Dynamics in Bose-Einstein Condensates

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Outline

- Mathematical Models
- Motivation
- Numerical Simulation
- Numerical Results

1 Mathematical Models

(i) Time-dependent Gross-Pitaevskii equation

$$i\,u_t = -\Delta u + V_\epsilon(x,y)\,u + \frac{1}{\epsilon^2}(|u|^2 - 1)\,u, \quad t > 0, \tag{1.1}$$

with the initial data $u|_{t=0} = u_0(x,y)$ and $(x,y) \in \mathbb{R}^2$.

u: a complex-valued order parameter,

$\epsilon > 0$: a small parameter,

$V_\epsilon(x,y) = \alpha_\epsilon\, x^2 + \beta_\epsilon\, y^2$: a harmonic trap potential,

$\alpha_\epsilon,\ \beta_\epsilon > 0$: depending on ϵ.

This time-dependent Gross-Pitaevskii equation was introduced as a phenomenological equation for the order parameter in superfluids.

(ii) Coupled Gross-Pitaevskii equations

$$\begin{cases} \boldsymbol{\iota}\hbar\frac{\partial\psi_1(\boldsymbol{x},t)}{\partial t} = -\frac{\hbar^2}{2m_a}\nabla^2\psi_1 + V_1\psi_1 + \mu_{11}|\psi_1|^2\psi_1 + \mu_{12}|\psi_2|^2\psi_1, \\ \boldsymbol{\iota}\hbar\frac{\partial\psi_2(\boldsymbol{x},t)}{\partial t} = -\frac{\hbar^2}{2m_a}\nabla^2\psi_2 + V_2\psi_2 + \mu_{22}|\psi_2|^2\psi_2 + \mu_{21}|\psi_1|^2\psi_2. \end{cases} \tag{1.2}$$

$$\boldsymbol{x} \in \Omega \in \mathbb{R}^{2,3},\ \psi_j(\boldsymbol{x},t) = 0,\ \boldsymbol{x} \in \partial\Omega, j = 1,2.$$

ψ_j: macroscopic wave fts, V_j: trap potential,

μ_{jj}: intra-comp., μ_{ij} $(i \neq j)$: inter-comp. scattering lengths.

Goal

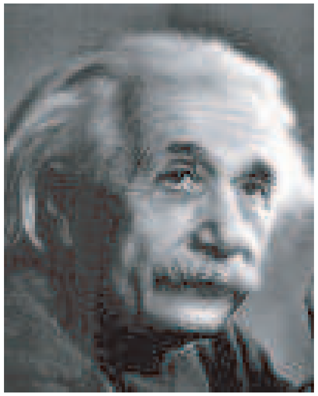
(i) Time-dependent Gross-Pitaevskii equation:
to observe the motion of vortices.

(ii) Coupled Gross-Pitaevskii equations:
to solve Ground state & bound states.

2 Motivation

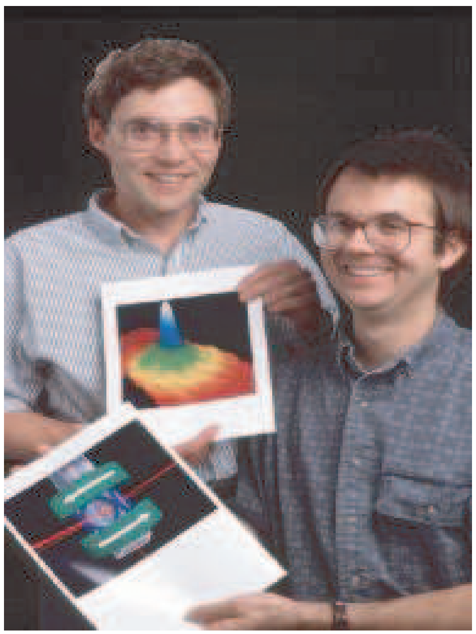
- Bose-Einstein Condensates (BEC)

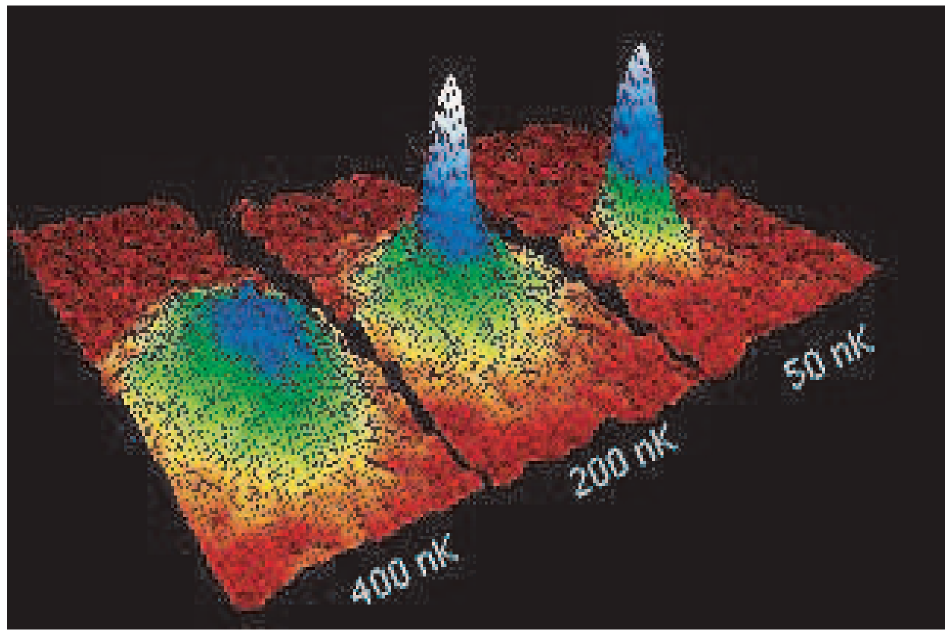
- Theoretical prediction 1924 ...
  - S. Bose: derived Planck's black body radiation law from considering the cavity radiation as an ideal photon gas and worked out Bose statistics for photons.
  - A. Einstein: generalized Bose statistics to other Bosonic particles and atoms (Bose-Einstein statistics) and predicted if the atoms were cold enough, almost all of the particles would congregate in the ground states (BEC).
  - Since 1924, BEC is the Holy Grail in physics.



A. Einstein (1879 $\sim$ 1955)   S. Bose (1894 $\sim$ 1974)

- – E. A. Cornell & C. E. Wieman (JILA, 1995):
  first observed BEC of rubidium (^{87}Rb) atoms at 20 nK, i.e. 0.000 000 02 K.

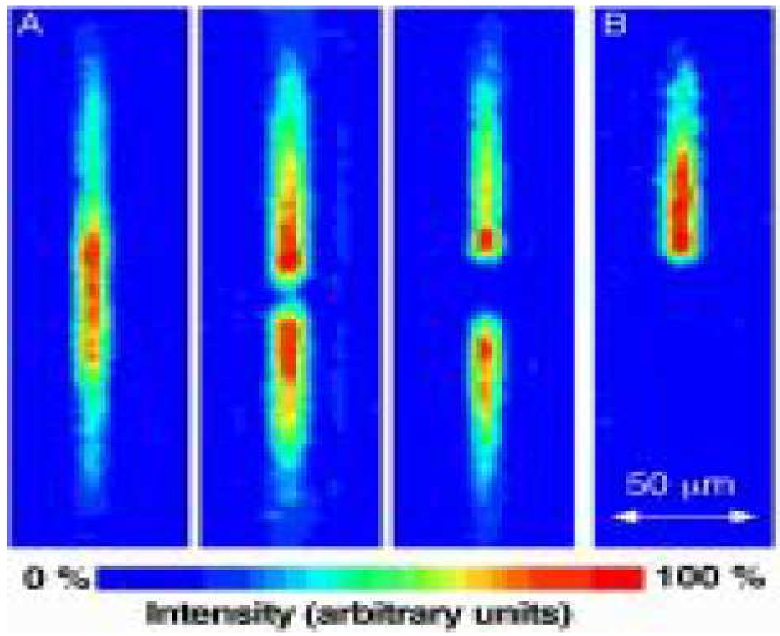




C. E. Wieman & E. A. Cornell   BEC at 400, 200, and 50 nK

– W. Ketterle (MIT, 1995):
observed BEC of sodium (^{23}Na) atoms.

W. Ketterle

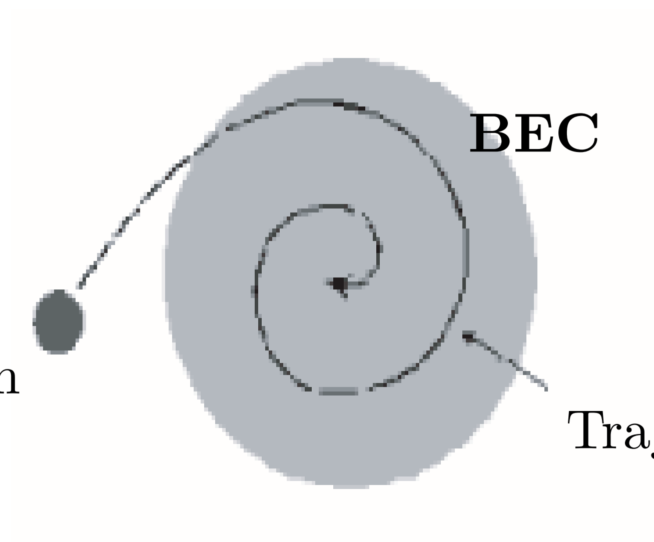


Two-Component BEC

Vortices in BEC

- How do vortices happen?
  - Idea 1: rotation (standard way in fluid mechanics).
  - Idea 2: laser beam moving slowly through the condensate (without rotation), by B. Jackson et al. (1998, theoretical); K. Staliunas (1999, experiment).

