Nonlinear Schrödinger Solitary Waves

Shu-Ming Chang

Department of Mathematics
National Tsing Hua University

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Outline

1. Introduction
2. Mathematical model
3. Numerical algorithms and methods
4. Numerical results

Nonlinear Schrödinger equation (NLS) with focusing power nonlinearity

$$i\partial_t\psi = -\Delta\psi - |\psi|^{p-1}\psi, \tag{1}$$

where $\psi(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ and $1 < p < \infty$.

- Goal ▸≫
- Motivation ▸≫

Well-posedness in $H^1(\mathbb{R}^n)$-norm

The Cauchy (initial value) problem for Eq. (1):

- local

  $$1 < p < p_{\max}, \text{ where } p_{\max} = \begin{cases} \infty & \text{if } n = 1, 2, \\ 1 + \frac{4}{n-2} & \text{if } n \geq 3; \end{cases}$$

- global

  $1 < p < p_c$, where $p_c = 1 + \frac{4}{n}$.

For $p \geq p_c$, $\exists$ sol.s whose H^1-norms go to ∞ in finite time. (*blow up*)

Solitary waves

$$\psi(t,x) = Q(x)\, e^{it}. \tag{2}$$

- Special solutions of the NLS (1) for a certain range of the power p.

$Q(x)$ in Eq. (2) satisfies the nonlinear elliptic equation

$$-\Delta Q - |Q|^{p-1} Q = -Q. \tag{3}$$

Non-trivial radial solution $Q(x)$

- For $p \in (1, p_{\max})$ and $n \in \mathbb{N}$, $\exists$ at least one non-trivial radial solution $Q(x) = Q(|x|)$ of Eq. (3).
- $\exists!$ pos. sol., ground state, i.e., smooth, decreases monotonically as a function of $|x|$, decays exponentially at ∞, and can be taken to be pos.: $Q(x) > 0$.

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Non-radial solutions $Q_{m,\kappa,p}$

In $\mathbb{R}^n$, $n \geq 2$, $Q_{m,\kappa,p}$ with non-zero angular momenta, $p \in (1, p_{\max})$, $\kappa = 0, 1, 2, \ldots$, each with exactly κ pos. zeros as a function of $|x|$. (those suggested by P. L. Lions)

- $n = 2$, $Q = \phi(r)\, e^{im\theta}$: polar coord.s r, θ;
- $n = 3$, $Q = \phi(r, x_3)\, e^{im\theta}$: cylindrical coord.s r, θ, x_3,

and similarly defined for $n \geq 4$.

Goal

To study the spectra of the *linearized operators* which arise when the NLS (1) is linearized around the solitary waves.

Case 1: $\psi(t, x) = \phi(r)\, e^{it}$
with $Q(x)$: non-trivial radial sol..

Case 2: $\psi(t, x) = \phi(r)\, e^{im\theta}\, e^{it}$
with $Q(x)$: non-radial & non-zero angular momenta sol..

Linearized operator $\mathcal{L}$

To study the stability of a solitary wave sol. (2) w.r.t. the NLS (1):

$$\psi(t,x) = [Q(x) + h(t,x)]\, e^{it}. \tag{4}$$

Therefore, the perturbation $h(t,x)$ satisfies

$$\partial_t h = \mathcal{L}h + (\text{nonlinear terms}), \tag{5}$$

where $\mathcal{L}$ is the linearized operator around Q.

Case 1: $Q(x) = Q_{0,0,p} = \phi_{0,0,p}(r)$ radial

$$\mathcal{L}h = -i\left\{(-\Delta + 1 - Q^{p-1})h - \tfrac{p-1}{2}Q^{p-1}(h+\bar{h})\right\}. \quad (6)$$

$\mathcal{L}$ as a matrix operator acting on $\begin{bmatrix} \mathrm{Re}\, h \\ \mathrm{Im}\, h \end{bmatrix}$,

$$\mathcal{L} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}, \quad (7)$$

where

$$L_+ = -\Delta + 1 - pQ^{p-1}, \qquad L_- = -\Delta + 1 - Q^{p-1}. \quad (8)$$

Case 2: $Q(x) = Q_{m,0,p} = \phi_{m,0,p}(r)\, e^{im\theta}$ non-radial

$$\mathcal{L}h = i\left(\Delta h - h + \frac{p+1}{2}|Q|^{p-1}h + \frac{p-1}{2}|Q|^{p-3}Q^2\bar{h}\right). \tag{9}$$

Case 2-1: $\rho(\mathcal{L})$ in 2-dimensional form

$Q = \phi(r)\cos(m\theta) + i\phi(r)\sin(m\theta)$, then

$$\mathcal{L} \sim \begin{bmatrix} 0 & -\Delta + 1 \\ \Delta - 1 & 0 \end{bmatrix} + |\phi(r)|^{p-1} \begin{bmatrix} -(p-1)\cos\sin & -\cos^2 - p\sin^2 \\ p\cos^2 + \sin^2 & (p-1)\cos\sin \end{bmatrix}(m\theta). \tag{10}$$

By restricting the problem to some invariant subspaces of $\mathcal{L}$, we reduce the problem to 1-dimension.

