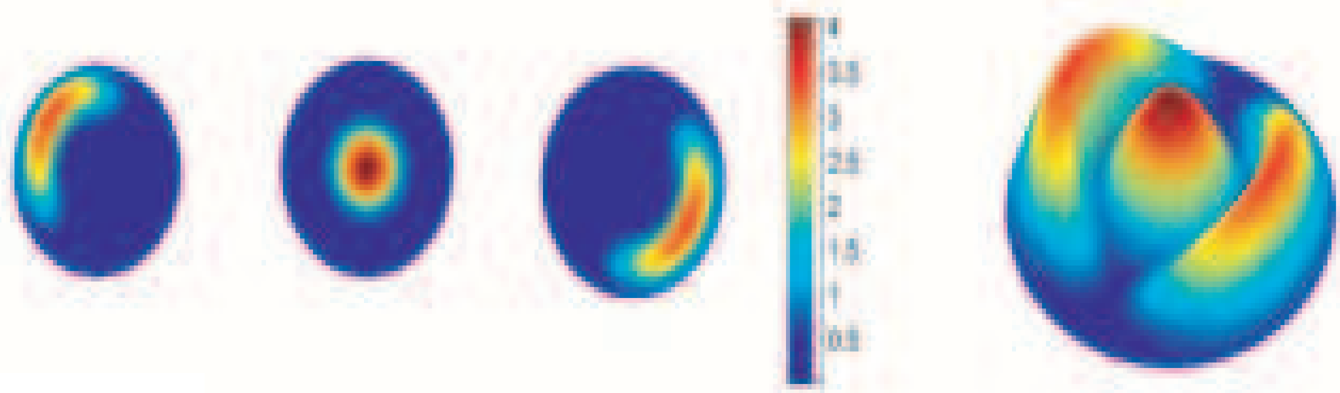


Figure 5.1: (a): Eigenvalue curves, (b): energy curves, vs β.

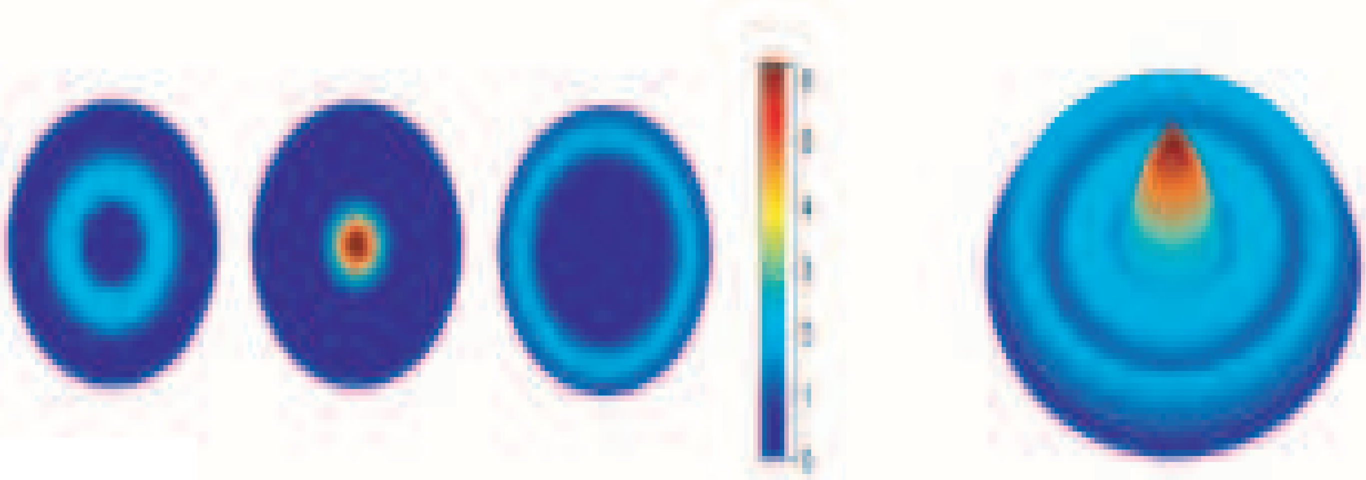
Three-component BEC



(a) green: $\beta^* = 1000$, $\lambda_1^* = \lambda_2^* = \lambda_3^* = 9.57$, $E(\boldsymbol{u}^*) = 9.52$



(b) red: $\beta^* = 1000$, $\lambda_1^* = \lambda_3^* = 18.36$, $\lambda_2^* = 20.85$, $E(\boldsymbol{u}^*) = 19.09$



(c) blue: $\beta^* = 1000$, $\lambda_1^* = 20.84$, $\lambda_2^* = 24.84$, $\lambda_3^* = 32.14$, $E(\boldsymbol{u}^*) = 25.85$

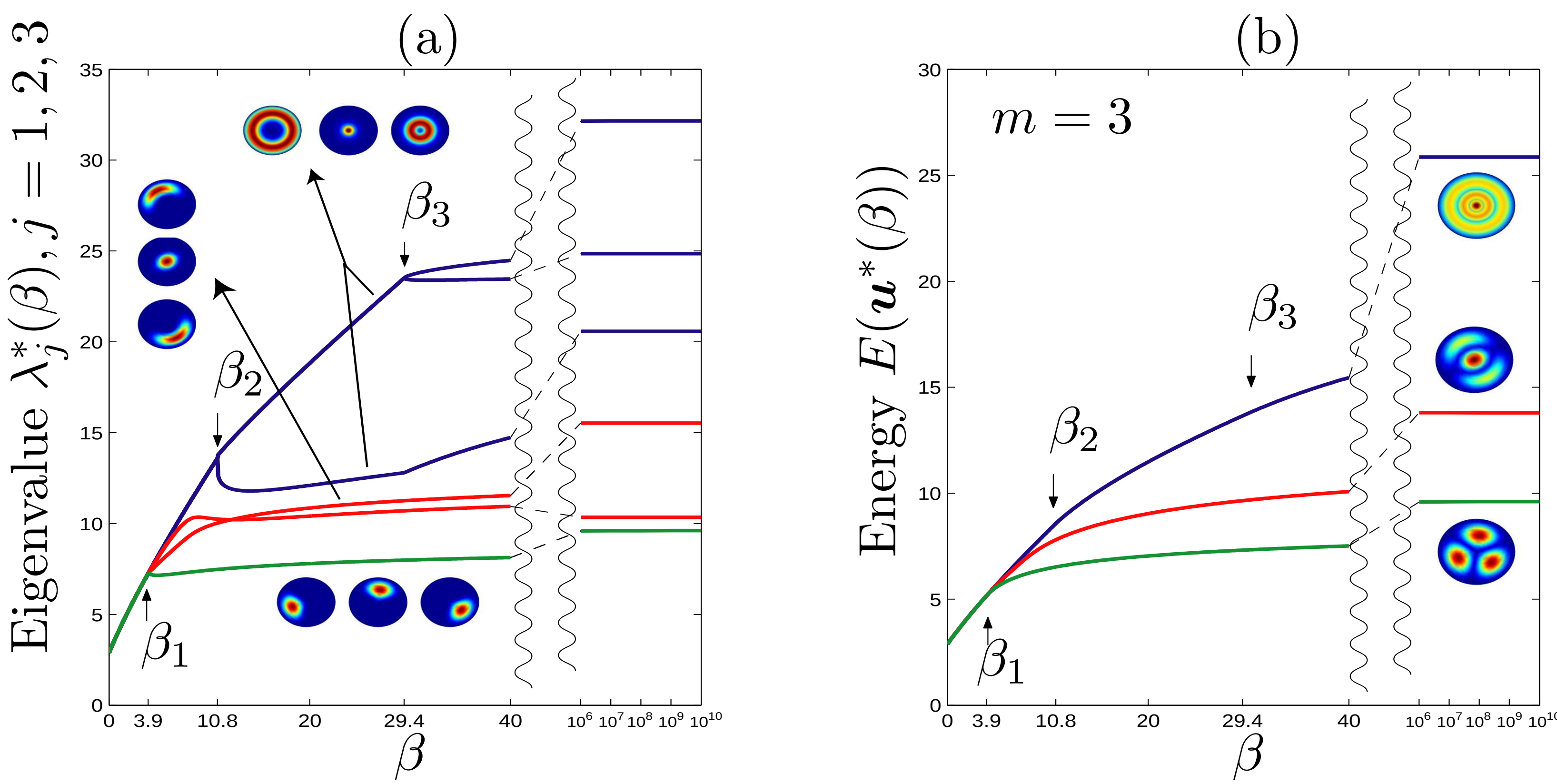
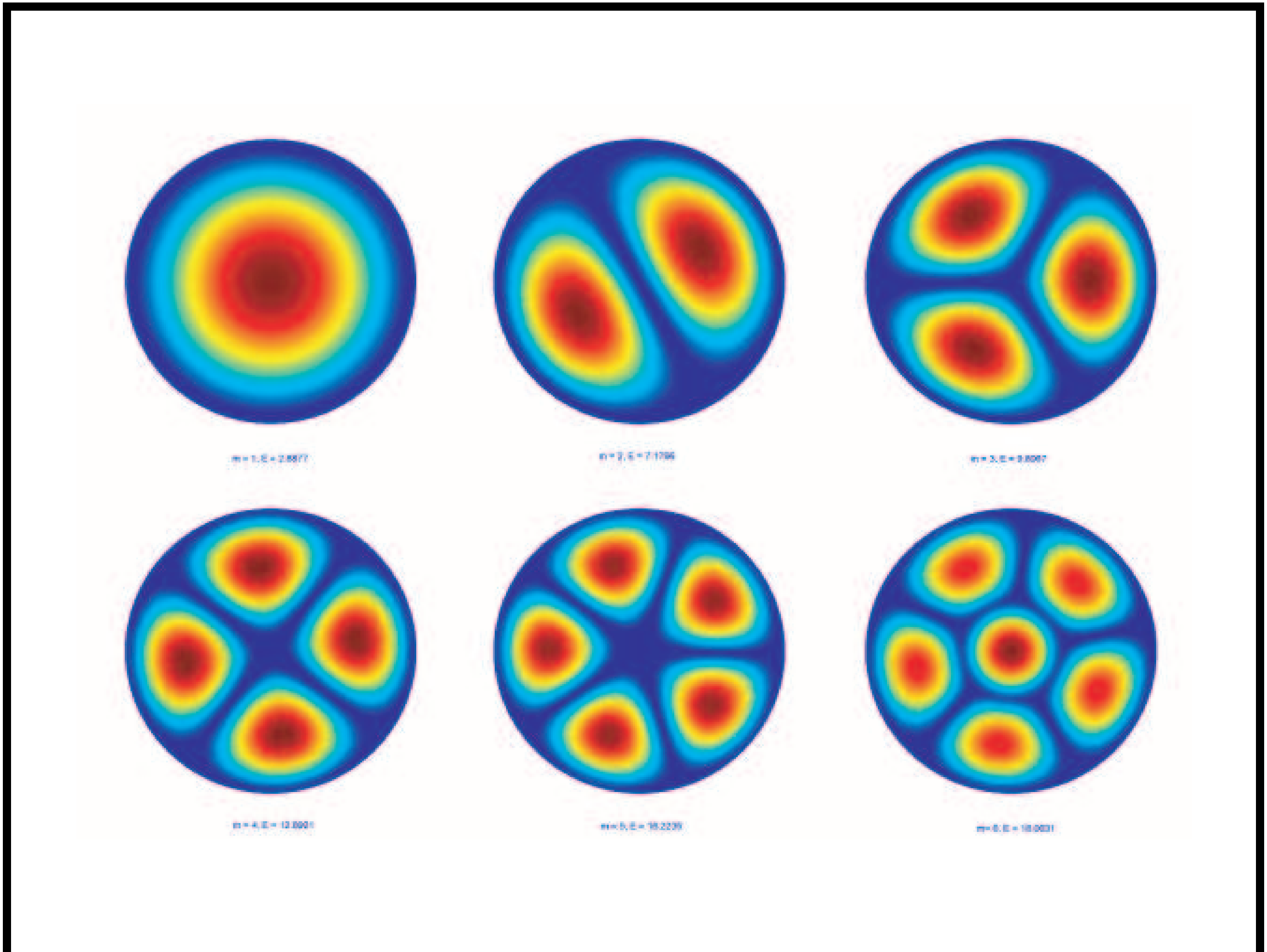
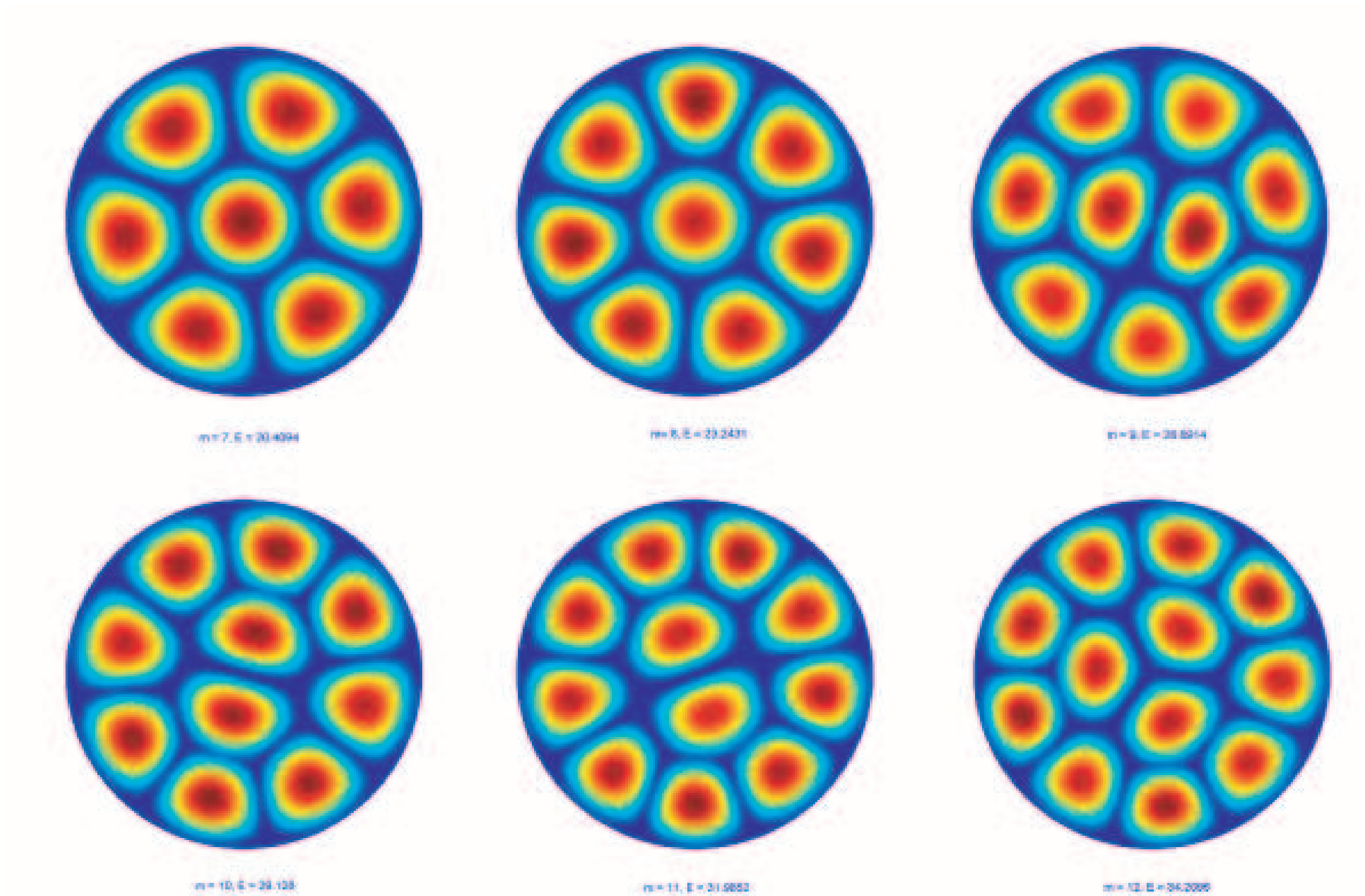