- Idea of K. Staliunas – stirred Bose-Einstein Condensates:
  (1) Create one component BEC.
  (2) The laser beam enters the condensate spiraling clockwise.
  (3) Reaching the center of the condensate it is switched off.



3 Numerical Simulation – Vortices in BEC

- Make a study of vortices's behavior in a two-dimensional trapped Bose-Einstein Condensates.
  - PDE: time-dependent Gross-Pitaevskii equation.
  - ODE: asymptotic motion equations of vortices.

- Dynamics of vortices in trapped BEC

  Suppose u_0 has d vortex centers at $q_j(0) = (q_{jx}(0), q_{jy}(0))^\top$. Under some specific assumptions on u_0, we obtain the asymptotic motion equations of d vortices q_j's in the following: (T. C. Lin done)

$$\left\{\begin{array}{rcl} \dot{q}_{jx} & = & -\sum_{\substack{k=1 \\ k \neq j}}^{d} n_k \frac{q_{jy}-q_{ky}}{|q_j-q_k|^2} - \omega_1 \, q_{jy} \, , \\ \dot{q}_{jy} & = & \sum_{\substack{k=1 \\ k \neq j}}^{d} n_k \frac{q_{jx}-q_{kx}}{|q_j-q_k|^2} + \omega_2 \, q_{jx} \, , \end{array}\right. \tag{3.1}$$

  where $q_j = q_j(t) = (q_{jx}(t), q_{jy}(t))$, n_j: winding numbers and $\omega_1 = -\omega + 2\beta_0$, $\omega_2 = -\omega + 2\alpha_0$. For the stability of the vortex structure in u, we require $n_j \in \{\pm 1\}, j = 1, \dots, d$.

Numerical Study of 3 Vortices in ODE

- Characterize the motion:
  - Lyapunov exponent,
  - Poincaré map,
  - Spectrums of waveforms.
- Indicator for ratio topologically synchronized chaotic regimes (Afraimovich et al. (1999, 2000)):
  - the Poincaré dimension for Poincaré recurrences.

4 Numerical Results – Vortices in BEC

We consider $d = 3$, then obtain

(1) the bounded and collisionless trajectories of three vortices form chaotic, quasi 2- or quasi 3-periodic orbits,

(2) a new phenomenon of $1:1$-topological synchronization is observed in the chaotic trajectories of vortices with the same sign of winding numbers..

Case $(n_1, n_2, n_3) = (1, -1, -1)$

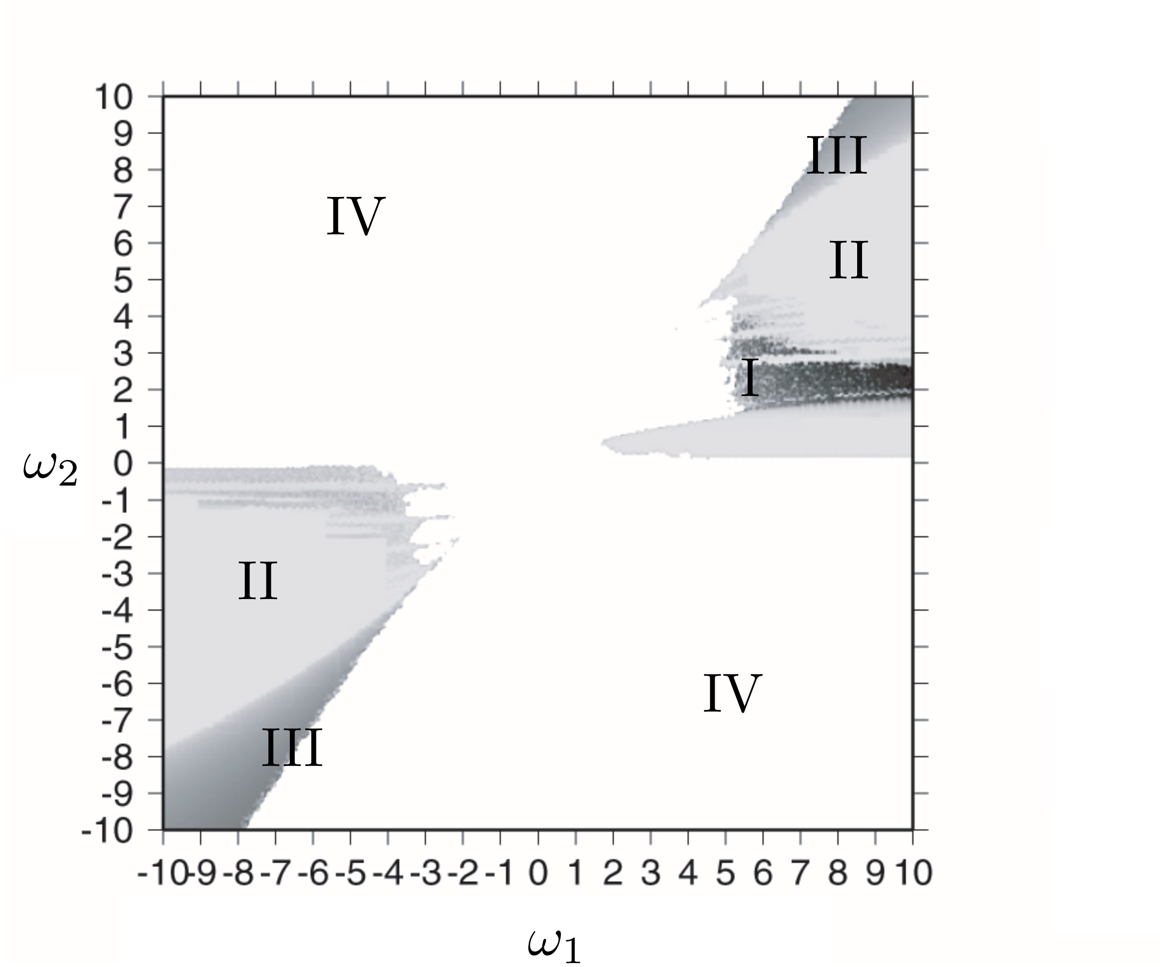


Figure 4.1: The first Lyapunov exponent.

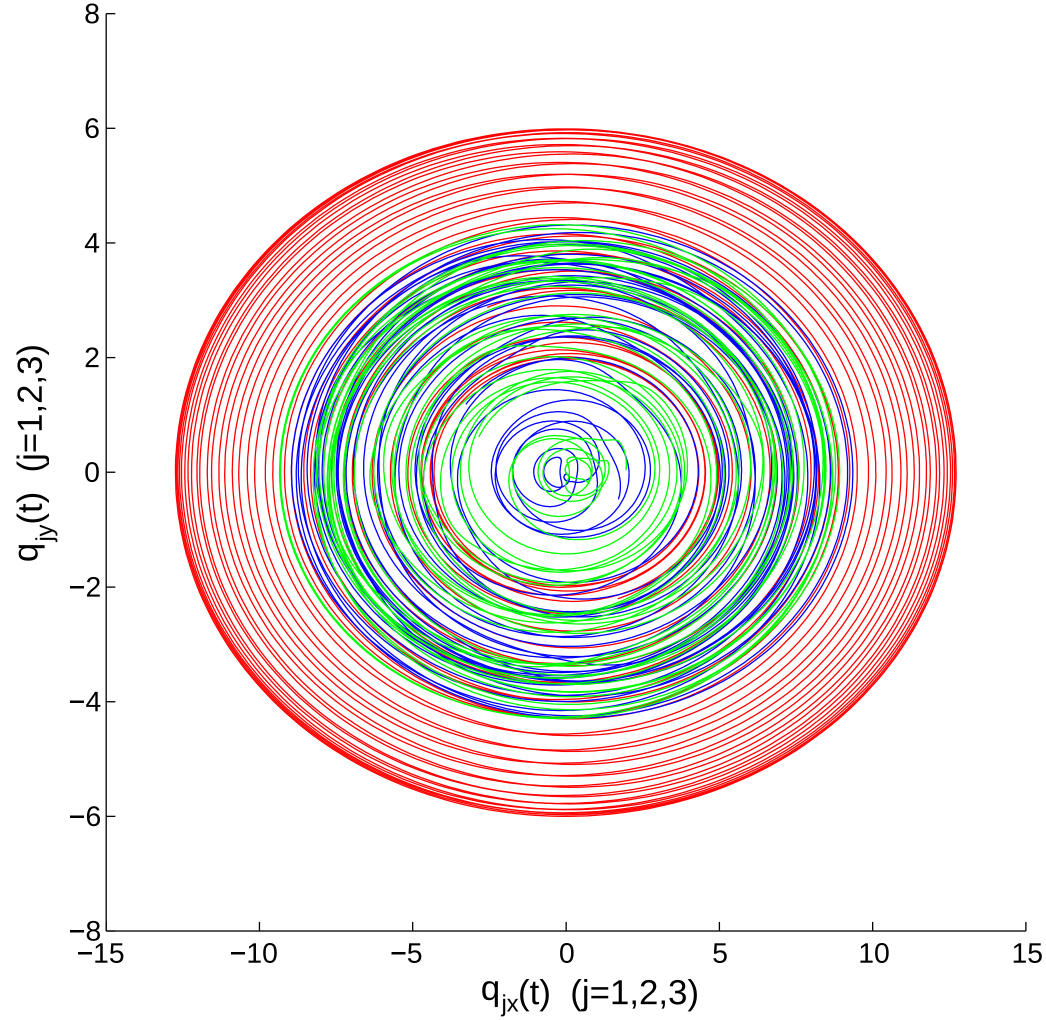


Figure 4.2: Chaotic trajectories: $(\omega_1, \omega_2) = (9.88, 2.24), t = 25,050 \sim 25,100$ sec.

