Outline

- Introduction of Bose-Einstein Condensates (BEC)
- Introduction of Vortices in BEC
- Mathematical Model
- Numerical Study of Three Vortices
- Conclusion

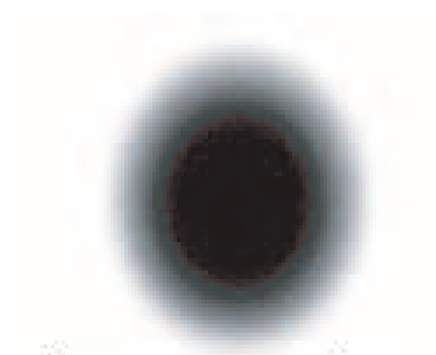
1 Introduction of BEC

- What is BEC?

(a) Cold atom: an atom in the lowest energy level is spread out a little, so it looks like a very small fuzzy ball.

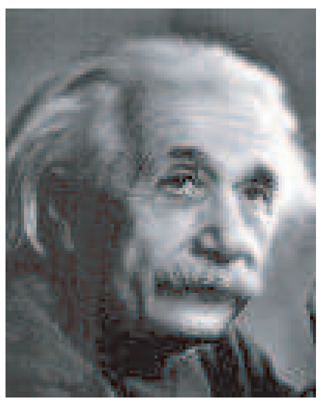
(b) Super atom: at the special incredibly low temperatures needed for BEC that they lose their individual identities and coalesce into a single blob.

(a)



(b)

- Theoretical prediction 1924 ...
  - S. Bose: derived Planck's black body radiation law from considering the cavity radiation as an ideal photon gas and worked out Bose statistics for photons.
  - A. Einstein: generalized Bose statistics to other Bosonic particles and atoms (Bose-Einstein statistics) and predicted if the atoms were cold enough, almost all of the particles would congregate in the ground state. (BEC)



A. Einstein
(1879 $\sim$ 1955)

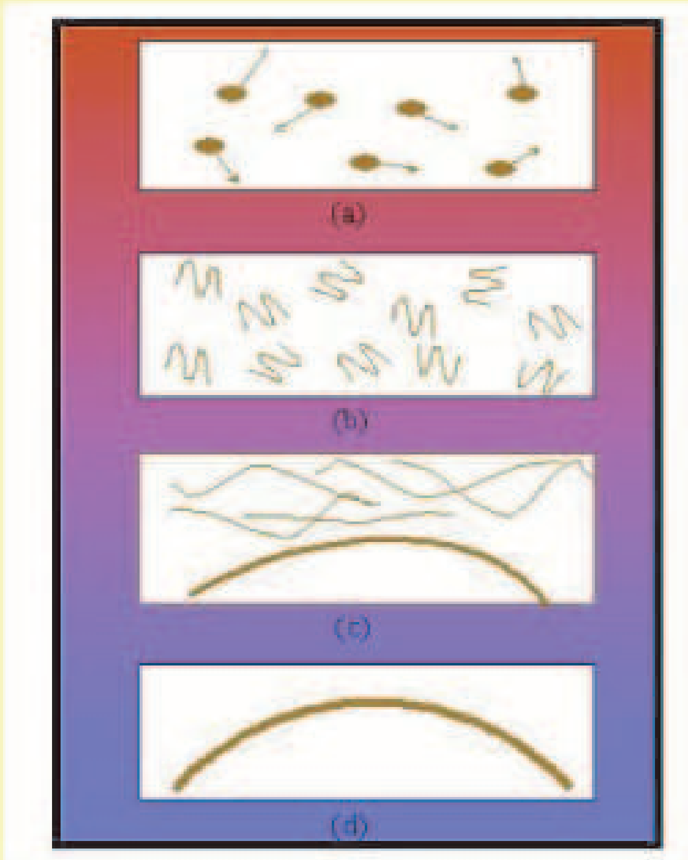
S. Bose
(1894 $\sim$ 1974)

- How does BEC happen?

$T \downarrow$

$T = T_c$

$T < T_c$



$$\lambda = \frac{\hbar}{p}, \quad p \propto \sqrt{m_a \mathrm{k} T}$$

$$\lambda \propto \frac{\hbar}{\sqrt{m_a \mathrm{k} T}}$$

Eg: ^{23}Na,

$$T = 300\mathrm{K},$$

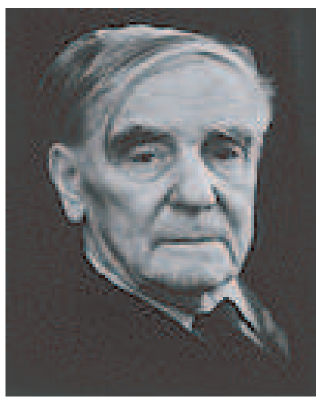
$$\lambda = 0.04\mathrm{nm}.$$

$$T = 0.0003\mathrm{K},$$

$$\lambda = 40\mathrm{nm}.$$

Note: $0\mathrm{K} = -273.15°\mathrm{C}.$

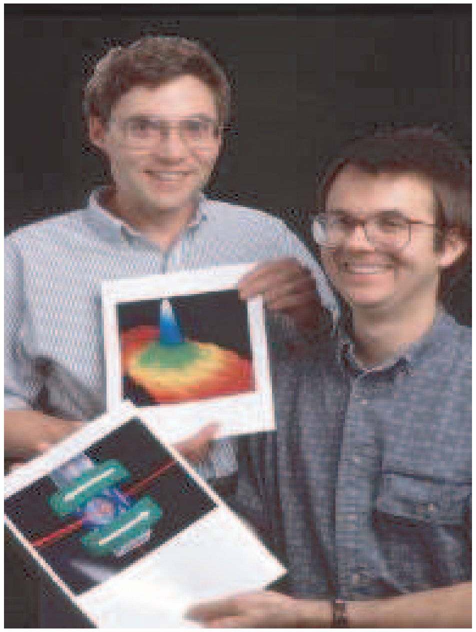
- Physical experiments
  - Superfluid He^4 1938:
    P. L. Kapitza, Allen and Misener: discovered the superfluidity of liquid helium.
    F. London: proposed that the superfluid fraction consisting of those atoms which have "condensed" to the ground state.