Case 2-2: $\rho(\mathcal{L}) = \cup\, \rho(\mathcal{L}|_{X_k}) = \cup\, \rho(L_{X_k})$

For $k = 0$, $\mathcal{L}|_{X_0}$ has the matrix form

$$L_{X_0} = \begin{bmatrix} 0 & H_0 + V \\ -H_0 + V & 0 \end{bmatrix}.$$

For $k > 0$, $\mathcal{L}|_{X_k}$ has the matrix form

$$L_{X_k} = \begin{bmatrix} 0 & H_k & 0 & V \\ -H_k & 0 & V & 0 \\ 0 & V & 0 & H_{-k} \\ V & 0 & -H_{-k} & 0 \end{bmatrix}.$$

The linearized operator acting on $[\mathrm{Re}\, h, \mathrm{Im}\, h]^\top$ and it is invariant on subspaces $Z_k = \left\{[a_1(r), a_2(r)]^\top e^{ik\theta}\right\}$ with integers k.

Case 2-3: $\rho(\mathcal{L}) = \cup\, \rho(\mathcal{L}|_{Z_k}) = \cup\, \rho(L_{m,k})$

$$\mathcal{L} \sim \begin{bmatrix} -2m/r^2 \partial_\theta & -\Delta + 1 + m^2/r^2 - \phi^{p-1} \\ -(-\Delta + 1 + m^2/r^2 - p\phi^{p-1}) & -2m/r^2 \partial_\theta \end{bmatrix}.$$

$$L_{m,k} := \begin{bmatrix} -\frac{2imk}{r^2} & -\Delta_r + 1 + \frac{m^2+k^2}{r^2} - \phi^{p-1} \\ -(-\Delta_r + 1 + \frac{m^2+k^2}{r^2} - p\phi^{p-1}) & -\frac{2imk}{r^2} \end{bmatrix},$$

$k = 0, \pm 1, \pm 2, \ldots.$

Aim

To get a more detailed understanding of the spectrum of $\mathcal{L}$, using both analytical and numerical techniques.

- Determine (or estimate) the number and locations of the ew.s of the linearized operator $\mathcal{L}$.
- Bifurcations, as p varies, of pairs of purely imaginary ew.s into pairs of ew.s with non-zero real part (a stability/instability transition).

The spectrum of $\mathcal{L}$

Step I. Compute $\phi(r) = \phi_{m,0,p}(r)$.

Case 1: $-\Delta Q - |Q|^{p-1}Q = -Q$,
where $Q = \phi_{0,0,p}(r)$.

Case 2: $-\phi'' - \frac{1}{r}\phi' + \frac{m^2}{r^2}\phi - |\phi|^{p-1}\phi = -\phi$.

Step II. Compute the spectra of the linearized operator $\mathcal{L}$ (L_{X_k}, $L_{m,k}$).

Discretization

$\Omega = \{x \in \mathbb{R}^n : |x| \leq R, R \in \mathbb{R}\}$

- Polar coordinate system.
- Dirichlet boundary condition.
- Standard central finite difference method.

Numerical methods

$\mathbf{A} \in \mathbb{R}^{N \times N}$, $\mathbf{q} = (q_1, \ldots, q_N)^\top \in \mathbb{R}^N$, $\mathbf{q}^{\textcircled{p}} = \mathbf{q} \circ \cdots \circ \mathbf{q}$: p-time Hadamard product of $\mathbf{q}$.

Step I. Compute the nonlinear ground state by iteration and renormalization:
after discretizing, we obtain the following nonlinear algebraic equation,

$$\mathbf{A}\mathbf{q} + \mathbf{q} - \mathbf{q}^{\textcircled{p}} = 0. \tag{11}$$

$$\mathbf{A}\widetilde{\mathbf{q}}_{j+1} + \widetilde{\mathbf{q}}_{j+1} = \mathbf{q}_j^{\textcircled{p}}. \tag{12}$$

$[\![\mathbf{q}]\!] := \mathrm{diag}(\mathbf{q})$, the diagonal matrix of $\mathbf{q}$.

Step II. Compute the spectra of $\mathbf{L}$:
after discretizing $\mathcal{L}$, we obtain the following large-scale linear algebraic eigenvalue problem,

$$\mathbf{L}\begin{bmatrix}\mathbf{u}\\ \mathbf{w}\end{bmatrix} = \lambda\begin{bmatrix}\mathbf{u}\\ \mathbf{w}\end{bmatrix}. \tag{13}$$

For Case 1

$$\mathbf{L} = \begin{bmatrix} 0 & \mathbf{A} + \mathrm{I} - [\![\mathbf{q}^{\gamma}]\!] \\ -\mathbf{A} - \mathrm{I} + [\![p\mathbf{q}^{\gamma}]\!] & 0 \end{bmatrix},$$

$\gamma = p - 1$, and $\mathbf{q}$ from Step I, and satisfies in (11). To use implicitly restarted Arnoldi method to deal with this problem.