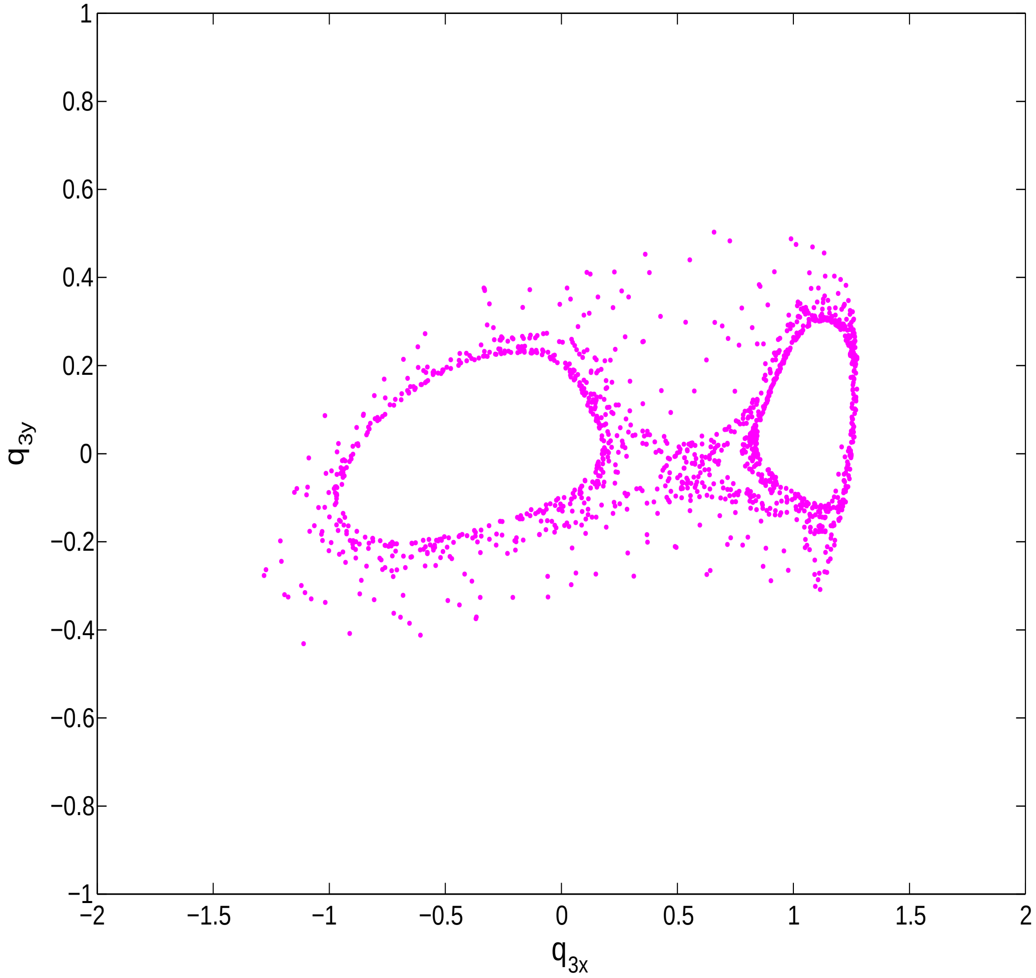


Figure 4.3: Chaotic second-Poincaré maps (4 dim.), $t = 1,393 \sim 5,000,000$ sec.

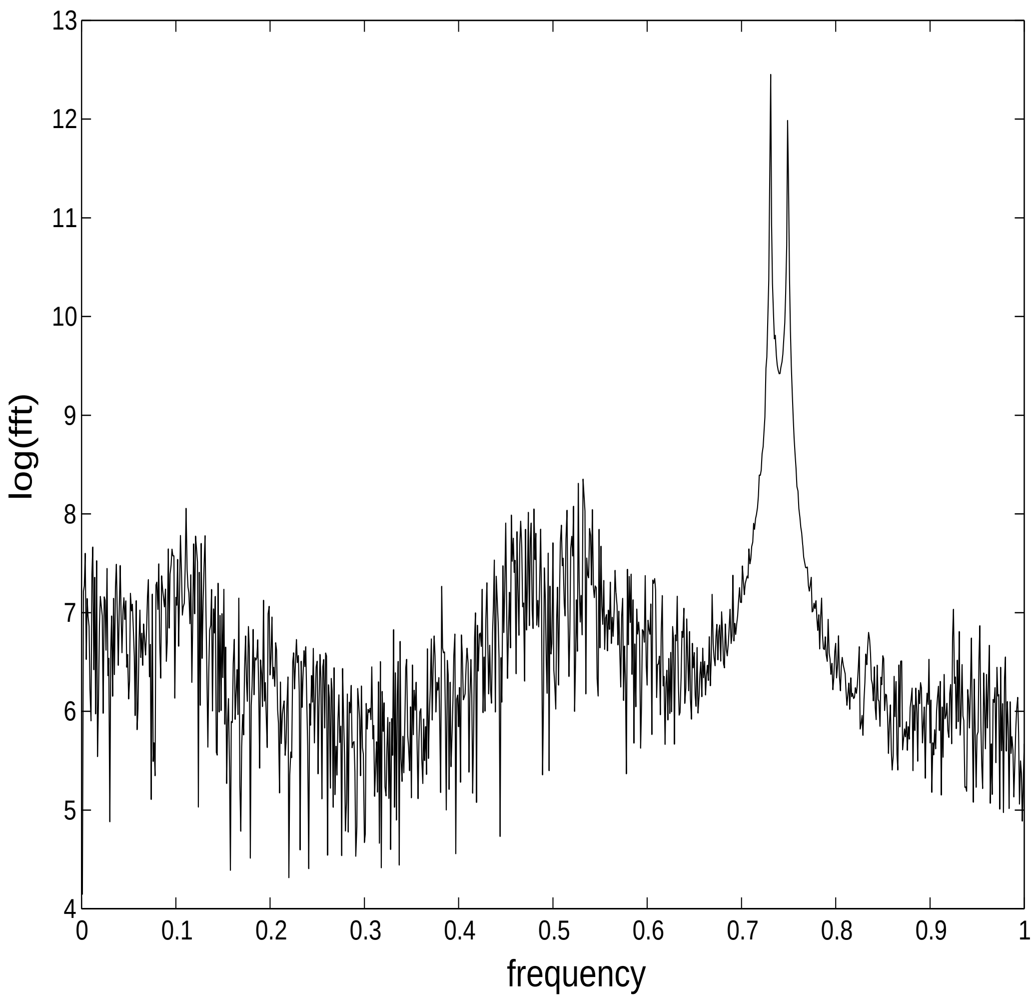


Figure 4.4: Chaotic spectrum of waveforms, $t = 1,050 \sim 25,500$ sec.