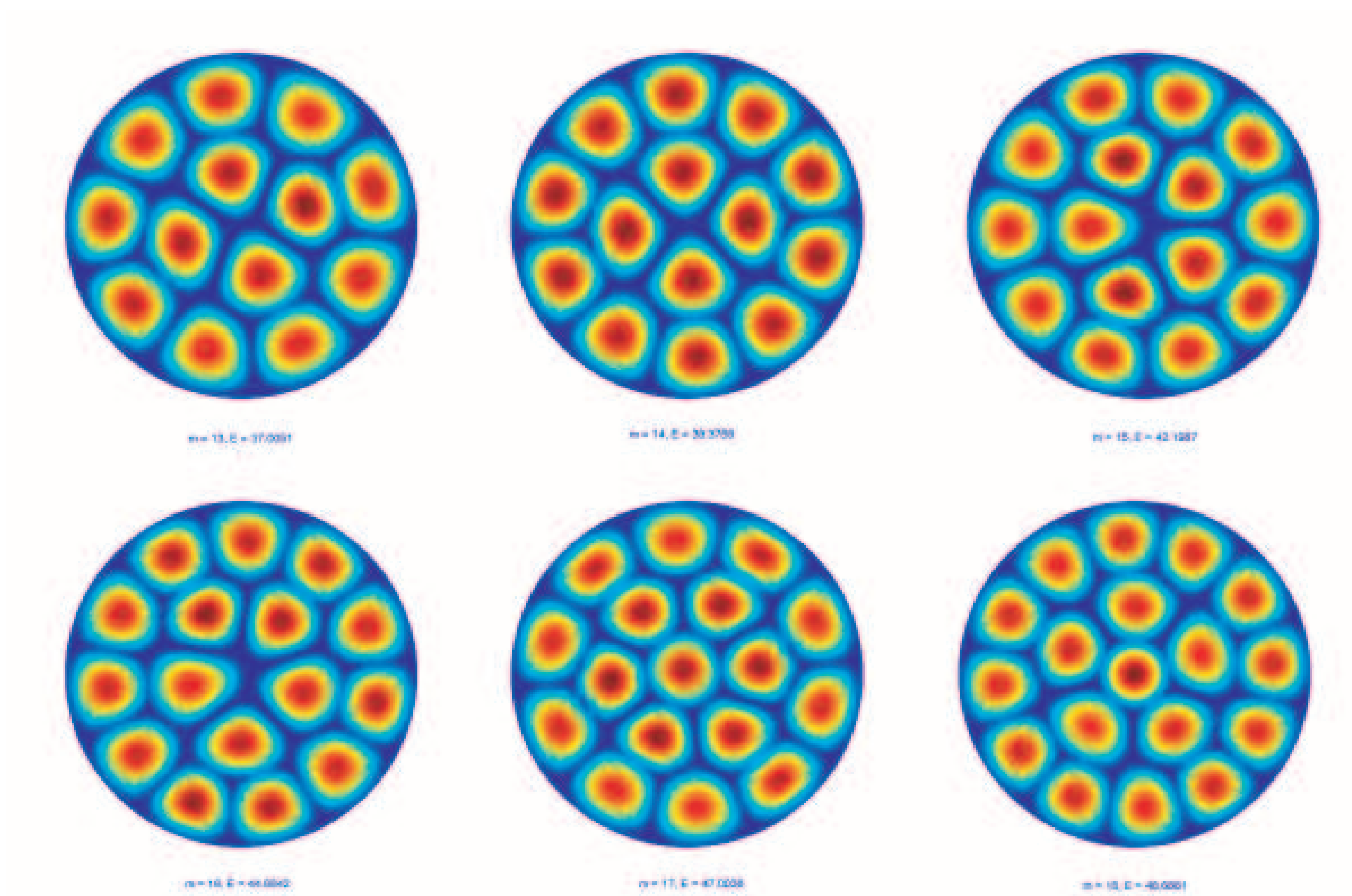
Figure 5.2: (a): Eigenvalue curves, (b): energy curves, vs β.