For Case 2

We develop 3 algorithms for computing the spectrum of $\mathcal{L}$ in Case 2-1, 2-2 & 2-3.

Alg. 1: 2-dim. mesh, $r = 0 : \delta_r : R$, $\theta = 0 : \delta_\theta : 2\pi$. The discretized matrix has size NT by NT with $N = R/\delta_r$ and $T = 2\pi/\delta_\theta$, where $R = 15$, $\delta_r = 0.04$, and $T = 160$.

For Case 2

Alg. 2: To discretize the operator, we use the 1-dim. mesh, $r = 0 : \delta_r : R$, $N = R/\delta_r$.

- The matrix corresponding to X_0 has size $2N$ by $2N$. The matrix for X_k with $k > 0$ has size $4N$ by $4N$.
- Counting multiplicity, the ew.s of $\mathcal{L}$ is the union of ew.s of $\mathcal{L}|_{X_k}$ with $k = 0, 1, 2, \ldots$.

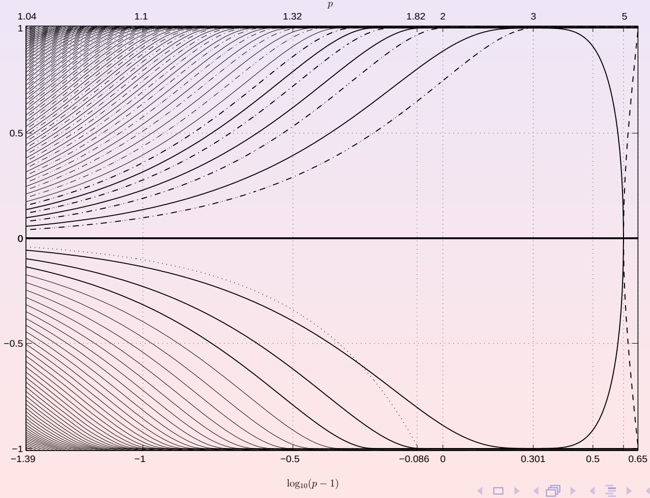
Alg. 3: Similar to Alg. 2 but the matrix size is only half.