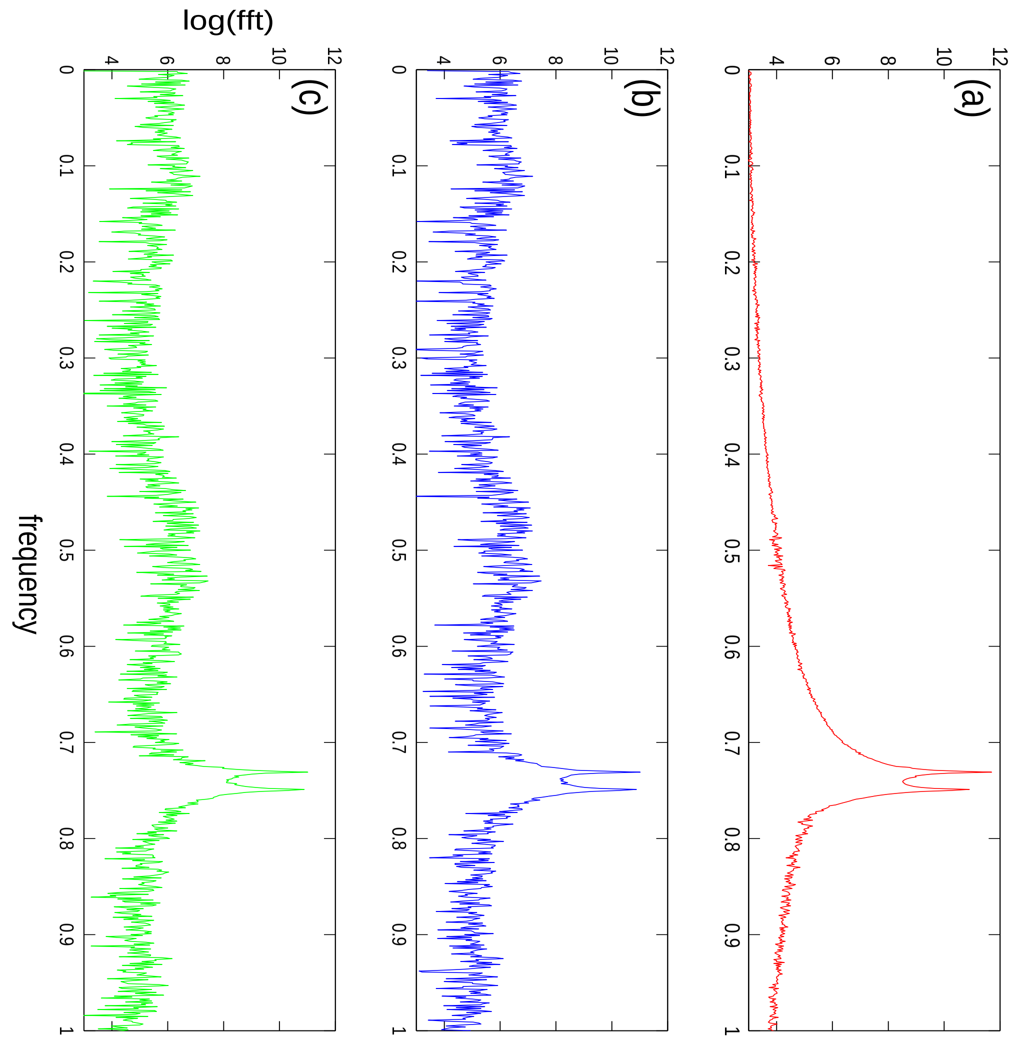


Figure 4.5: Chaotic individual spectrum, $t = 1,000 \sim 25,500$ sec.

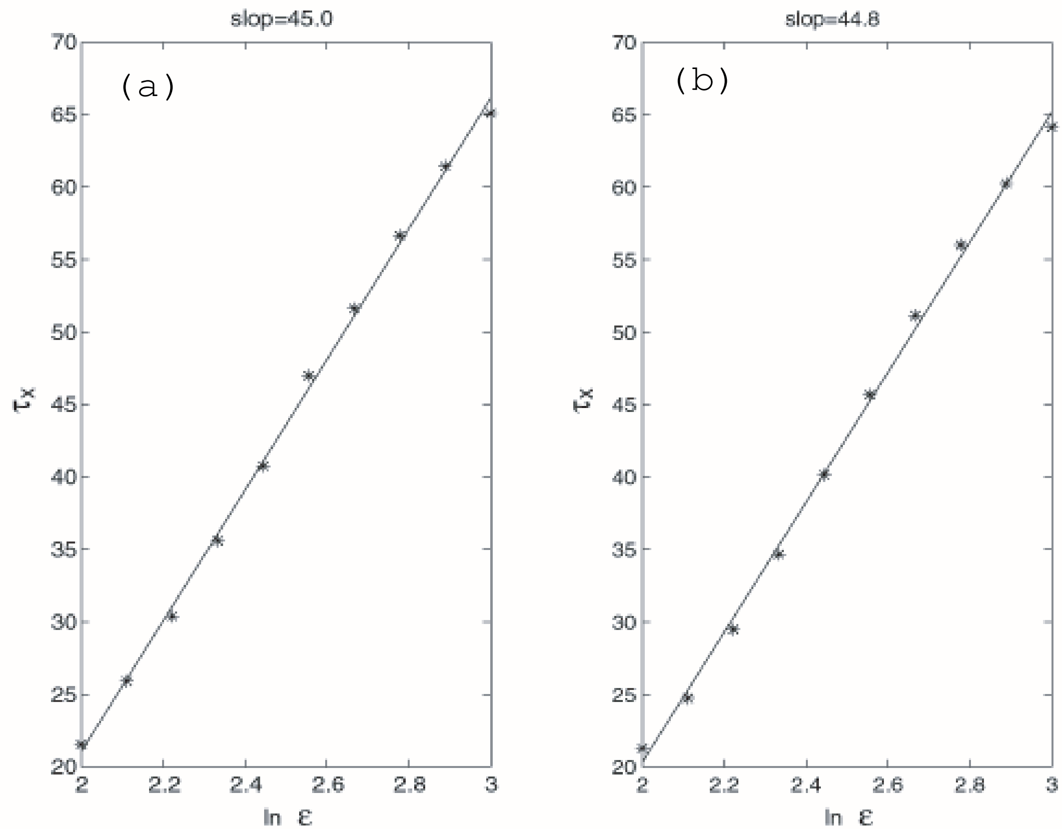


Figure 4.6: The ratio of slopes $= 45.0/44.8 \approx 1.006$

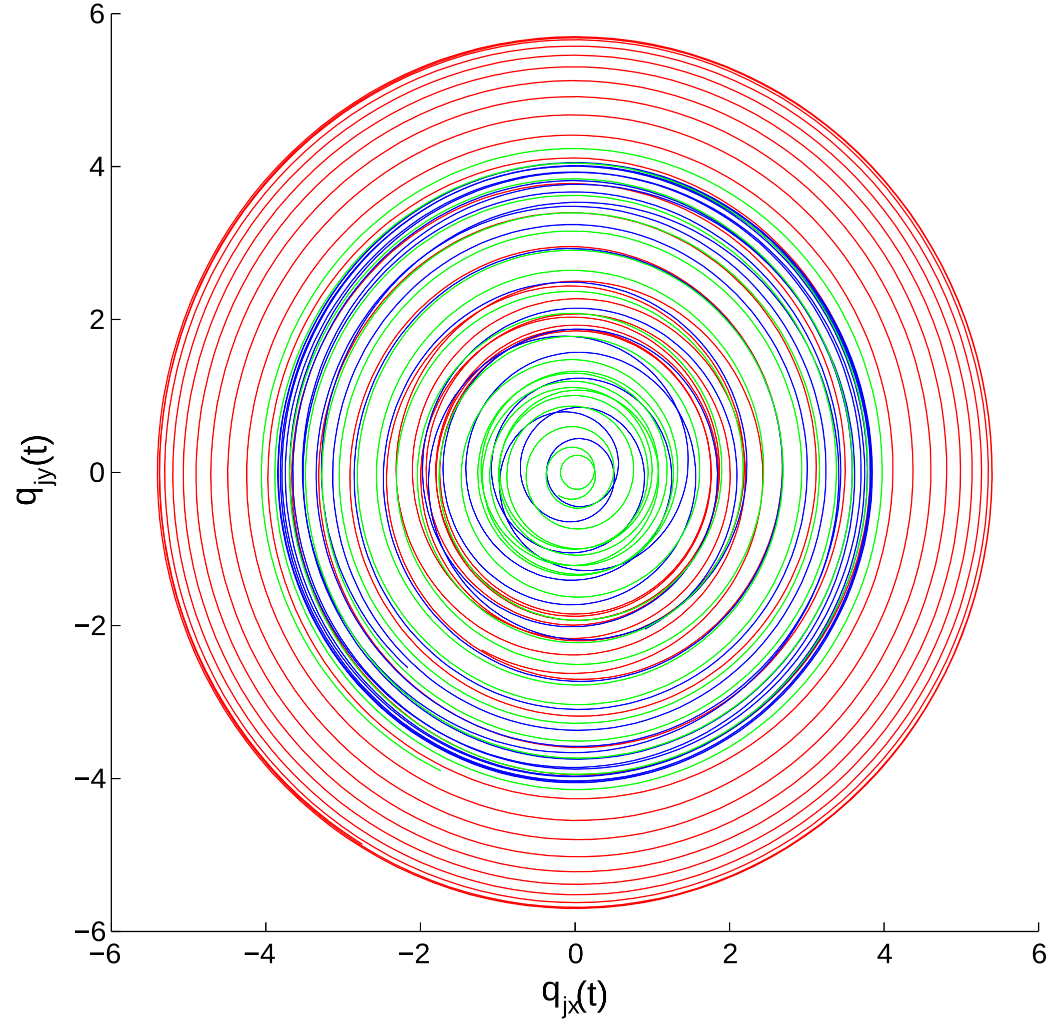


Figure 4.7: Quasi 3-periodic trajectories: $(\omega_1, \omega_2) = (9, 10), t = 25,080 \sim 25,095$ sec.