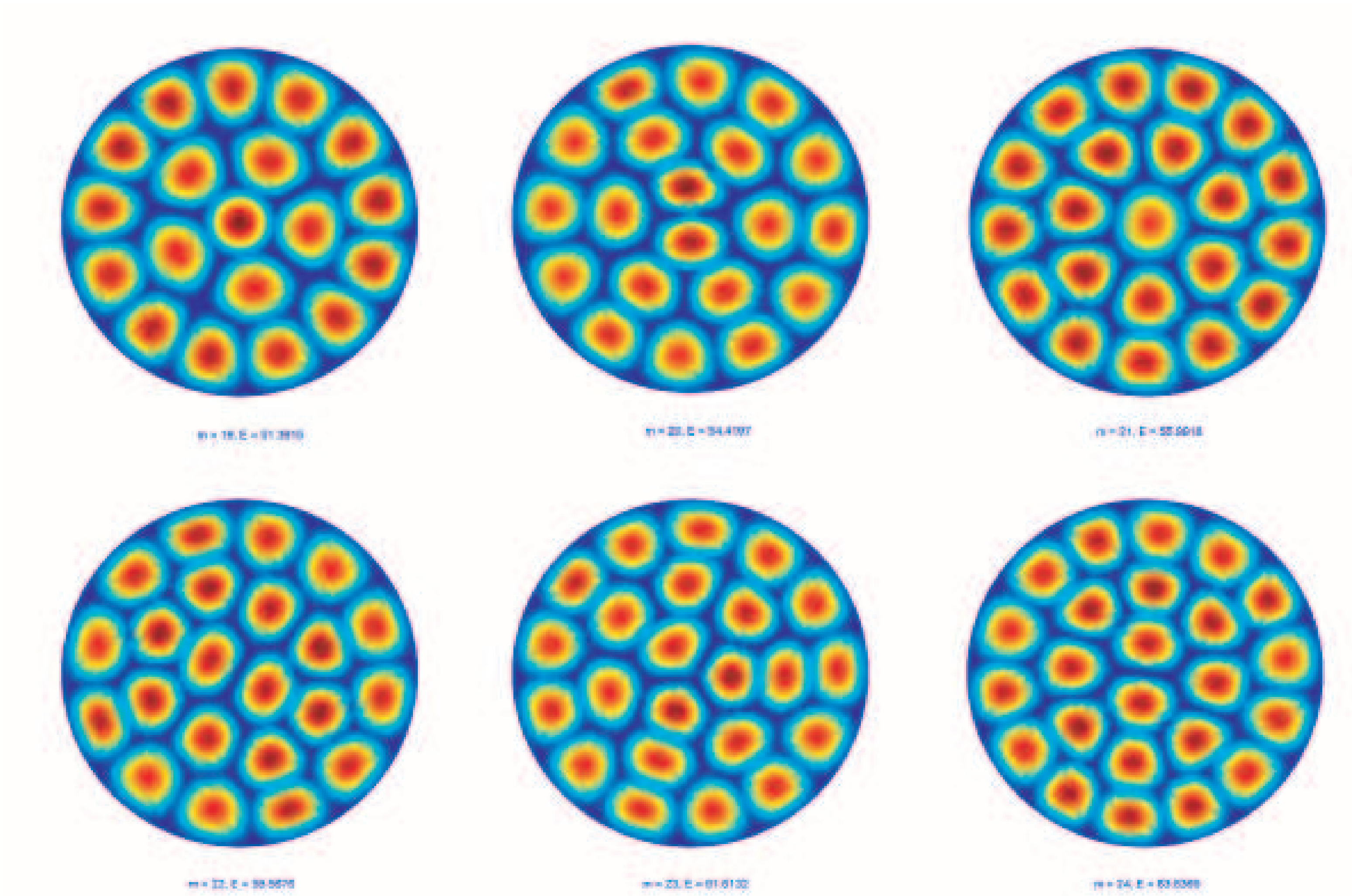
Verticillate Structures

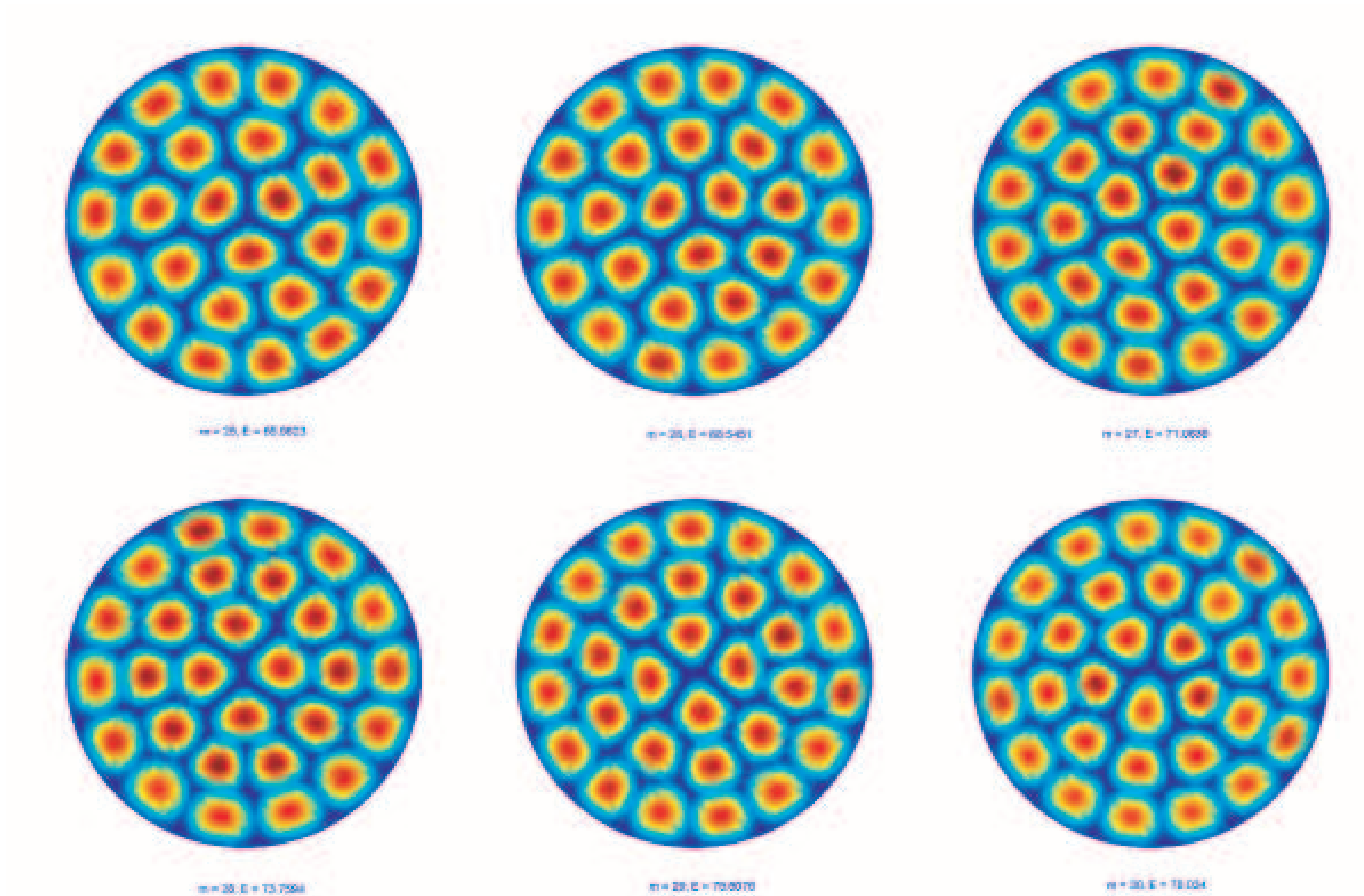
- How to distribute in multi-component BEC when the scattering length is sufficiently large?
- All positive bound state solutions may repel each other and form finitely segregated nodal domains when scattering length approaches to infinity.
- Verticillate: [Botany] leaf, arranged in verticils.

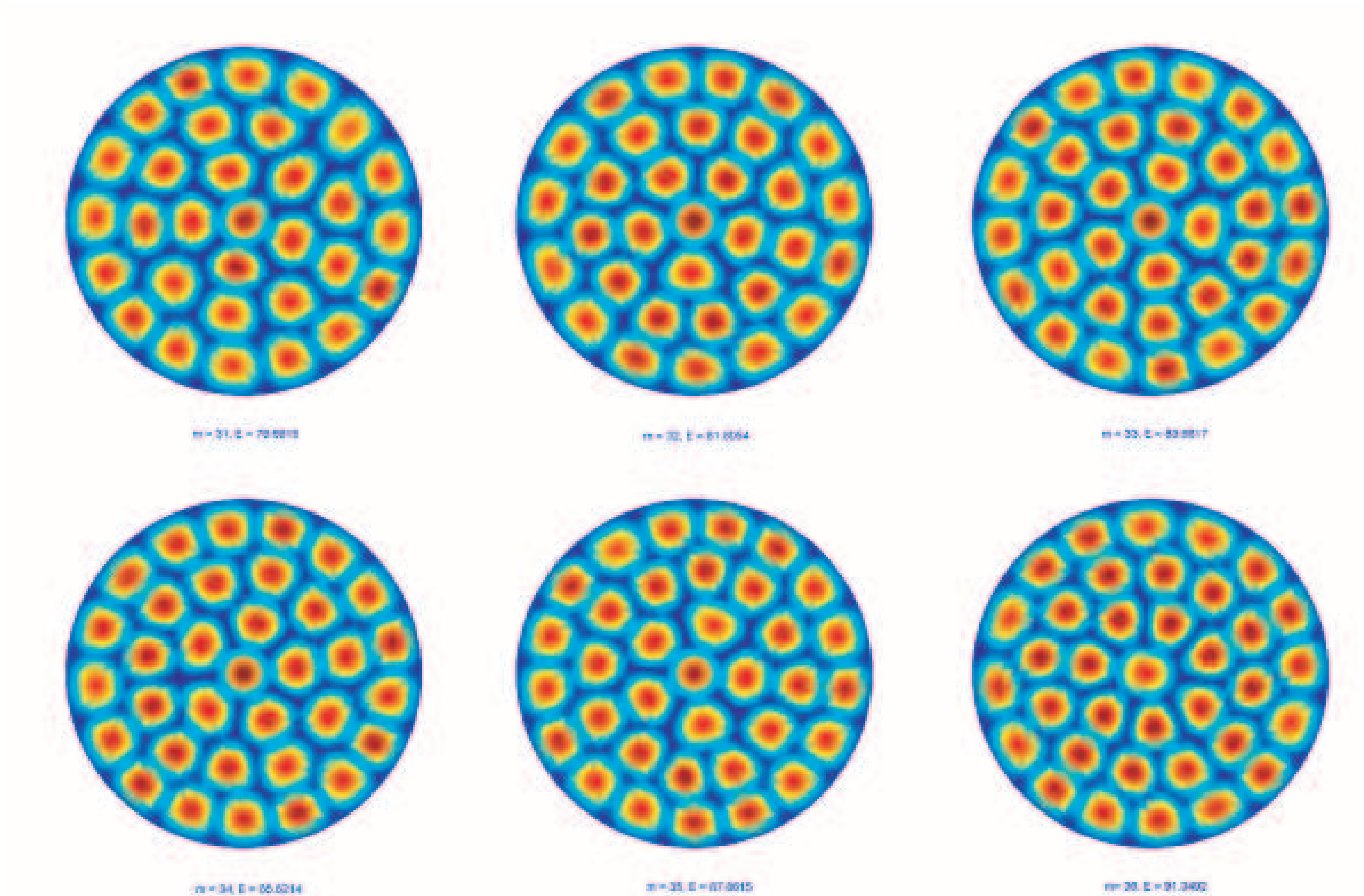



m = 7, E = 20.4094
m = 8, E = 23.2431
m = 9, E = 26.8914
m = 10, E = 29.138
m = 11, E = 31.9852
m = 12, E = 34.2086









We observe that verticillate or multiple verticillate structure

(i) $(n_1, \ldots, n_\gamma)$ depends on m and $\sum_{i=1}^{\gamma} n_i = m$ $(\beta \gg 1)$,

(Single, Double, Triple, Quadruple verticillate, ...)

(ii) $1 \leq n_1 \leq 5$.