Properties of these algorithms

1. Equivalence of Algorithms 2 and 3.
2. Numerical efficiency: Alg. 3 $\simeq$ Alg. 2 $\succ$ Alg. 1.

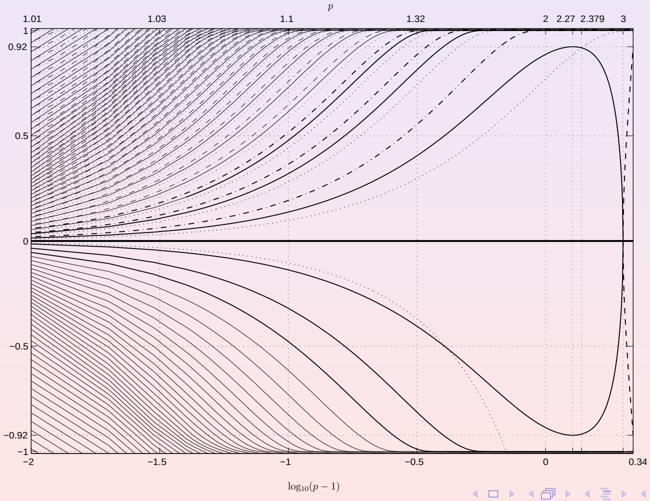
Case 1: radial

$n = 1$



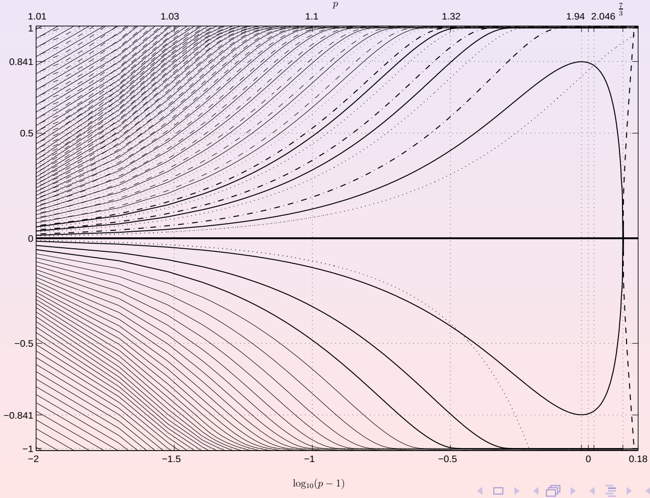
Case 1: radial

$n = 2$



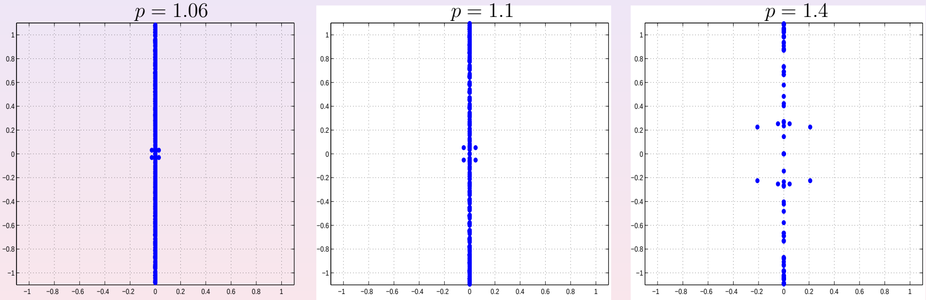
Case 1: radial

$n = 3$



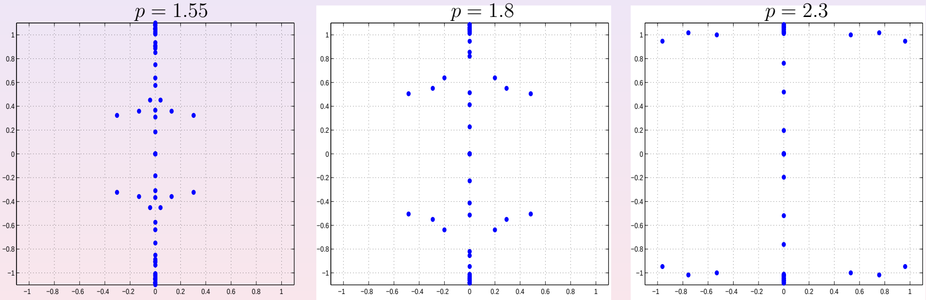
Case 2: non-radial

$n = 2, m = 1$ by Alg. 1



Case 2: non-radial

$n = 2, m = 1$ by Alg. 1



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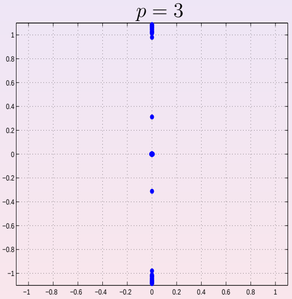
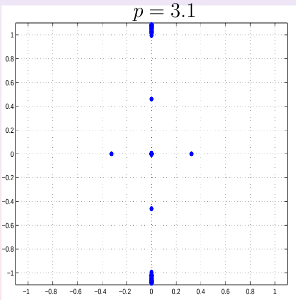
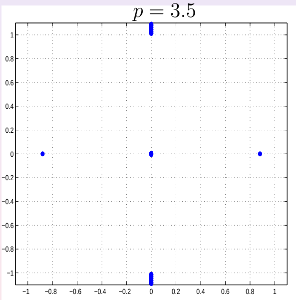
$n = 2, m = 1$ by Alg. 1