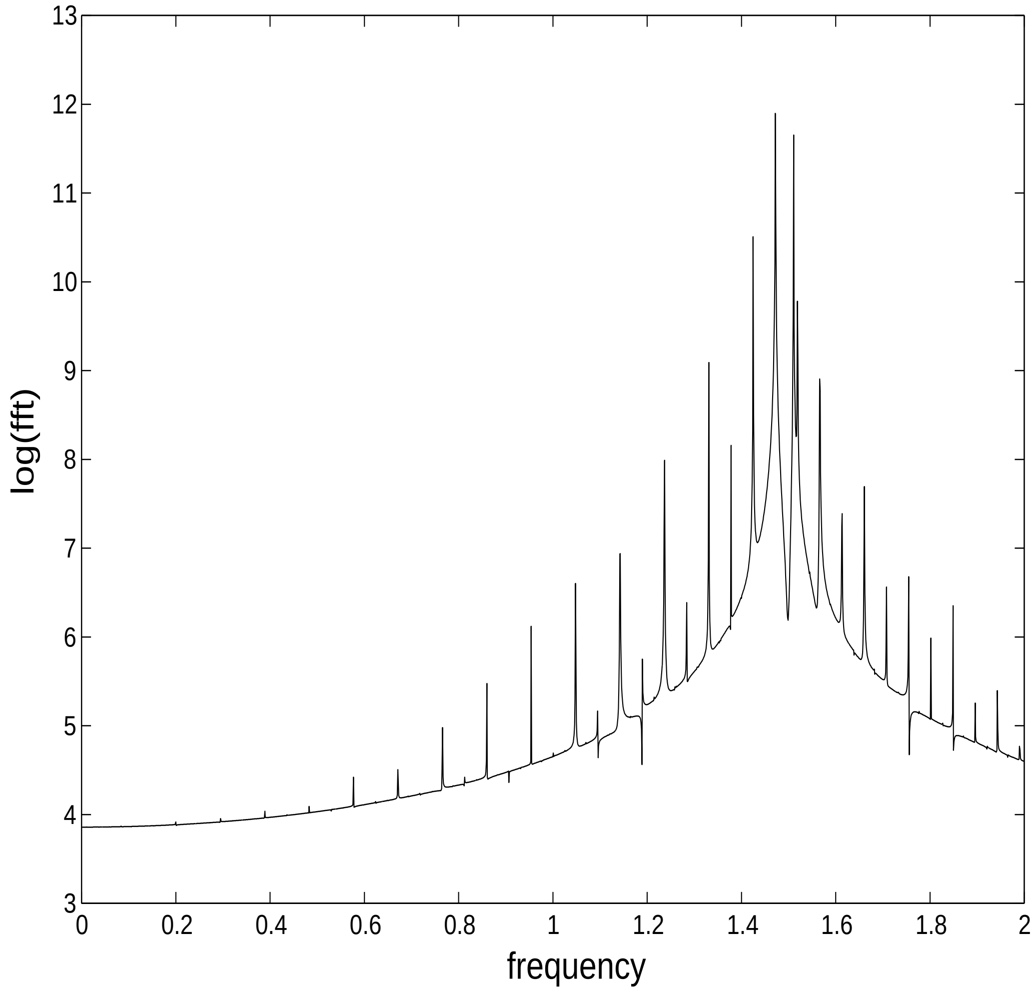


Figure 4.8: Quasi 3-periodic spectrum, $t = 1,000 \sim 25,500$ sec.

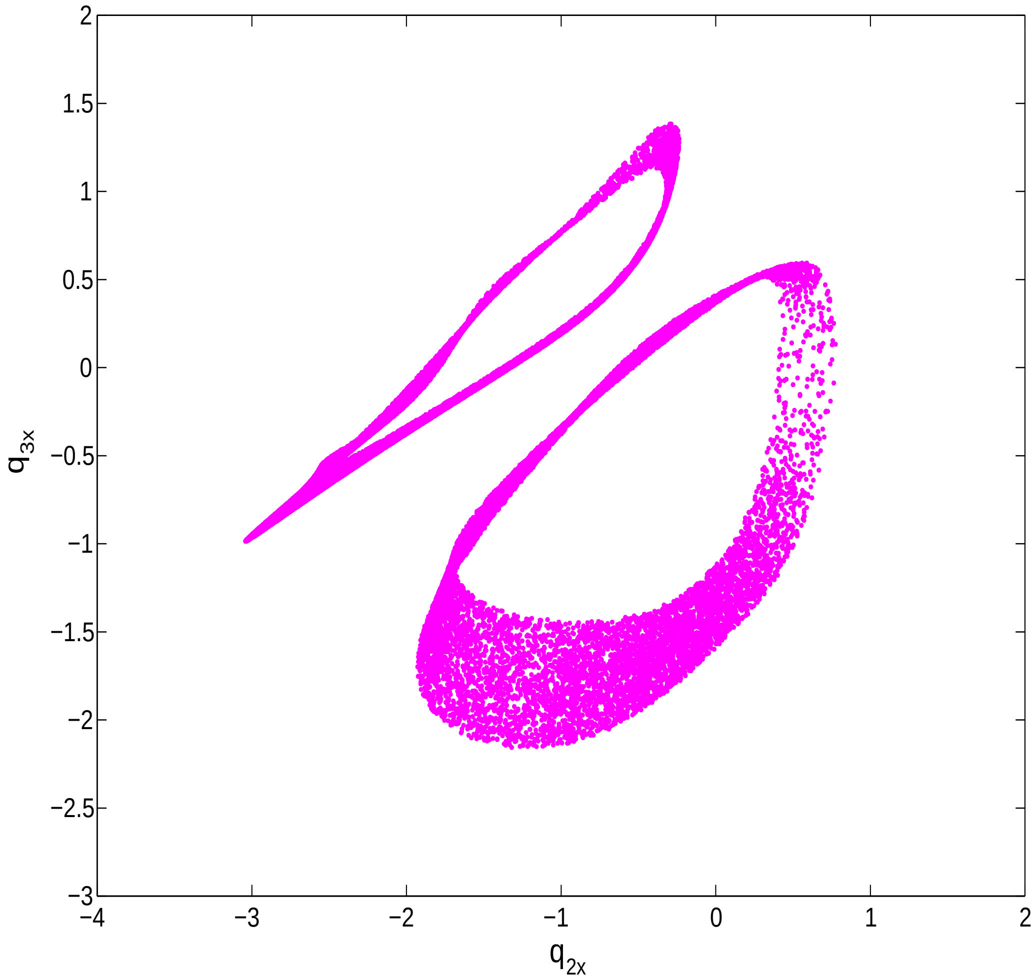


Figure 4.9: Quasi 3-periodic second-order Poincaré maps (4 dim.), $t = 41,179 \sim 4,000,000$ sec.

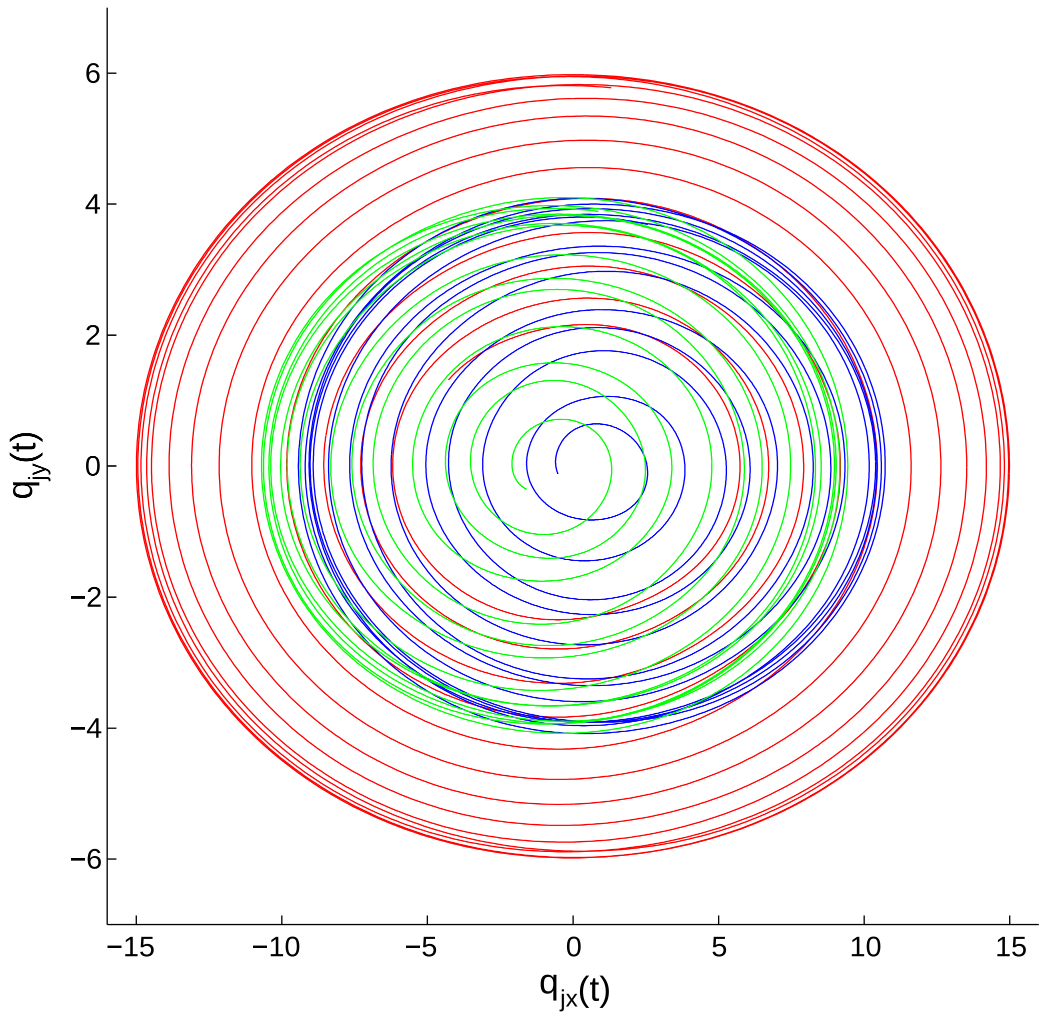


Figure 4.10: Quasi 2-periodic trajectories: $(\omega_1, \omega_2) = (6, 1), t = 25,155 \sim 25,190$ sec.