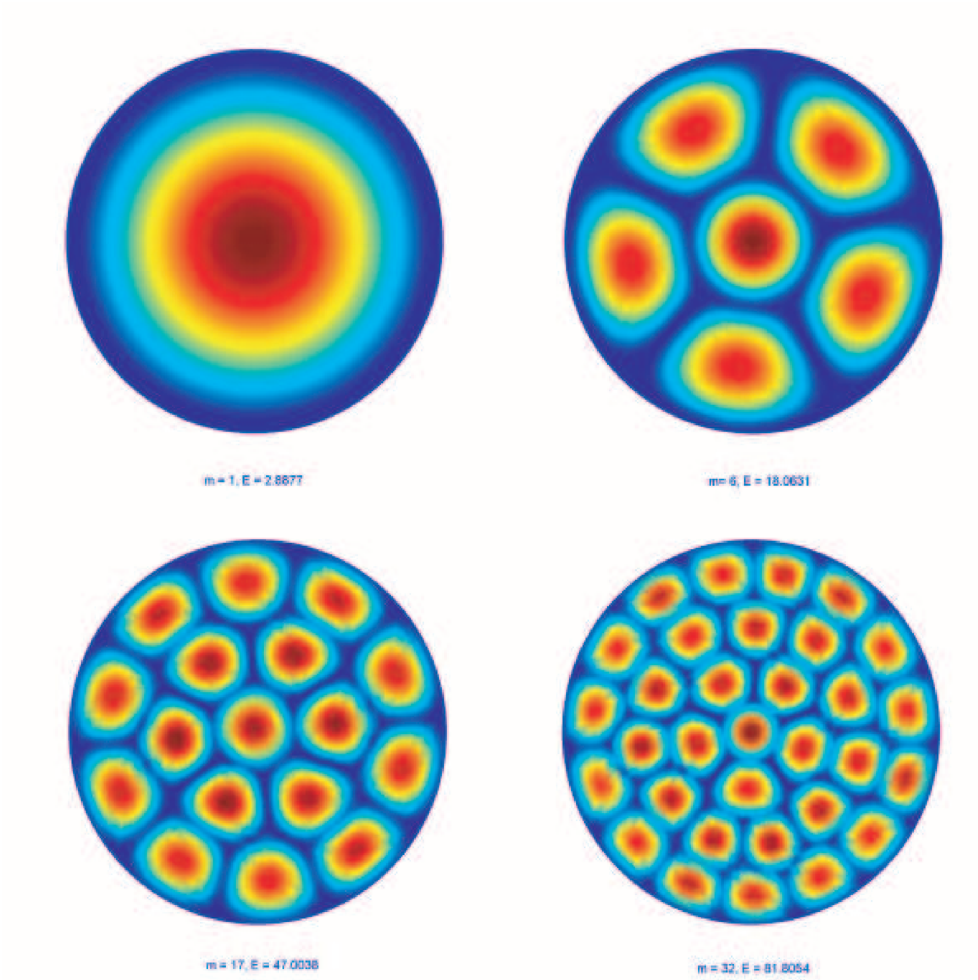
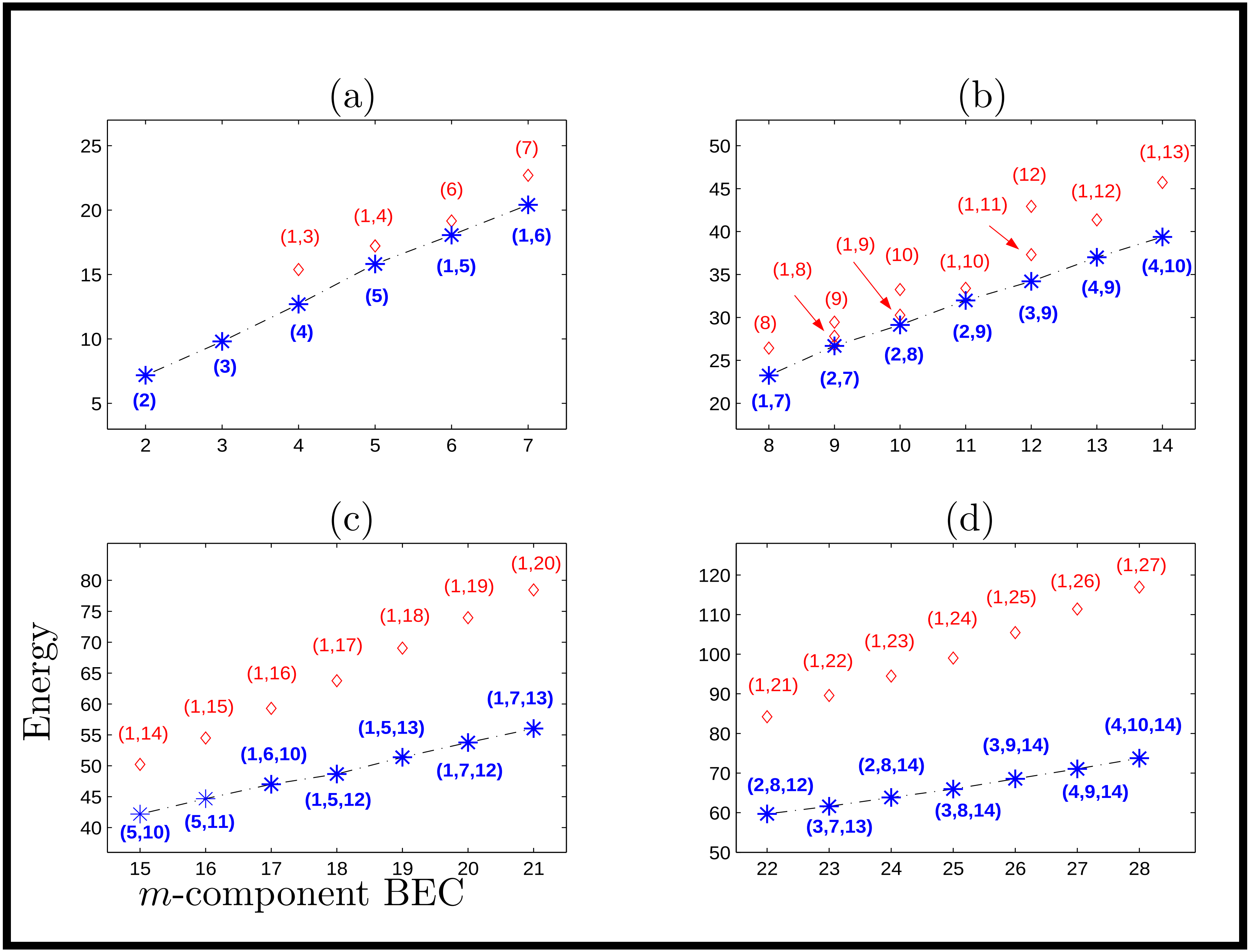
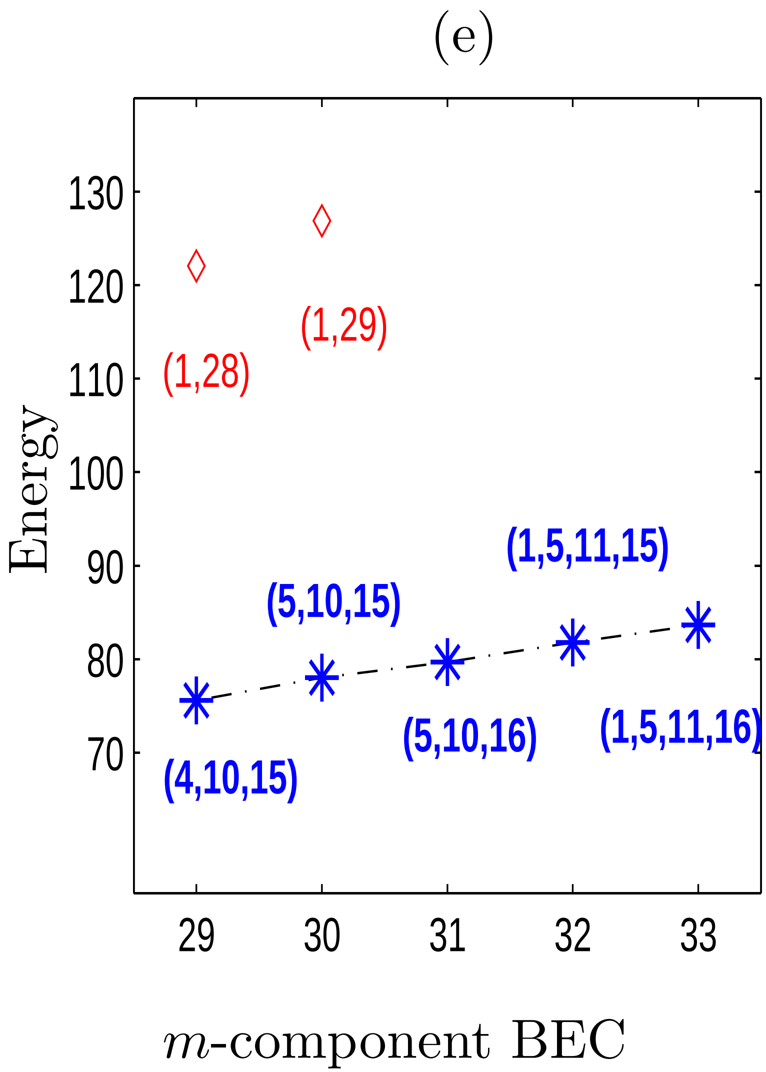


Figure 5.3: Single, Double, Triple, Quadruple verticillate: (1), (1,5), (1,6,10), (1,5,11,15).



(a)
(b)
(c)
(d)
Energy
m-component BEC
(2)
(3)
(4)
(5)
(1,5)
(1,6)
(1,3)
(1,4)
(6)
(7)
(8)
(9)
(1,8)
(1,9)
(10)
(1,10)
(1,11)
(12)
(1,12)
(1,13)
(1,7)
(2,7)
(2,8)
(2,9)
(3,9)
(4,9)
(4,10)
(1,14)
(1,15)
(1,16)
(1,17)
(1,18)
(1,19)
(1,20)
(5,10)
(5,11)
(1,6,10)
(1,5,12)
(1,5,13)
(1,7,12)
(1,7,13)
(1,21)
(1,22)
(1,23)
(1,24)
(1,25)
(1,26)
(1,27)
(2,8,12)
(3,7,13)
(2,8,14)
(3,8,14)
(3,9,14)
(4,9,14)
(4,10,14)



(i) Single verticillate (2) occurring at $m = 2$,

(ii) Double verticillate (1,5) occurring at $m = 6$,

(iii) Triple verticillate (1,6,10) occurring at $m = 17$,

(iv) Quadruple verticillate (1,5,11,15) occurring at $m = 32$.