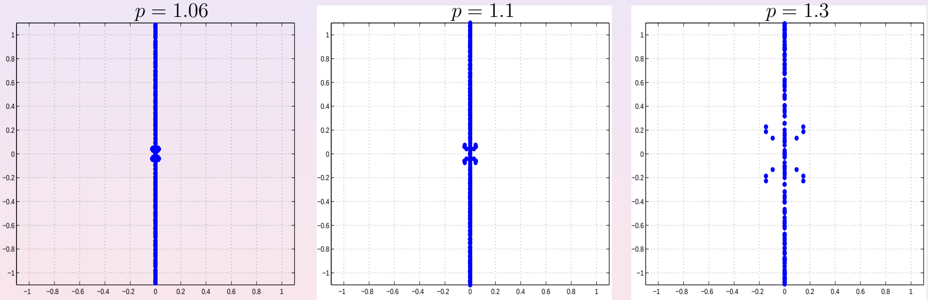






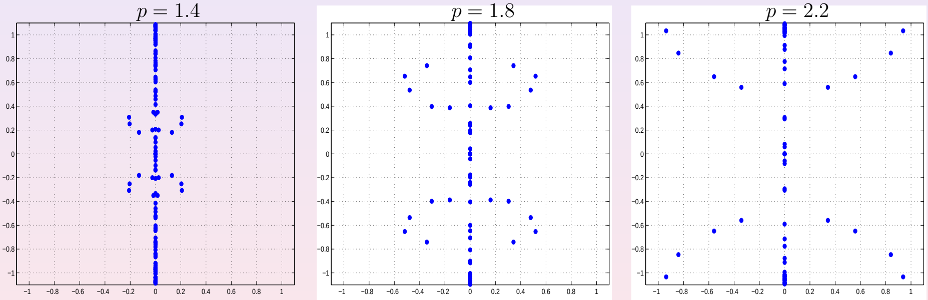
Case 2: non-radial

$n = 2, m = 2$ by Alg. 1



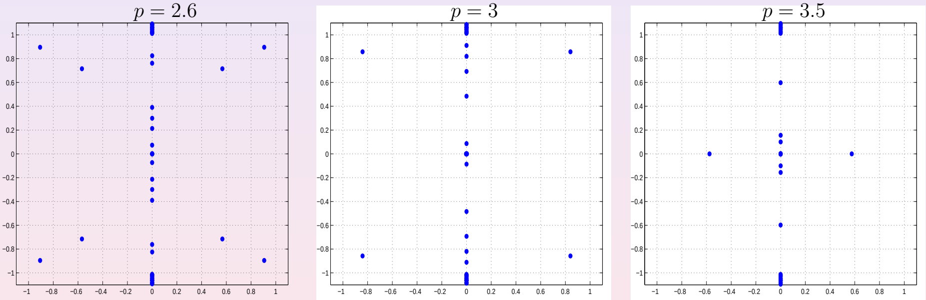
Case 2: non-radial

$n = 2$, $m = 2$ by Alg. 1



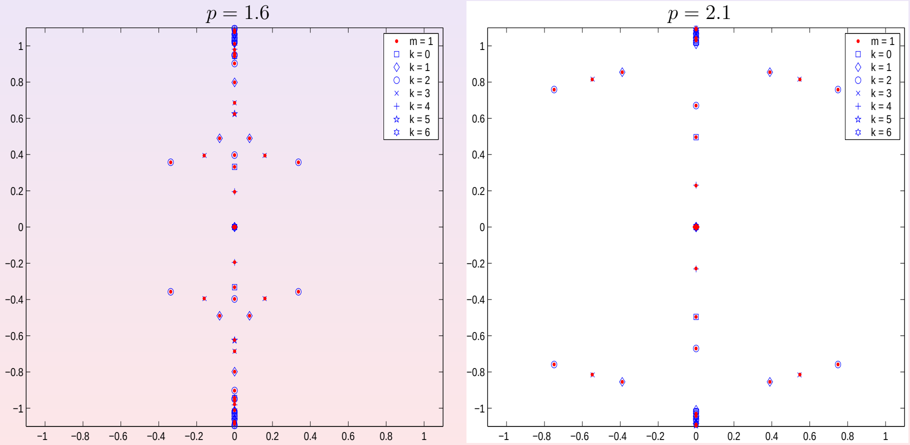
Case 2: non-radial

$n = 2, m = 2$ by Alg. 1



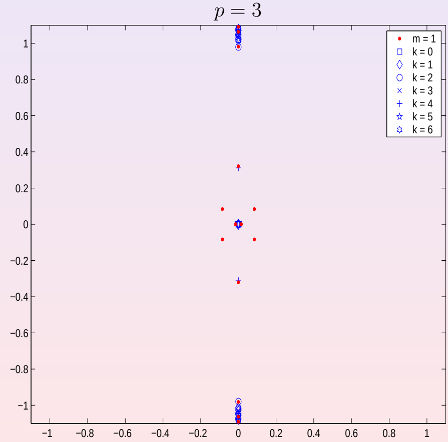
Case 2: non-radial

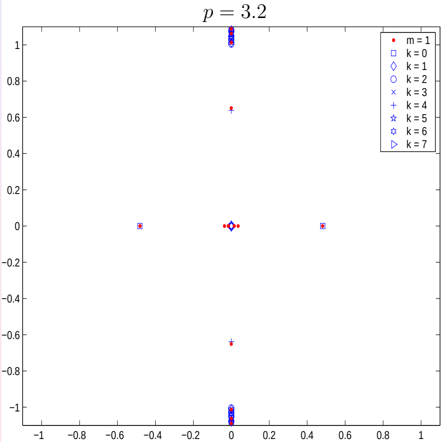
$n = 2, m = 1$ comparison between Alg. 1 and Alg. 2,3



Case 2: non-radial

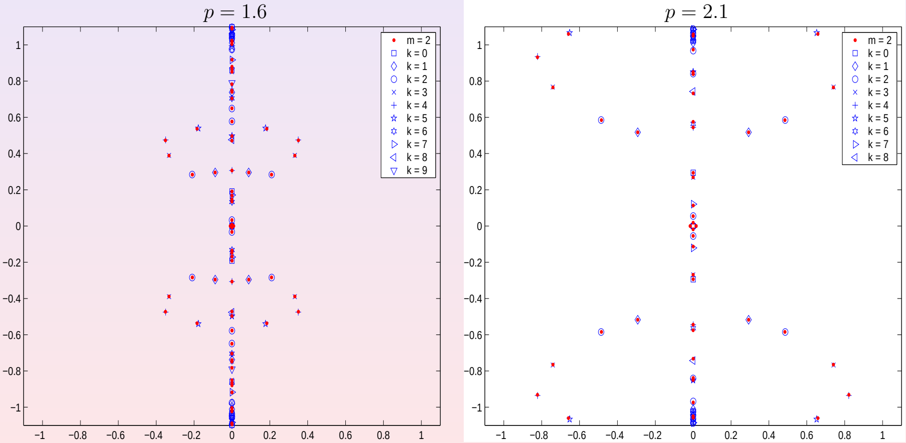
$n = 2, m = 1$ comparison between Alg. 1 and Alg. 2,3