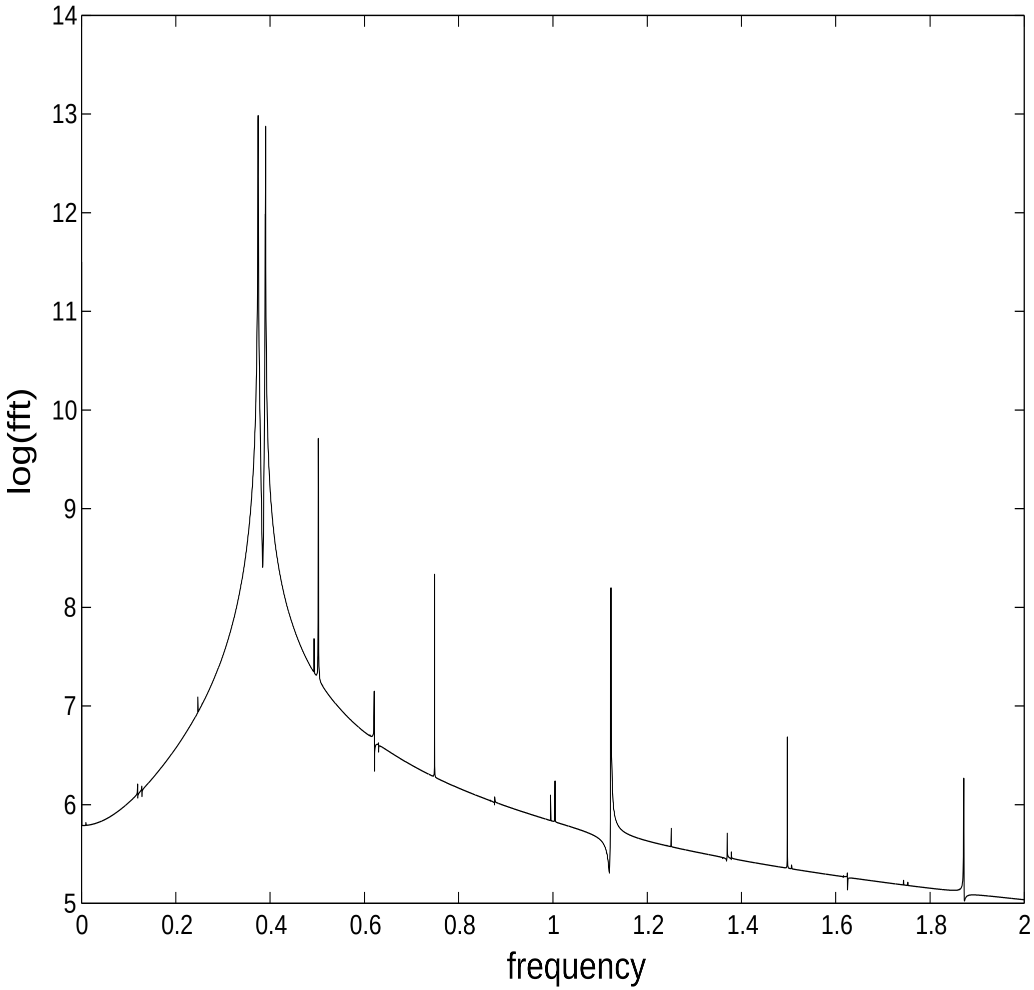


Figure 4.11: Quasi 2-periodic spectrum, $t = 2,000 \sim 25,500$ sec.

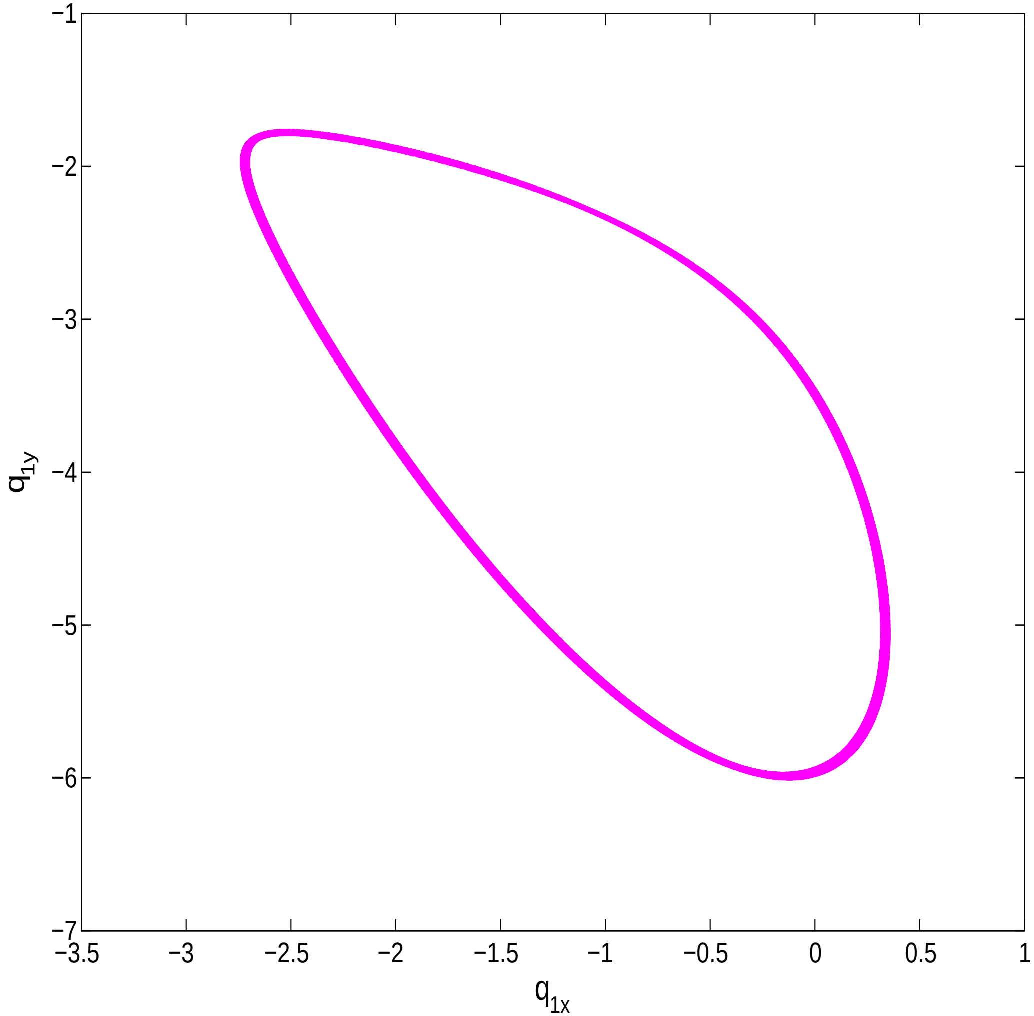


Figure 4.12: Quasi 2-periodic first-order Poincaré maps (5 dim.), $t = 37,193 \sim 1,000,000$ sec.

Case $(n_1, n_2, n_3) = (1, 1, 1)$

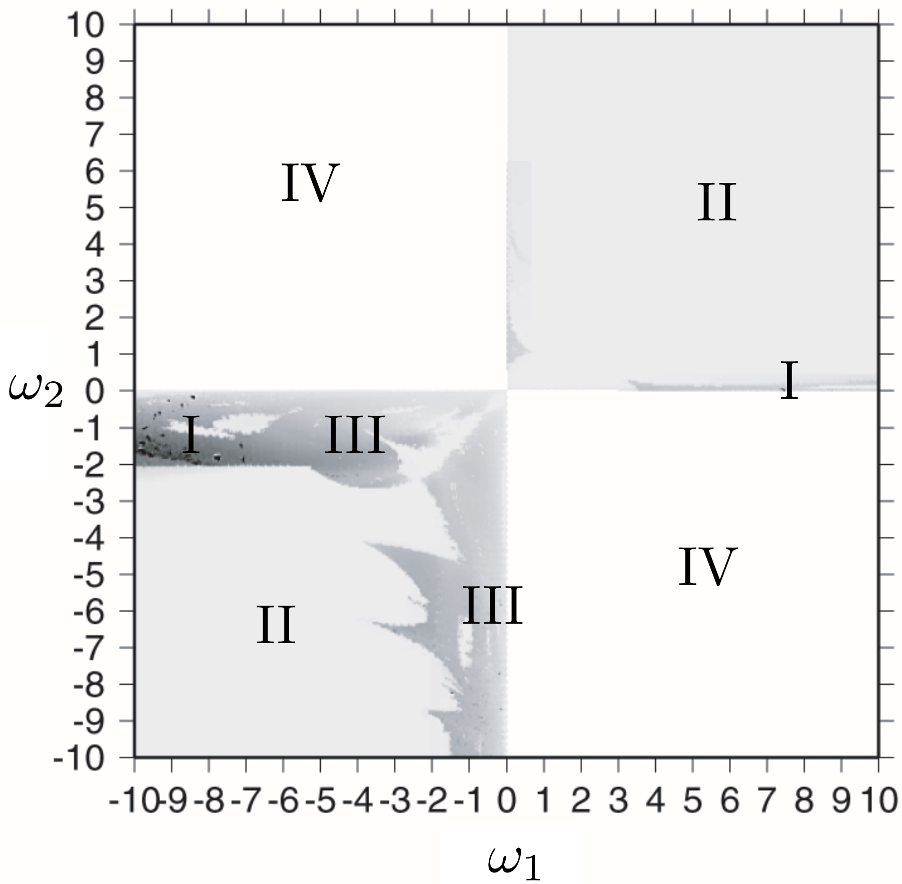


Figure 4.13: The first Lyapunov exponent.