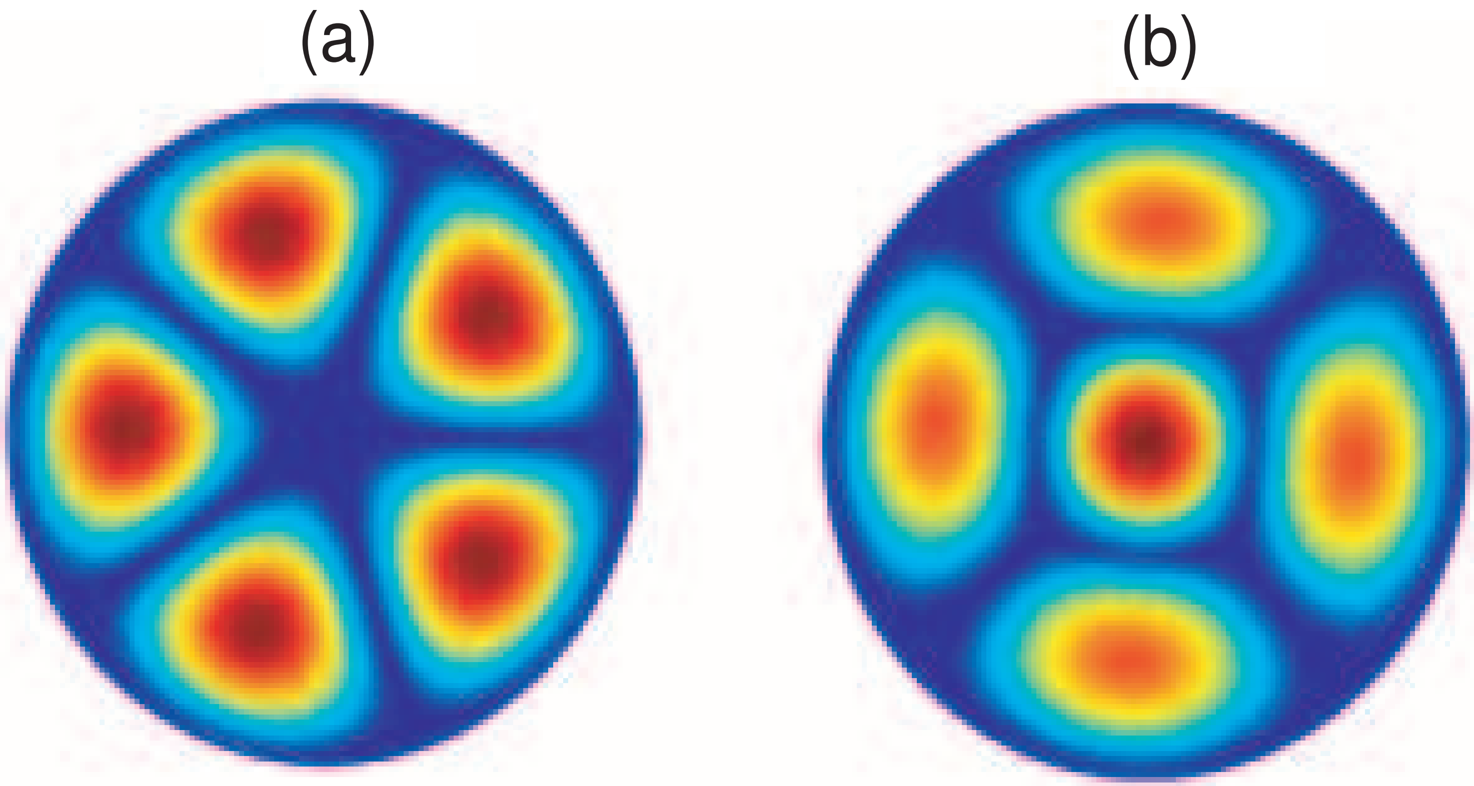


Figure 5.4: $m = 5$: (a) Ground state solutions, (b) bound state solutions.

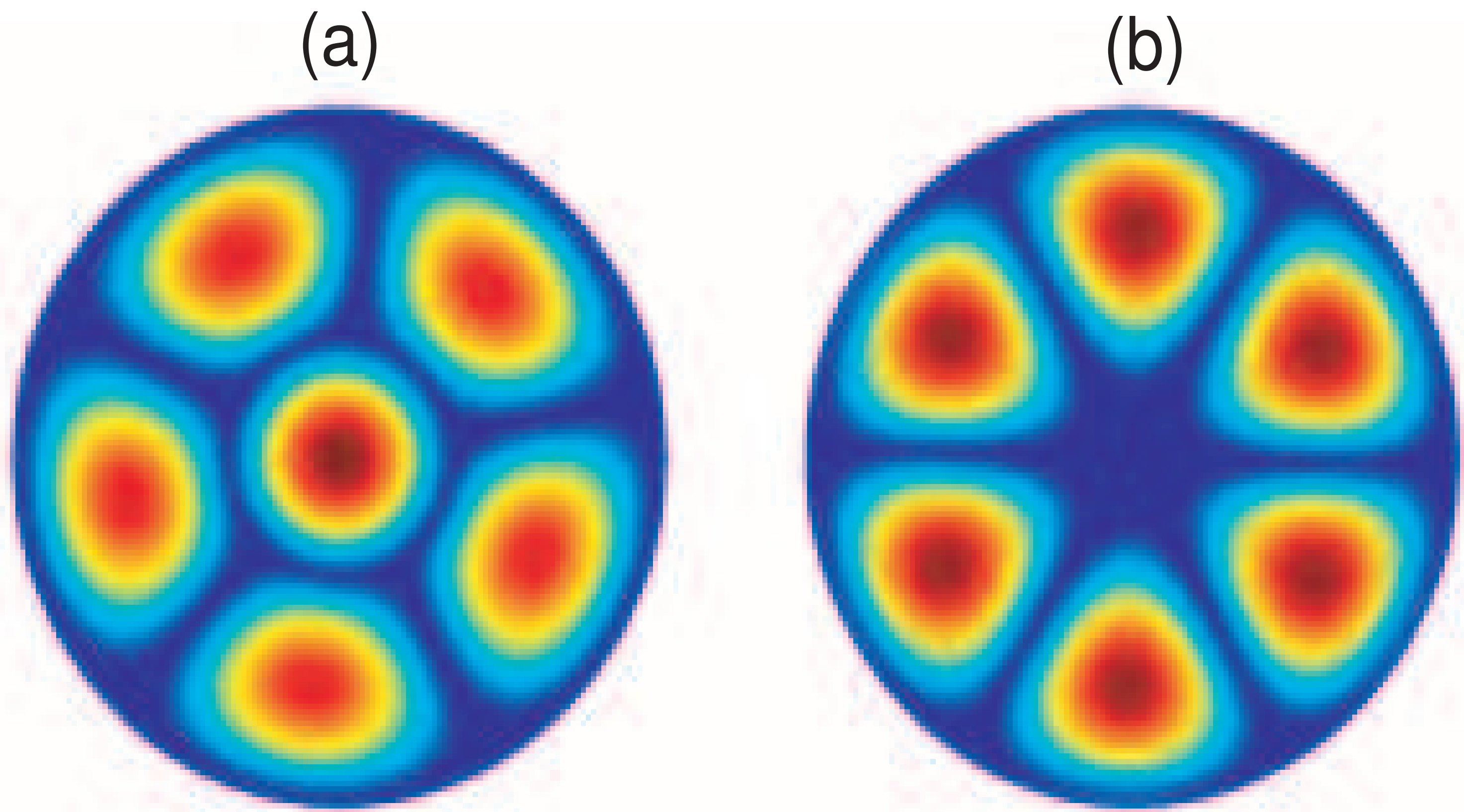


Figure 5.5: $m = 6$: (a) Ground state solutions, (b) bound state solutions.

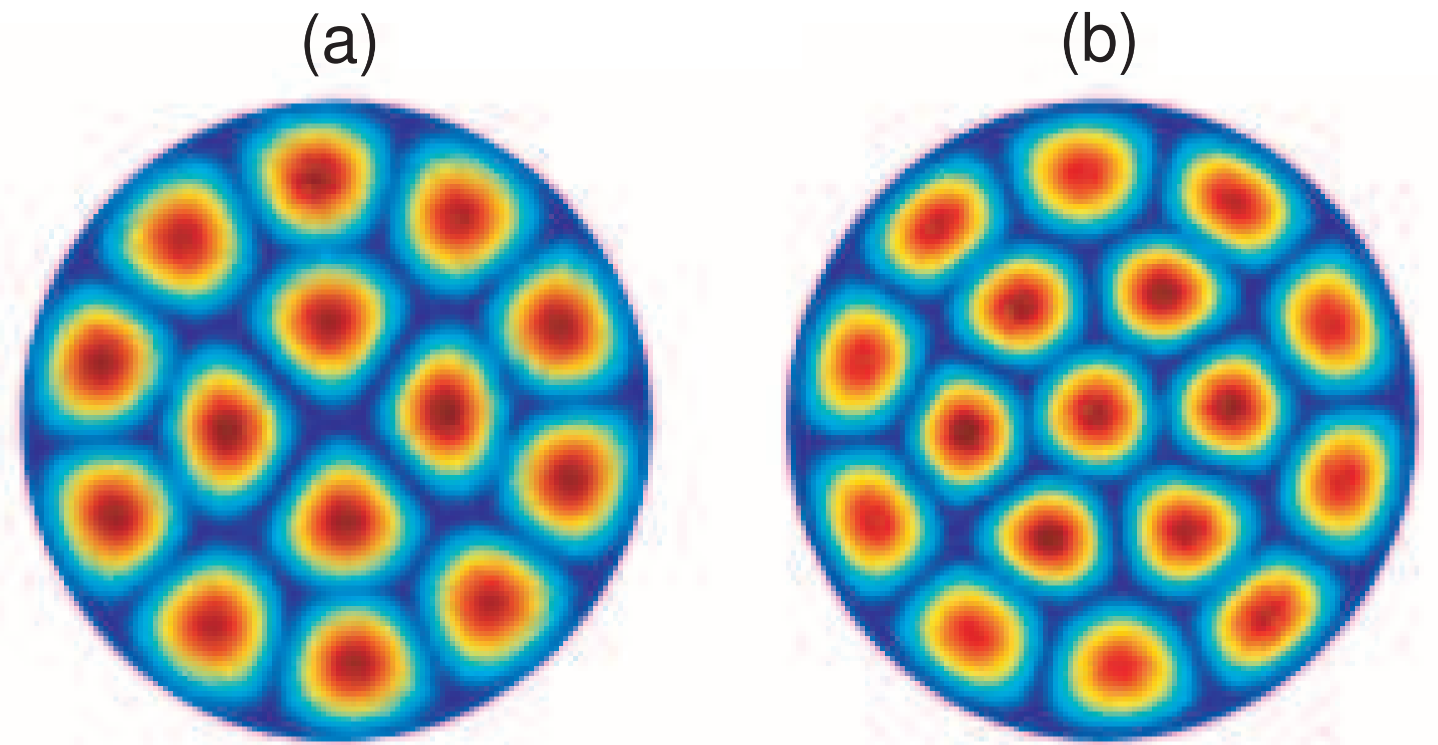


Figure 5.6: Ground state solutions, (a) $m = 14$, (b) $m = 17$.

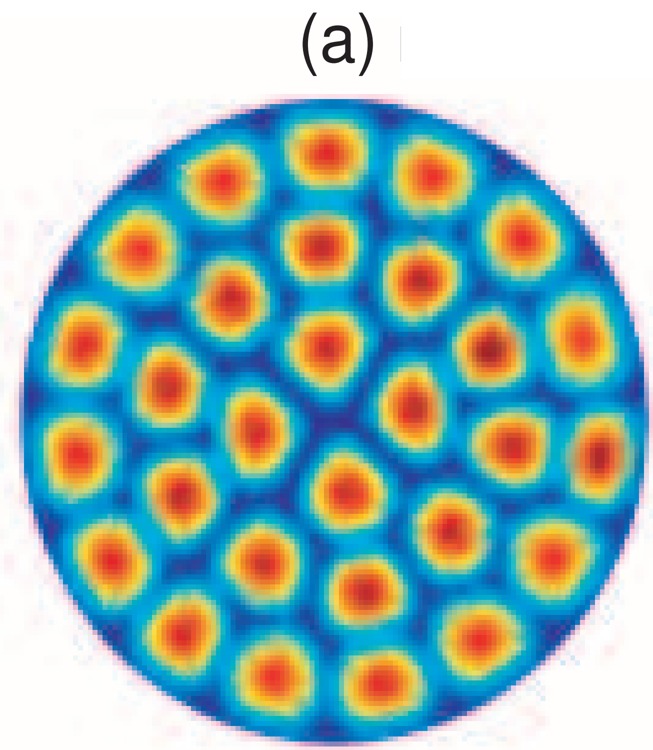


Figure 5.7: Ground state solutions, (a) $m = 29$, (b) $m = 32$.