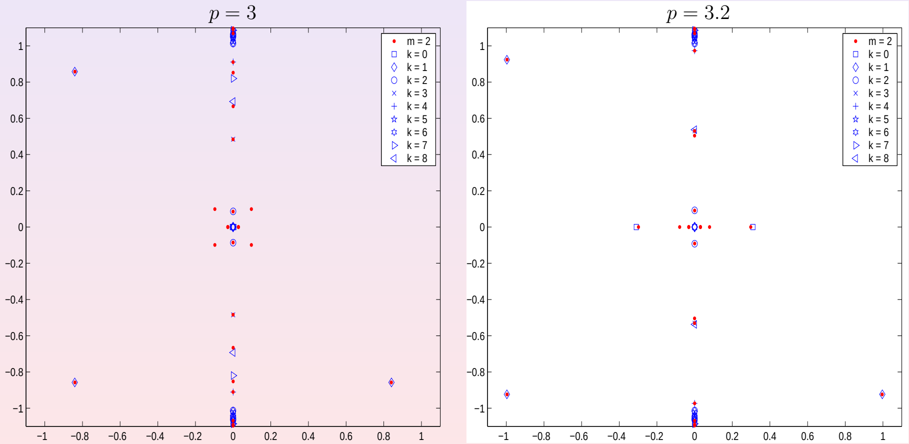
Case 2: non-radial

$n = 2, m = 2$ comparison between Alg. 1 and Alg. 2,3



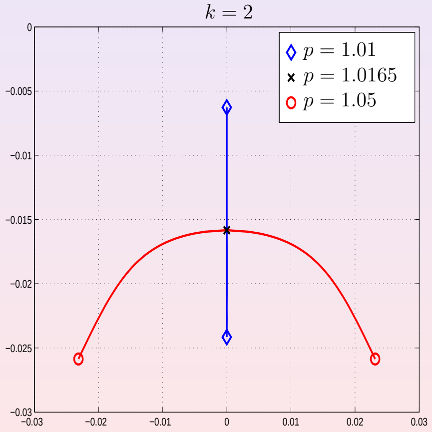
Case 2: non-radial

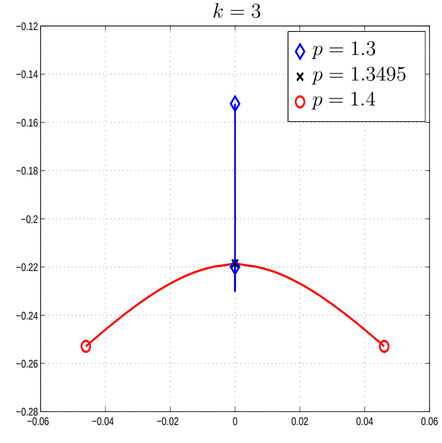
$n = 2$, $m = 2$ comparison between Alg. 1 and Alg. 2,3