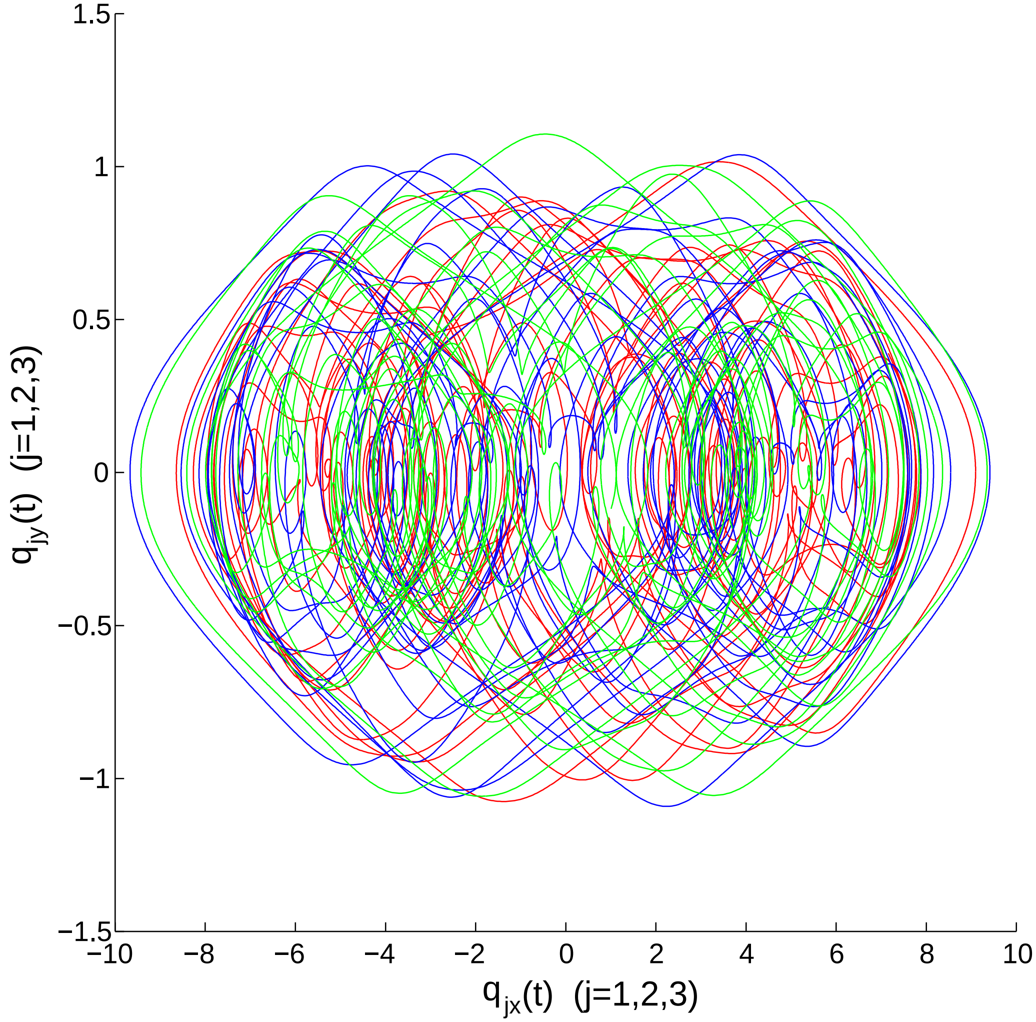


Figure 4.14: Chaotic trajectories: $(\omega_1, \omega_2) = (7.4, 0.025), t = 25,050 \sim 25,070$ sec.

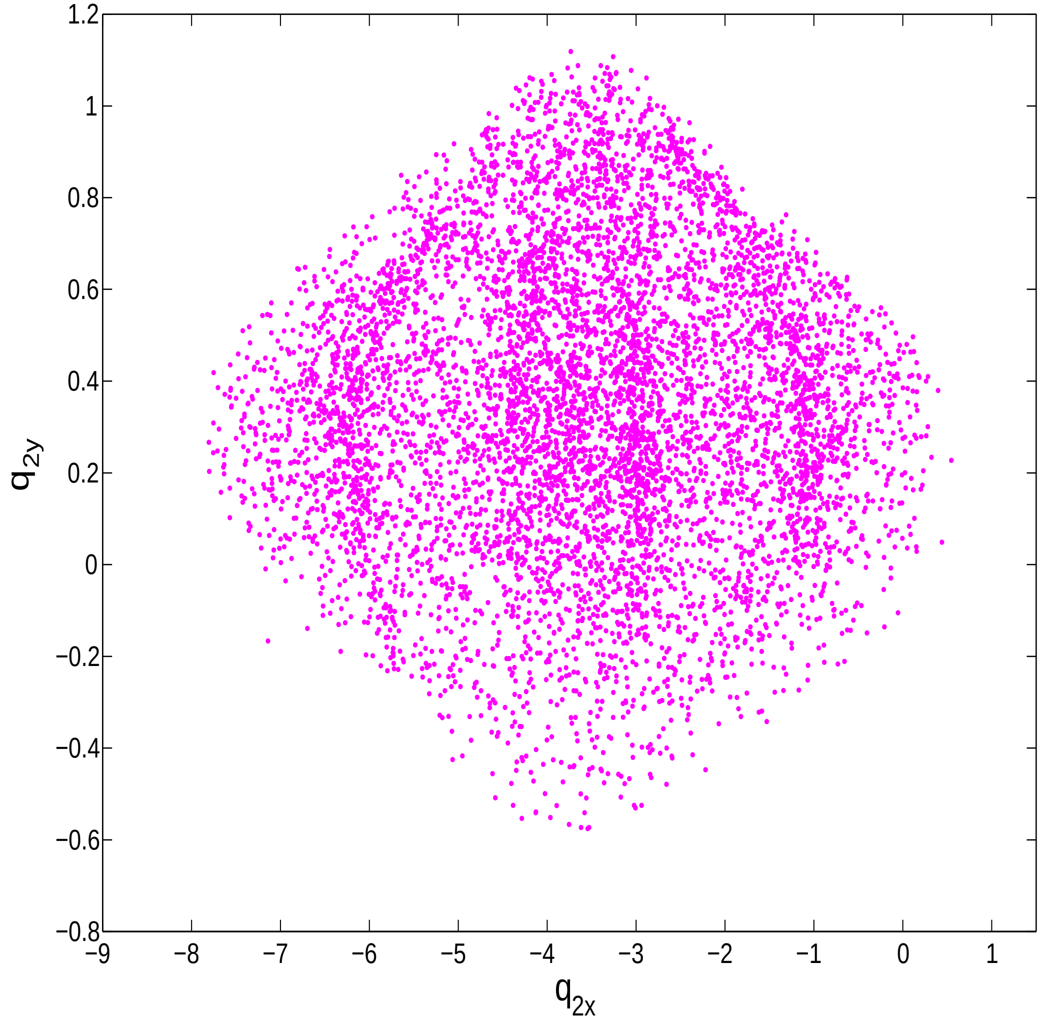


Figure 4.15: Chaotic first-order Poincaré maps (5 dim.), $t = 1,000 \sim 100,000$ sec.