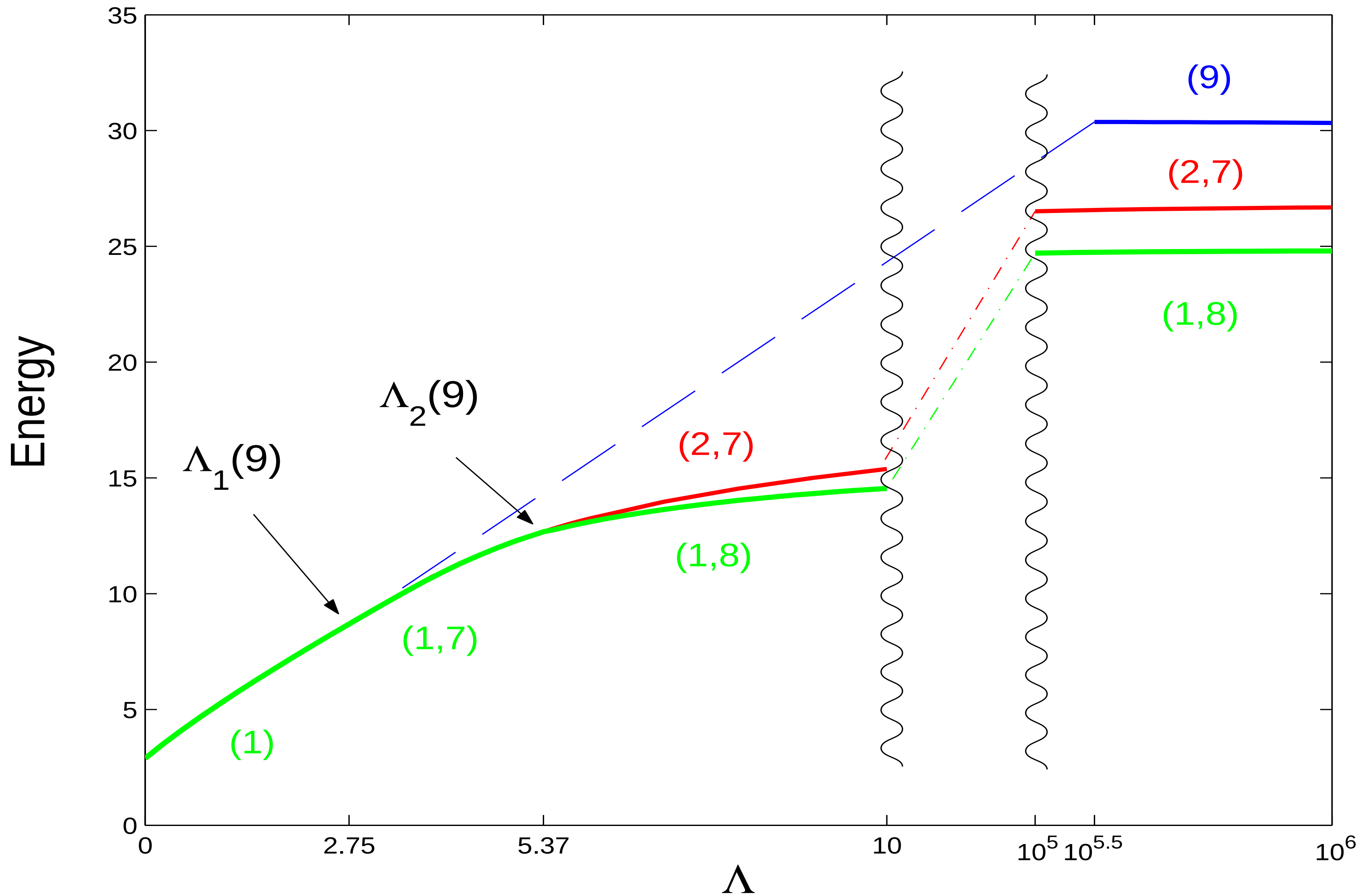


Figure 5.8: Energy curves vs Λ

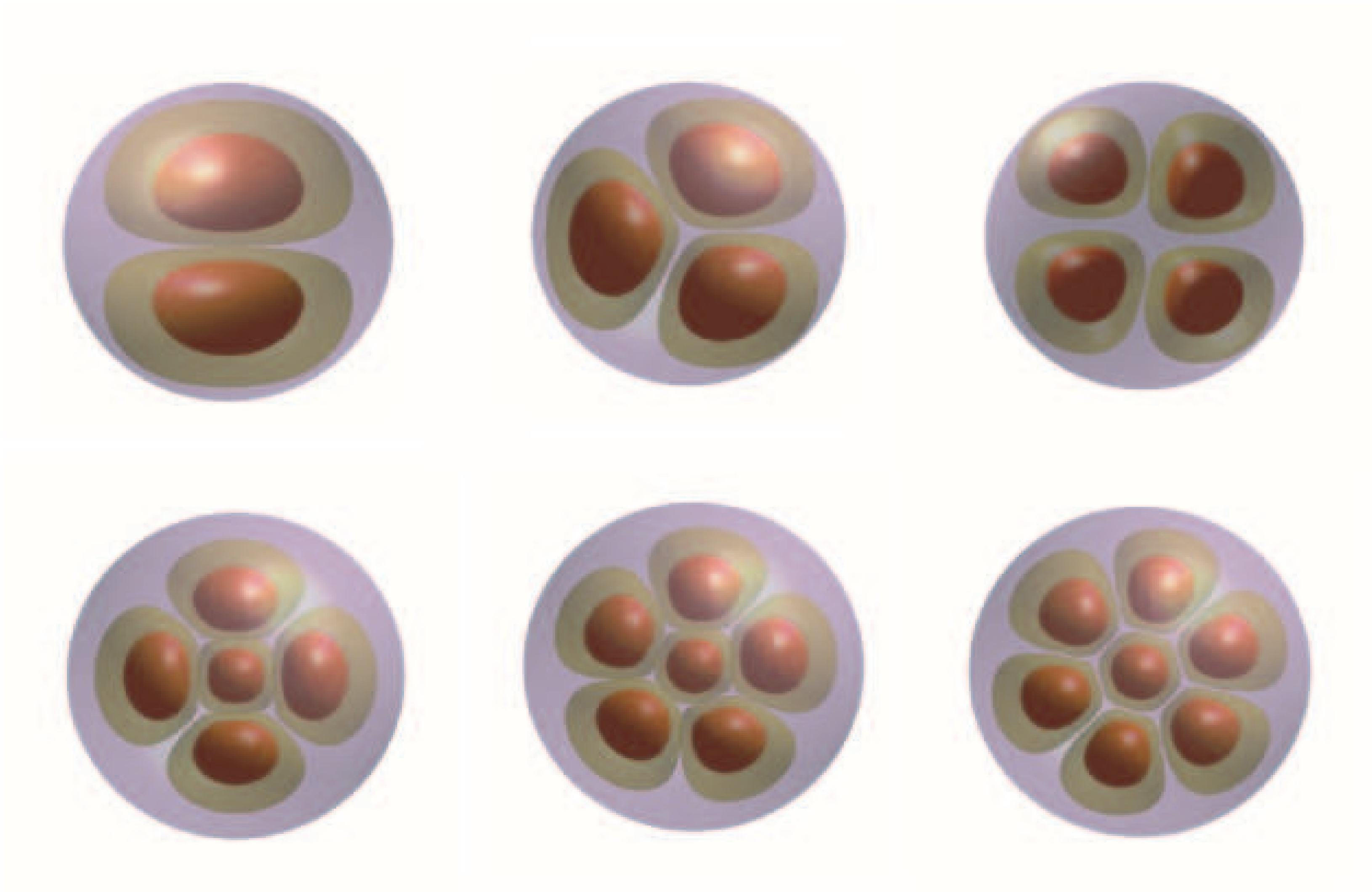


Figure 5.9: In three-dimensional domain from $m = 2$ to $m = 7$.

6 Conclusion

- Theoretical
  - The JI and GSI are proposed from the viewpoint of eigenvalue approach.
  - The necessary and sufficient conditions of convergence of the GSI method are proven that the energy functional has a strictly local minimum at the fixed point.
- Numerical
  - GSI method converges much faster than JI, globally and linearly between 10 to 20 steps.
  - New phenomenon: verticillate multipling.
- Future works
  - A Global convergence of GSI is still under investigation.
  - Study in different trap potentials.

7 Numerical Algorithms

Gauss-Seidel Type Iteration (GSI(m))

(i) Given $\boldsymbol{A}_j = \boldsymbol{A} + 2[\![\boldsymbol{V}_j]\!] + 2\beta_{jj}[\![\boldsymbol{u}_j^{(0)②}]\!]$, $\beta_{jj} \ll 0$, $\beta_{jk} = \beta_{kj} \geq 0$ $(j \neq k)$, $j,k = 1,\ldots,m$ and $\boldsymbol{u}_j^{(0)} > 0$ with $\|\boldsymbol{u}_j^{(0)}\|_2 = 1$, $j = 1,\ldots,m$. Let $n = 0$;

(ii) Repeat n: until convergence,

(ii) For $j = 1,\ldots,m$,

Use e.g., the Shift-Invert Arnoldi algorithm or the Jacobi-Davidson algorithm to solve the minimal positive EW. $\lambda_j^{(n+1)}$ of $\boldsymbol{A}_j^{(n+1)}$ and the assoc. EV $\boldsymbol{u}_j^{(n+1)}$ with $\|\boldsymbol{u}_j^{(n+1)}\|_2 = 1$, where

$$\boldsymbol{A}_j^{(n+1)} := \boldsymbol{A}_j + \sum_{k<j}[\![\beta_{jk}\boldsymbol{u}_j^{(n+1)}]\!] + \sum_{k\geq j}[\![\beta_{jk}\boldsymbol{u}_j^{(n)}]\!],$$

Endfor j;

Comment: we denote $\boldsymbol{u}_j^{(n+1)} = \boldsymbol{f}_j(\boldsymbol{u}_1^{(n+1)}, \ldots, \boldsymbol{u}_{j-1}^{(n+1)}, \boldsymbol{u}_{j+1}^{(n)}, \ldots, \boldsymbol{u}_m^{(n)})$;

(iv) Compute the residual,

$$\mathrm{res}_j^{(n+1)} = \boldsymbol{A}_j^{(n+1)} \boldsymbol{u}_j^{(n+1)} - \lambda_j^{(n+1)} \boldsymbol{u}_j^{(n+1)}, \quad j = 1, \ldots, m, \quad (7.1)$$

(v) If $\|\mathrm{res}_j^{(n+1)}\|_2 < \mathrm{Tol}$, $j = 1, \ldots, m$, then stop, else $n \leftarrow n+1$, go to Repeat.

Variant GSI(2)≡ V1-GSI(2)

(i) Given $\boldsymbol{A}_j = \boldsymbol{A} + 2[\![\boldsymbol{V}_j]\!] + 2\alpha_j[\![\boldsymbol{u}_j^{(0)②}]\!]$, $\boldsymbol{u}_j^{(0)} > 0$ with $\|\boldsymbol{u}_j^{(0)}\|_2 = 1$, $j = 1, 2$, $\alpha_j \ll 1$, $\beta > 0$; Let $n = 0$;

(ii) Repeat n: until convergence,

(iii) Compute $\boldsymbol{u}_1^{(n+1)} = \boldsymbol{f}_1(\boldsymbol{u}_2^{(n)})$, $\boldsymbol{u}_1^{(n+1)} \leftarrow \mathsf{nl}(\mathsf{ave}(\boldsymbol{u}_1^{(n+1)}))$,
Compute $\boldsymbol{u}_2^{(n+1)} = \boldsymbol{f}_2(\boldsymbol{u}_1^{(n+1)})$,

(iv) Compute the residuals as in (7.1),

(v) If converges, then stop; else $n \leftarrow n + 1$, go to Repeat (ii).