Case 2: non-radial

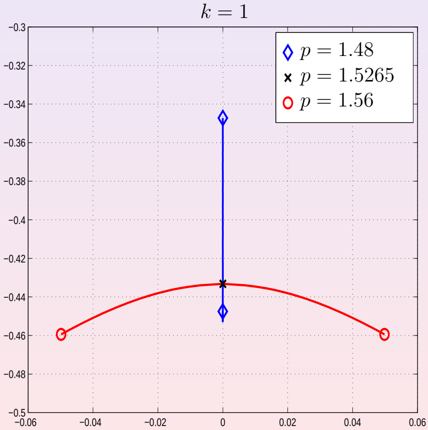
$n = 2, m = 1$ bifurcation

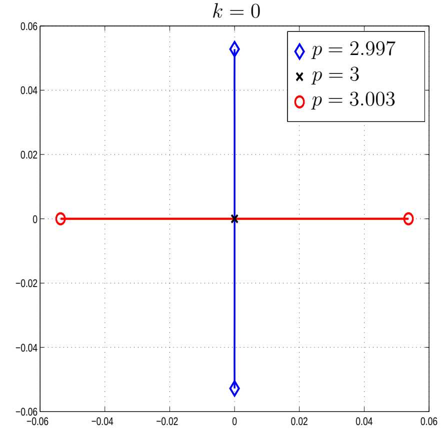




Case 2: non-radial

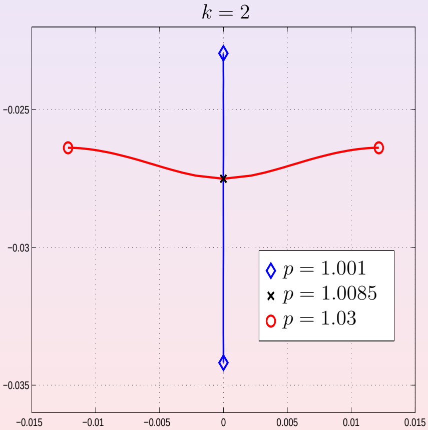
$n = 2, m = 1$ bifurcation

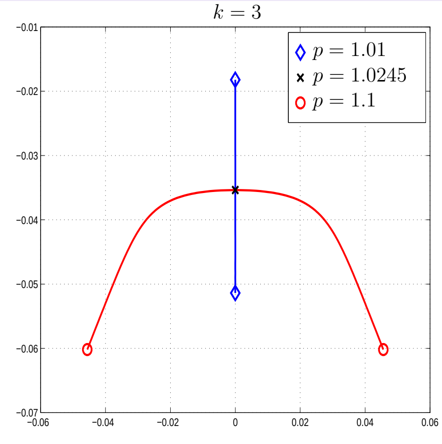




Case 2: non-radial

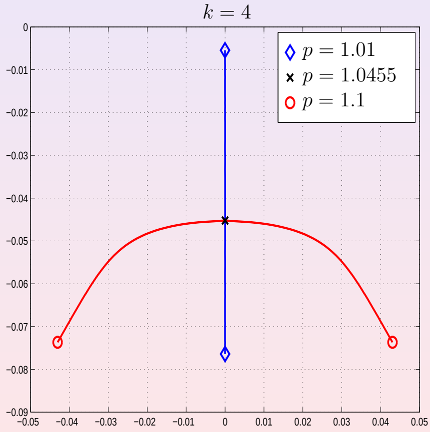
$n = 2$, $m = 2$ bifurcation

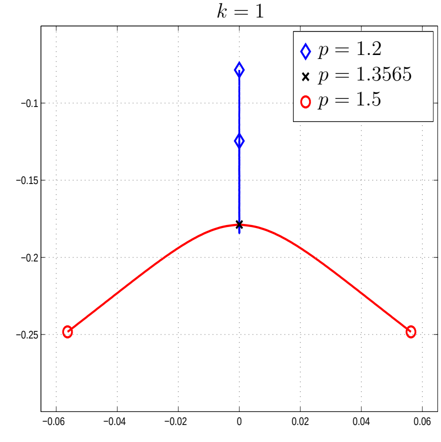




Case 2: non-radial

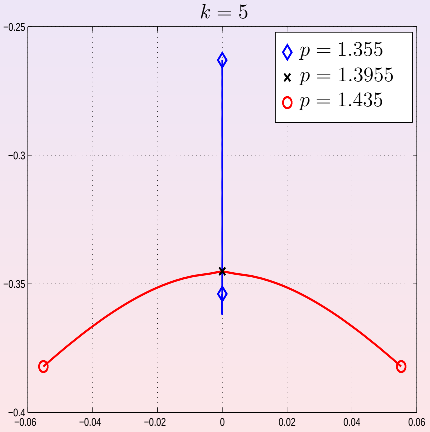
$n = 2, m = 2$ bifurcation

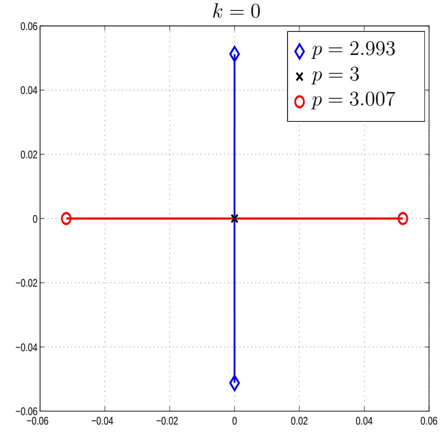




Case 2: non-radial

$n = 2$, $m = 2$ bifurcation





Thank you for your attention!

Authors

Stephen Gustafson & Tai-Peng Tsai

Department of Mathematics, University of British Columbia, Vancouver, Canada.

Kenji Nakanishi

Department of Mathematics, Kyoto University, Kyoto, Japan.

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Goal

To study the spectra of the *linearized operators* which arise when the NLS (1) is linearized around *solitary waves*.

Return

Motivation

Properties of these spectra are intimately related to the problem of the stability (orbital and asymptotic) of these solitary waves, and to the long-time dynamics of solutions of NLS.

Return

Reference

Existence

S. I. Pohozaev, *Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$*, Sov. Math. Doklady **5** (1965), 1408–1411.

Return

Ground state

See Sulem for the various existence & uniqueness results and various nonlinearities.

$\min J[u]$

For all $n \geq 1$ and $p \in (1, p_{\max})$, the ground state minimizes the Gagliardo-Nirenberg quotient

$$J[u] := \frac{\left(\int |\nabla u|^2\right)^a \left(\int u^2\right)^b}{\int u^{p+1}}$$

among nonzero $H^1(\mathbb{R}^n)$ radial functions.