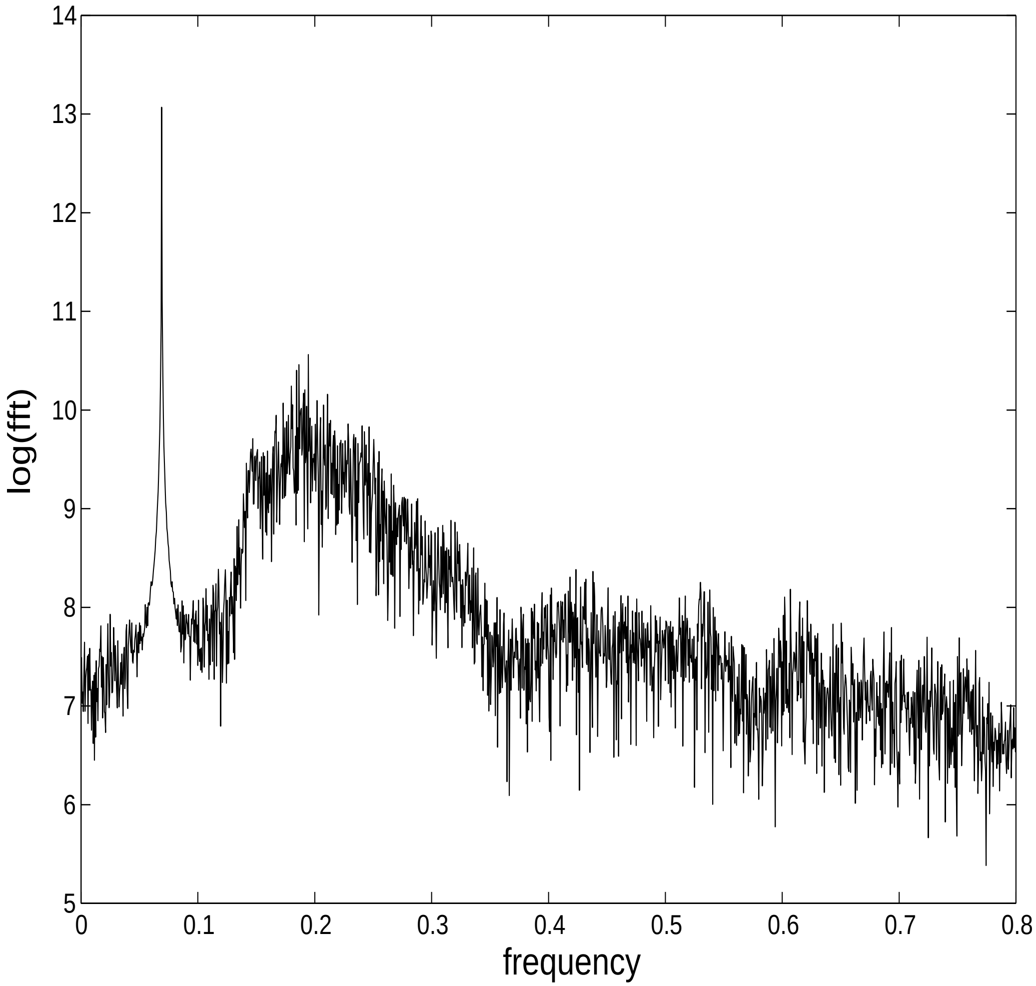


Figure 4.16: Chaotic spectrum, $t = 2,000 \sim 25,500$ sec.

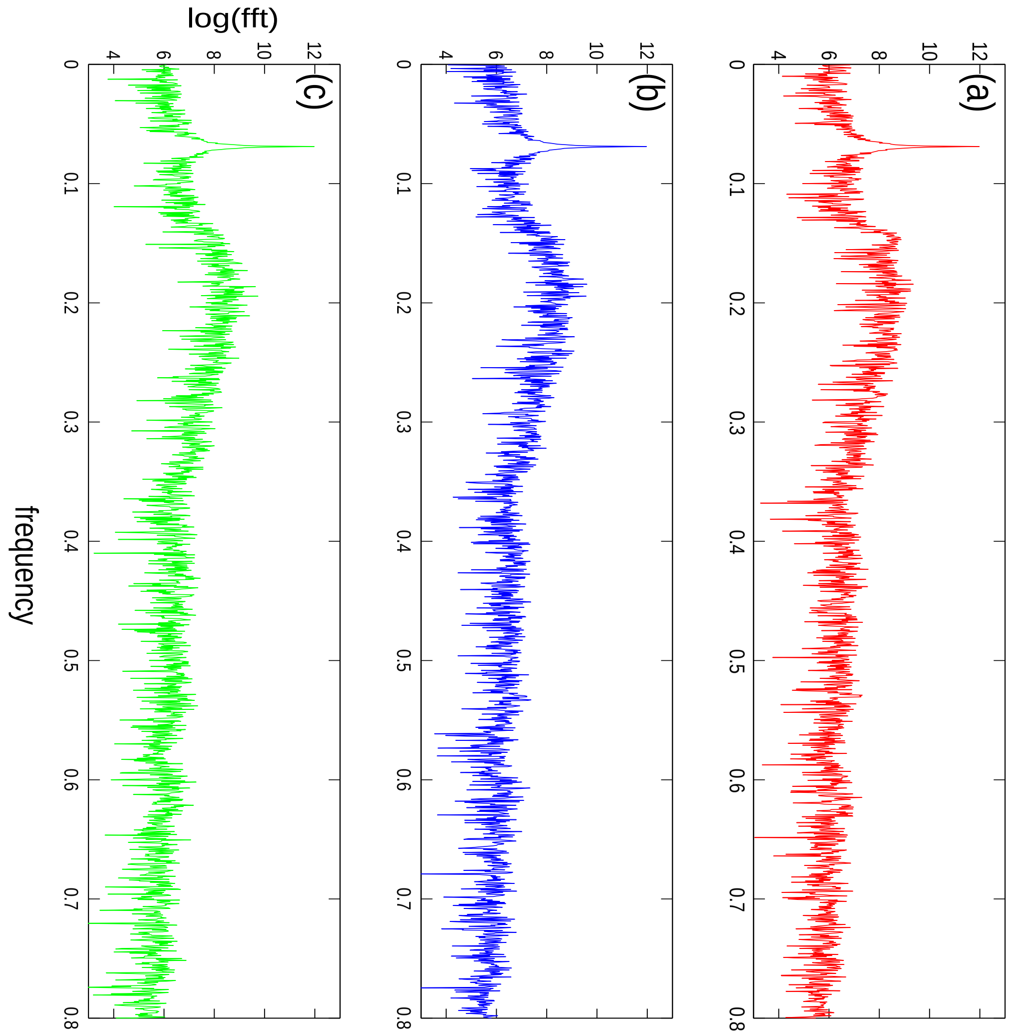


Figure 4.17: Chaotic individual spectrum, $t = 2,000 \sim 25,500$ sec.

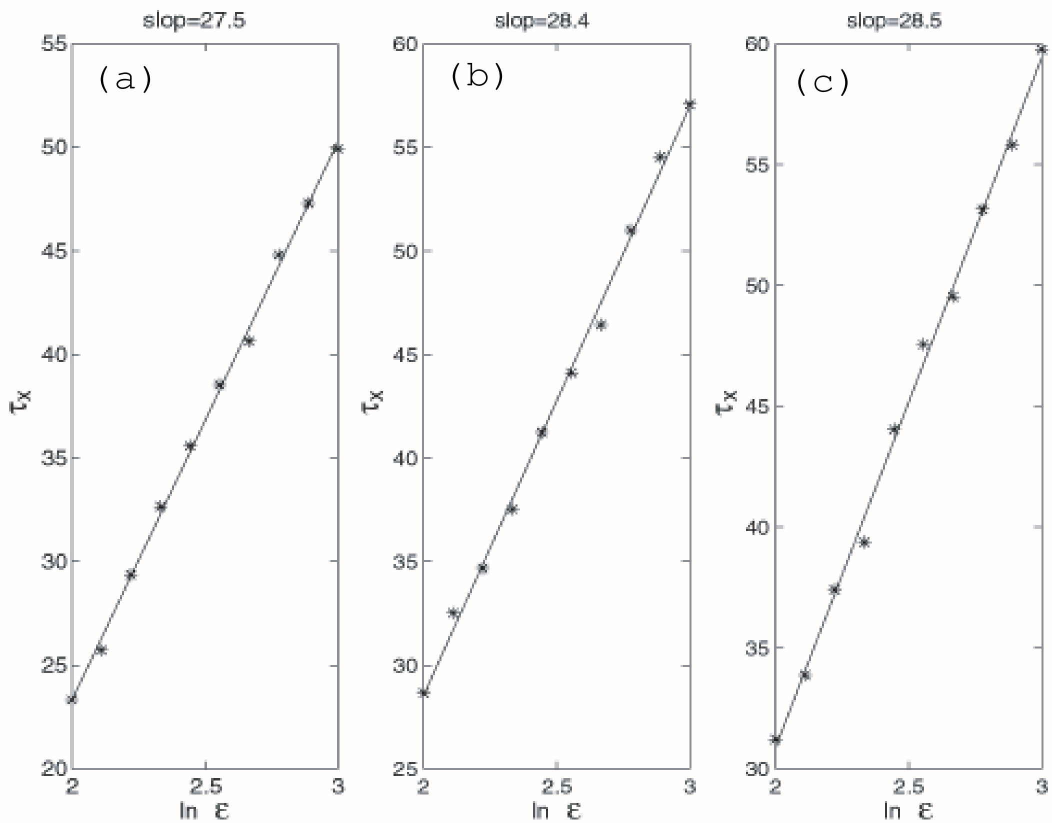


Figure 4.18: The ratio of slopes $= 27.5 : 28.4 : 28.5 \approx 0.97 : 0.996 : 1$.