Variant GSI(3)

(i) Given $\boldsymbol{A}_j := \boldsymbol{A} + 2[\![\boldsymbol{V}_j]\!] + 2\alpha_j [\![\boldsymbol{u}_j^{(0)②}]\!]$, $\boldsymbol{u}_j^{(0)} > 0$ with $\|\boldsymbol{u}_j^{(0)}\|_2 = 1$, $j = 1, 2, 3$, $\alpha_j \ll 1$, $\beta > 0$; Let $n = 0$;

(ii) Repeat n: until convergence,

V1-GSI(3)

(iii) Compute $\boldsymbol{u}_1^{(n+1)} = \boldsymbol{f}_1(\boldsymbol{u}_2^{(n)}, \boldsymbol{u}_3^{(n)})$, $\boldsymbol{u}_1^{(n+1)} \leftarrow \mathsf{nl}(\mathsf{ave}(\boldsymbol{u}_1^{(n+1)}))$,
Compute $\boldsymbol{u}_2^{(n+1)} = \boldsymbol{f}_2(\boldsymbol{u}_1^{(n+1)}, \boldsymbol{u}_3^{(n)})$,
$\boldsymbol{u}_3^{(n+1)} = \boldsymbol{f}_3(\boldsymbol{u}_1^{(n+1)}, \boldsymbol{u}_2^{(n+1)})$,

(iv) Compute the residuals as in (7.1),

(v) If converges, then stop, else $n \leftarrow n + 1$, go to Repeat (ii);

V2-GSI(3)

(iii) Compute $\boldsymbol{u}_1^{(n+1)} = \boldsymbol{f}_1(\boldsymbol{u}_2^{(n)}, \boldsymbol{u}_3^{(n)})$, $\boldsymbol{u}_1^{(n+1)} \leftarrow \mathsf{nl}(\mathsf{ave}(\boldsymbol{u}_1^{(n+1)}))$,
Compute $\boldsymbol{u}_2^{(n+1)} = \boldsymbol{f}_2(\boldsymbol{u}_1^{(n+1)}, \boldsymbol{u}_3^{(n)})$, $\boldsymbol{u}_2^{(n+1)} \leftarrow \mathsf{nl}(\mathsf{ave}(\boldsymbol{u}_2^{(n+1)}))$,
Compute $\boldsymbol{u}_3^{(n+1)} = \boldsymbol{f}_3(\boldsymbol{u}_1^{(n+1)}, \boldsymbol{u}_2^{(n+1)})$,

(iv) Compute the residuals as in (7.1),

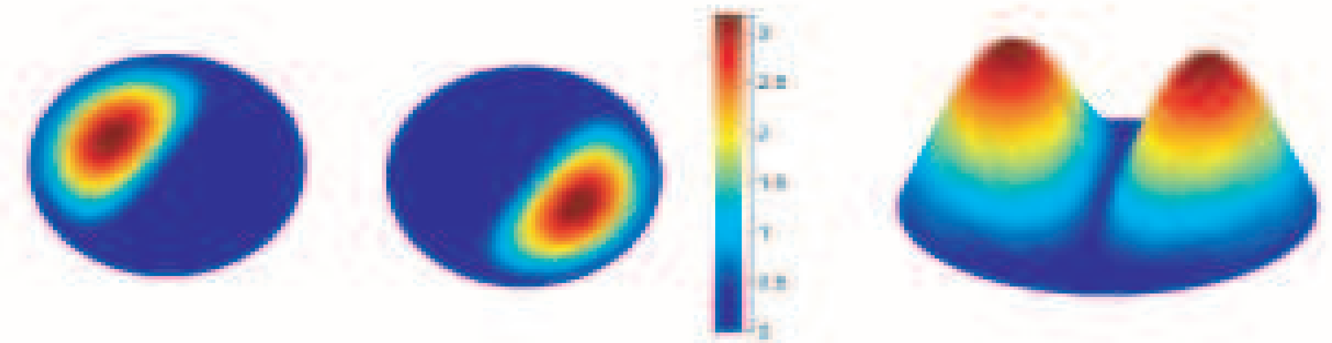
(v) If converges, then stop; else $n \leftarrow n + 1$, go to Repeat (ii).

The energy curves $E(\boldsymbol{u}^*(\beta))$ in Figure 7.10(b) and 7.11(b) are computed by

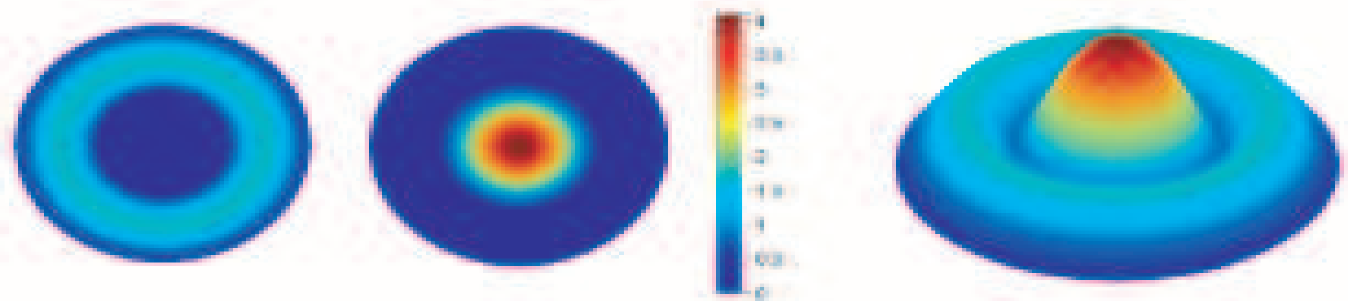
$$E(\boldsymbol{u}^*(\beta)) = \frac{1}{2}\sum_{j=1}^{m} \boldsymbol{u}_j^{*\top} \boldsymbol{A}_j \boldsymbol{u}_j^* + \frac{\beta}{2} \sum_{1\le j<k\le m} \boldsymbol{u}_k^{*②\top} \boldsymbol{u}_j^{*②}.$$

Table 7.1: (g): ground states, (b): bound states.

	$m = 2$	$m = 3$
green curves (g)	GSI(2)	GSI(3)
red curves (b)	V1-GSI(2)	V1-GSI(3)
blue curves (b)	—	V2-GSI(3)



(a) green: $\beta^* = 1000$, $\lambda_1^* = \lambda_2^* = 7.07$, $E(\boldsymbol{u}^*) = 7.02$



(b) red: $\beta^* = 1000$, $\lambda_1^* = 10.34$, $\lambda_2^* = 14.54$, $E(\boldsymbol{u}^*) = 12.43$

Table 7.2: Two-component BEC.

$\theta = \pi$, $m = 2$	green	red
$(0, \beta_1)$	$\lambda_1^* = \lambda_2^*$, $\boldsymbol{u}_1^* = \boldsymbol{u}_2^*$	—
(β_1, β_2)	$\lambda_1^* = \lambda_2^*$, $\boldsymbol{u}_2^* = R_\theta(\boldsymbol{u}_1^*)$	$\lambda_1^* = \lambda_2^*$, $\boldsymbol{u}_1^* = \boldsymbol{u}_2^*$
(β_2, β_∞)		$\lambda_1^* \neq \lambda_2^*$, $\boldsymbol{u}_j^* = \mathsf{ave}(\boldsymbol{u}_j^*)$, $j = 1, 2$

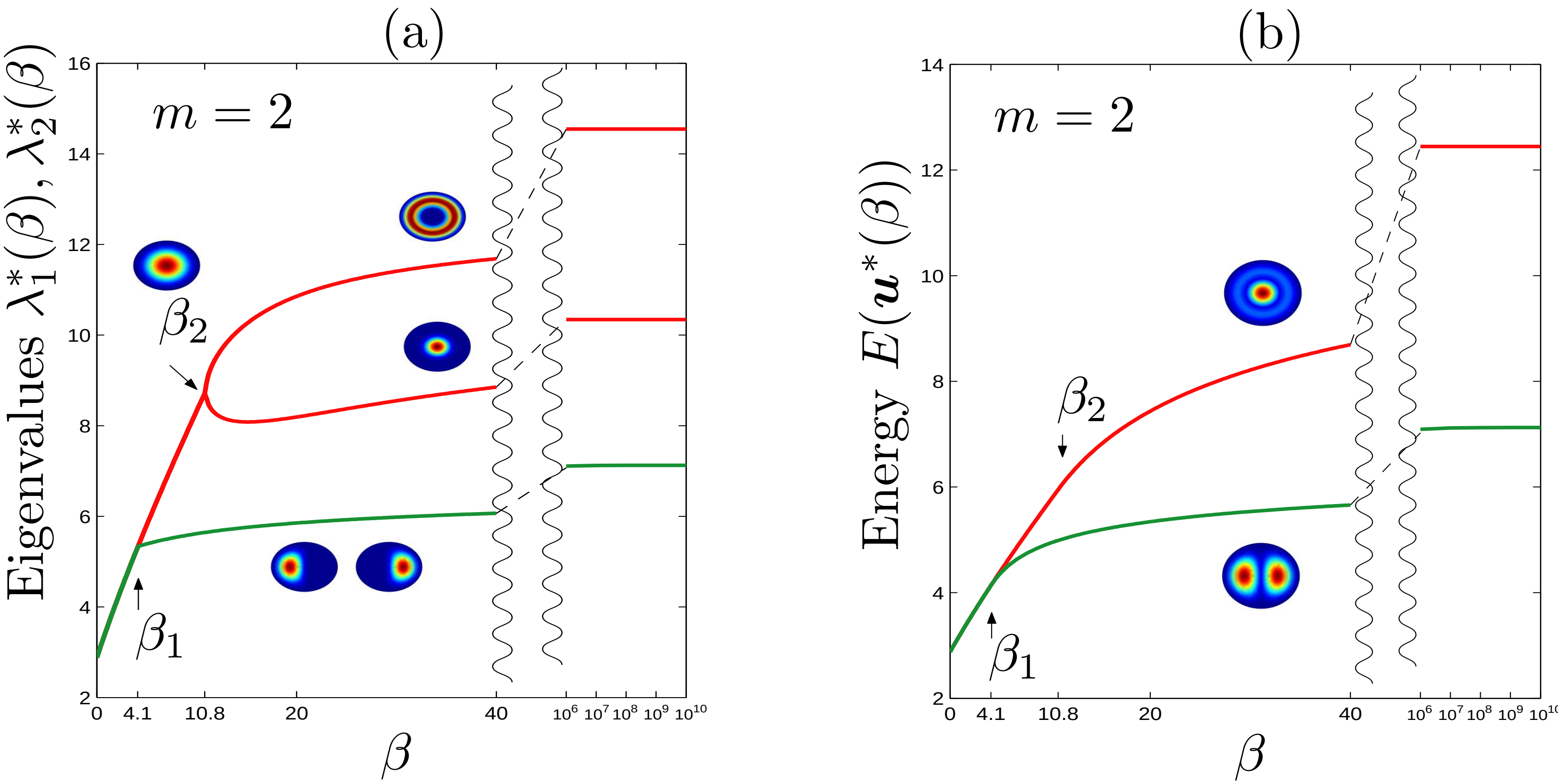
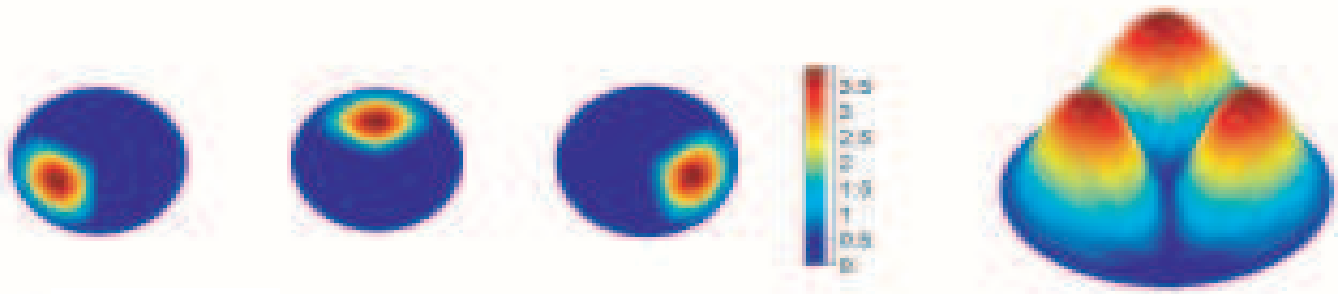
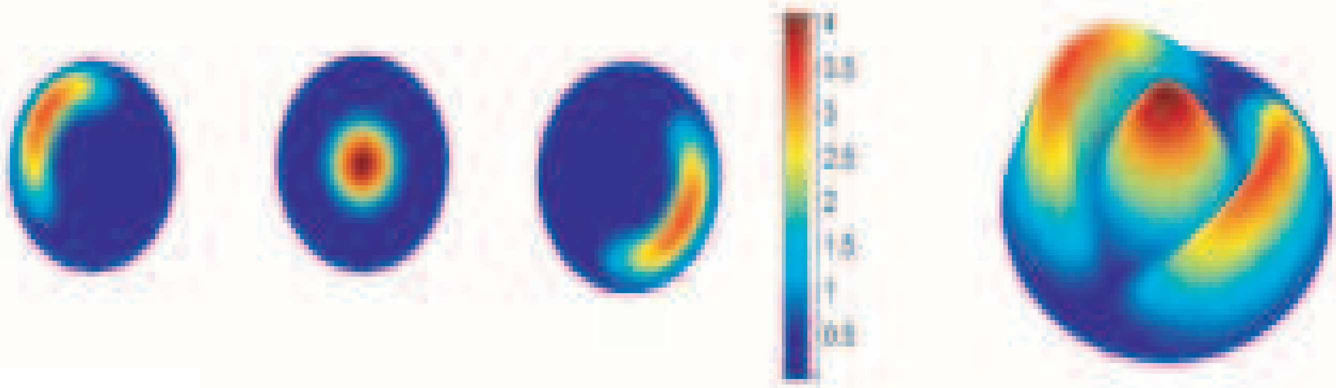