Return

Reference

Existence and uniqueness

C. Sulem and P. L. Sulem, *The nonlinear Schrödinger equations: self-focusing and wave collapse*, Springer, 1999.

Return

Non-zero angular momenta

In $\mathbb{R}^n$, $n \geq 2$ and let $\kappa = [n/2]$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, use polar coord.s r_j and θ_j for each pair x_{2j-1} and x_{2j}, $j = 1, \ldots, \kappa$. P. L. Lions considers sol.s of the form

$$Q(x) = \phi(r_1, r_2, \ldots, r_\kappa, x_n)\, e^{i(m_1\theta_1 + \cdots + m_\kappa\theta_\kappa)}, \qquad m_j \in \mathbb{Z}$$

and proves that $\exists$ energy minimizing sol.s.

Reference

P. L. Lions, *Solutions complexes d'équations elliptiques semilinéaires dans R^N*, C. R. Acad. Sci. Paris Sér. I Math. 302 (1986), No. 19, 673–676.

L_- and L_+

- Play a central role in the stability theory.
- Self-adjoint Schrödinger operators with continuous spectrum $[1, \infty)$, and with finitely many ew.s below 1.
- L_- is a nonnegative operator, L_+ has exactly one negative ew when Q is the ground state.

Return

Case 1: the spectra of $\mathcal{L}$

1. $\forall\, p \in (1, p_{\max})$, 0 is an ew of $\mathcal{L}$.
2. $\Sigma_c := \{ir : r \in \mathbb{R}, |r| \geq 1\}$ is the continuous spectrum of $\mathcal{L}$.
3. $p = p_c$ is critical for stability of the ground state solitary wave.
   - $p < p_c$ the ground state is orbitally stable.
   - $p \geq p_c$ it is unstable.
4. $p \in (1, p_c]$: all ew.s of $\mathcal{L}$ are purely imaginary.
5. $p \in (p_c, p_{\max})$: $\mathcal{L}$ has at least one ew with pos. real part.

Return

Reference

Stable and unstable

- M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry I*, J. Funct. Anal. **74** (1987), No. 1, 160–197.
- M. I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39** (1986), 51–68.

Return

Case 2-2

Define

$$V = \frac{p-1}{2}\phi^{p-1}, \quad H_k = -\Delta_r + 1 + \frac{(m+k)^2}{r^2} - \frac{p+1}{2}\phi^{p-1}.$$

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Solitary waves

For the simplest case $n = 2$, let

$$\psi(t, x) = \phi(r)\, e^{im\theta}\, e^{it},$$

then from NLS (1), $\phi(r)$ satisfies the nonlinear elliptic equation

$$-\phi'' - \frac{1}{r}\phi' + \frac{m^2}{r^2}\phi + \phi - |\phi|^{p-1}\phi = 0.$$

◂ Return

Reference

Discretization scheme

M. C. Lai, *A note on finite difference discretizations for poisson equation on a disk*, Numerical Methods for Partial Differential Equations **17** (2001), No. 3, 199–203.

Return

Reference

Iterative algorithm

T. M. Hwang and W. Wang, *Analyzing and visualizing a discretized semilinear elliptic problem with Neumann boundary conditions*, Numerical Methods for Partial Differential Equations **18** (2002), 261–279.

Return

Iterative algorithm

Step 0 Let $j = 0$.
Choose an initial solution $\widetilde{\mathbf{q}}_0 > 0$ and let $\mathbf{q}_0 = \frac{\widetilde{\mathbf{q}}_0}{\|\widetilde{\mathbf{q}}_0\|_2}$.

Step 1 Solve the equation (12), then obtain $\widetilde{\mathbf{q}}_{j+1}$.

Step 2 Let $\alpha_{j+1} = \frac{1}{\|\widetilde{\mathbf{q}}_{j+1}\|_2}$ and normalize $\widetilde{\mathbf{q}}_{j+1}$ to obtain
$\mathbf{q}_{j+1} = \alpha_{j+1}\widetilde{\mathbf{q}}_{j+1}$.

Step 3 If (convergent) then
Output the scaled solution $(\alpha_{j+1})^{\frac{1}{p-1}}\mathbf{q}_{j+1}$. Stop.
else
Let $j := j + 1$.
Goto Step 1.
end

Numerical efficiency

- Alg. 1 is 2-dim., and thus more expensive to compute and less accurate. Both Alg. 2 and 3 are 1-dim. and more accurate.
- The benefit of Alg. 3 than Alg. 2 is that it further decomposes the subspace of $L^2(\mathbb{R}^2, \mathbb{C}^4)$ corresponding to X_k to two subspaces.
- Although the matrix size of Alg. 3 is only half that of Alg. 2, its components are complex. It implies that Alg. 3 requires more storage space. Numerically these two algorithms are not very different.

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