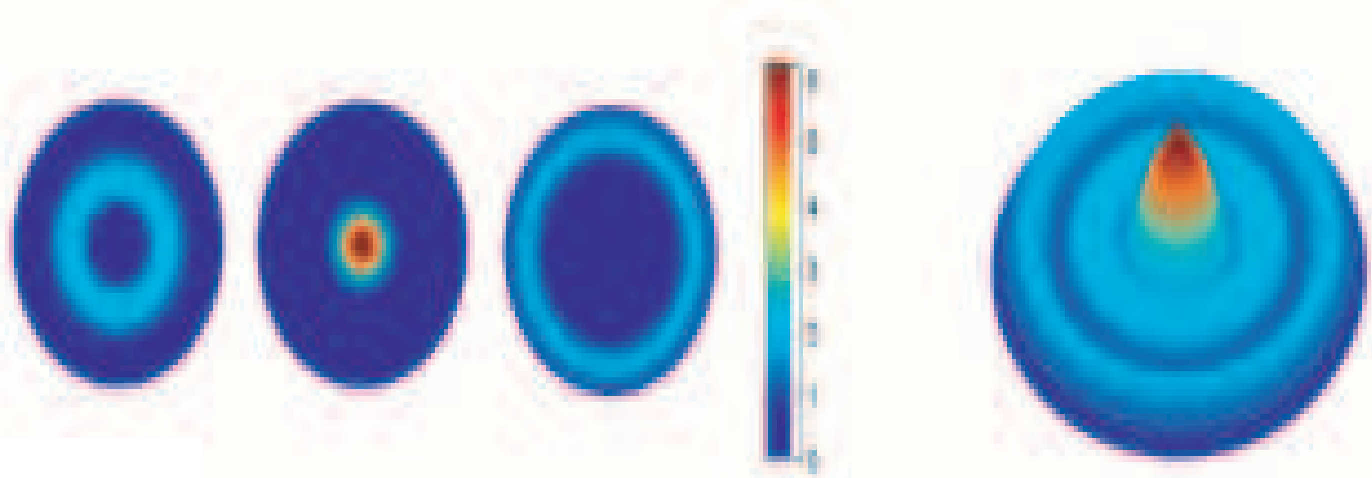
Figure 7.10: (a): Eigenvalue curves, (b): energy curves, vs β.



(a) green: $\beta^* = 1000$, $\lambda_1^* = \lambda_2^* = \lambda_3^* = 9.57$, $E(\boldsymbol{u}^*) = 9.52$



(b) red: $\beta^* = 1000$, $\lambda_1^* = \lambda_3^* = 18.36$, $\lambda_2^* = 20.85$, $E(\boldsymbol{u}^*) = 19.09$



(c) blue: $\beta^* = 1000$, $\lambda_1^* = 20.84$, $\lambda_2^* = 24.84$, $\lambda_3^* = 32.14$,
$E(\boldsymbol{u}^*) = 25.85$

Table 7.3: Three-component BEC.

<table>
<tr><th>$\theta = \frac{2\pi}{3}$, $m = 3$</th><th>green</th><th>red</th><th>blue</th></tr>
<tr><td>$(0, \beta_1)$</td><td>$\lambda_1^* = \lambda_2^* = \lambda_3^*$,
$\boldsymbol{u}_1^* = \boldsymbol{u}_2^* = \boldsymbol{u}_3^*$</td><td>—</td><td>—</td></tr>
<tr><td>(β_1, β_2)</td><td rowspan="3">$\lambda_1^* = \lambda_2^* = \lambda_3^*$,
$\boldsymbol{u}_2^* = R_\theta(\boldsymbol{u}_1^*)$,
$\boldsymbol{u}_3^* = R_\theta(\boldsymbol{u}_2^*)$</td><td rowspan="3">$\lambda_1^* \neq \lambda_2^* = \lambda_3^*$,
$\boldsymbol{u}_1^* = R_\pi(\boldsymbol{u}_1^*)$,
$\boldsymbol{u}_3^* = R_\pi(\boldsymbol{u}_2^*)$</td><td>—</td></tr>
<tr><td>(β_2, β_3)</td><td>$\lambda_1^* = \lambda_2^* \neq \lambda_3^*$,
$\boldsymbol{u}_1^* = \boldsymbol{u}_2^*$,
$\{\boldsymbol{u}_j^* = \mathsf{ave}(\boldsymbol{u}_j^*)\}_{j=1}^3$</td></tr>
<tr><td>(β_3, β_∞)</td><td>$\lambda_1^* \neq \lambda_2^* \neq \lambda_3^*$,
$\{\boldsymbol{u}_j^* = \mathsf{ave}(\boldsymbol{u}_j^*)\}_{j=1}^3$</td></tr>
</table>

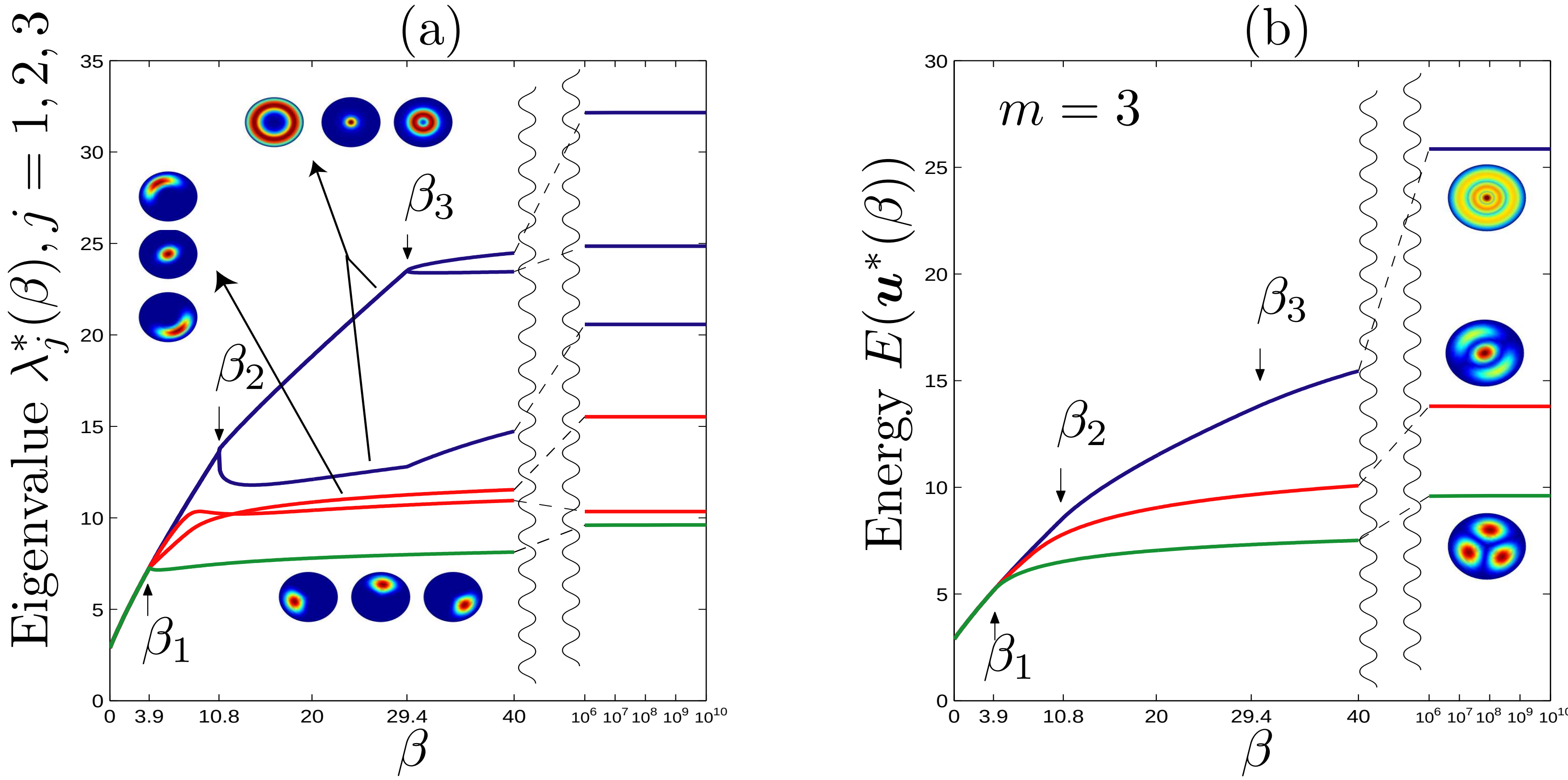


Figure 7.11: (a): Eigenvalue curves, (b): energy curves, vs β.