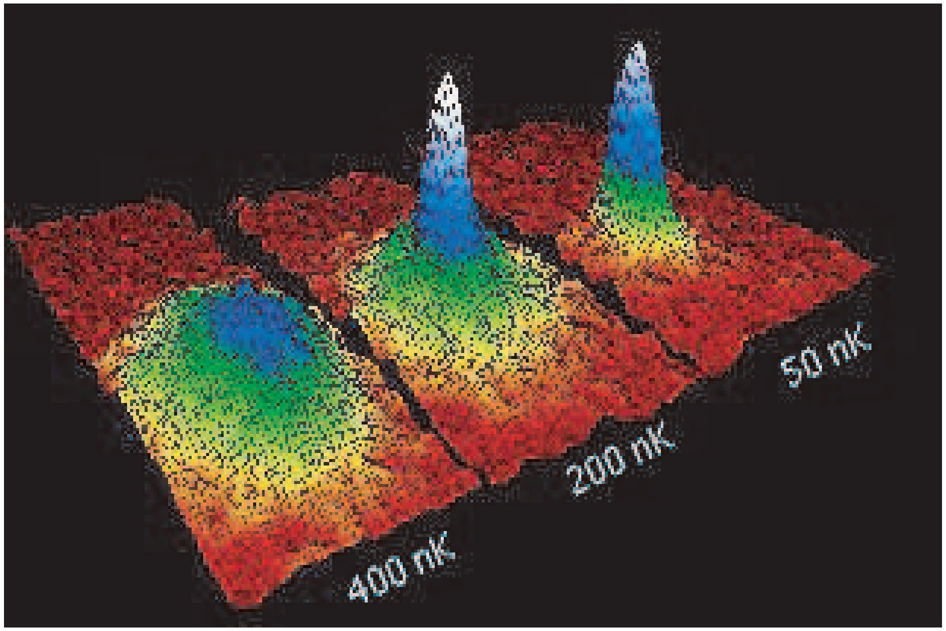


P. L.Kapitza
(1894 $\sim$ 1984)

F. London
(1900 $\sim$ 1954)

– E. A. Cornell & C. E. Wieman (JILA, 1995):
first observed BEC of rubidium (^{87}Rb) atoms at 20 nK, i.e. 0.000 000 02 K.

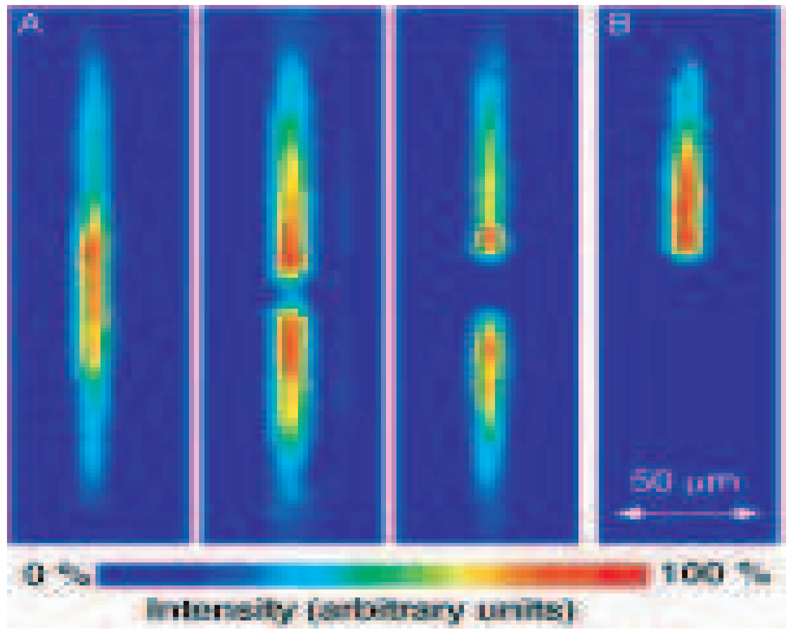




C. E. Wieman & E. A. Cornell BEC at 400, 200, and 50 nK

– W. Ketterle (MIT, 1995):
observed BEC of sodium (^{23}Na) atoms.

W. Ketterle



Two-Component BEC

- Experimental implementation
  - The BEC named Science Magazine's "Molecule of the Year 1995"!
  - Nobel Prize in Physics (2001), E. A. Cornell, C. E. Wieman (JILA), W. Ketterle (MIT):
    for the achievement of BEC in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates.
- Applications of BEC: atom laser, quantum computer, MEMS.
- Mathematical model: nonlinear Schrödinger equation, Gross-Pitaevskii equation (GPE), coupled nonlinear Schrödinger equations, vector Gross-Pitaevskii equations (VGPE).
- Numerical simulation: method, guide for experiment etc.

2 Introduction of Vortices in BEC

- How do vortices happen?
  - Idea 1: rotation (standard way in fluid mechanics).
  - Idea 2: laser beam moving slowly through the condensate (without rotation), by B. Jackson et al. (1998, theoretical); K. Staliunas (1999, experiment).

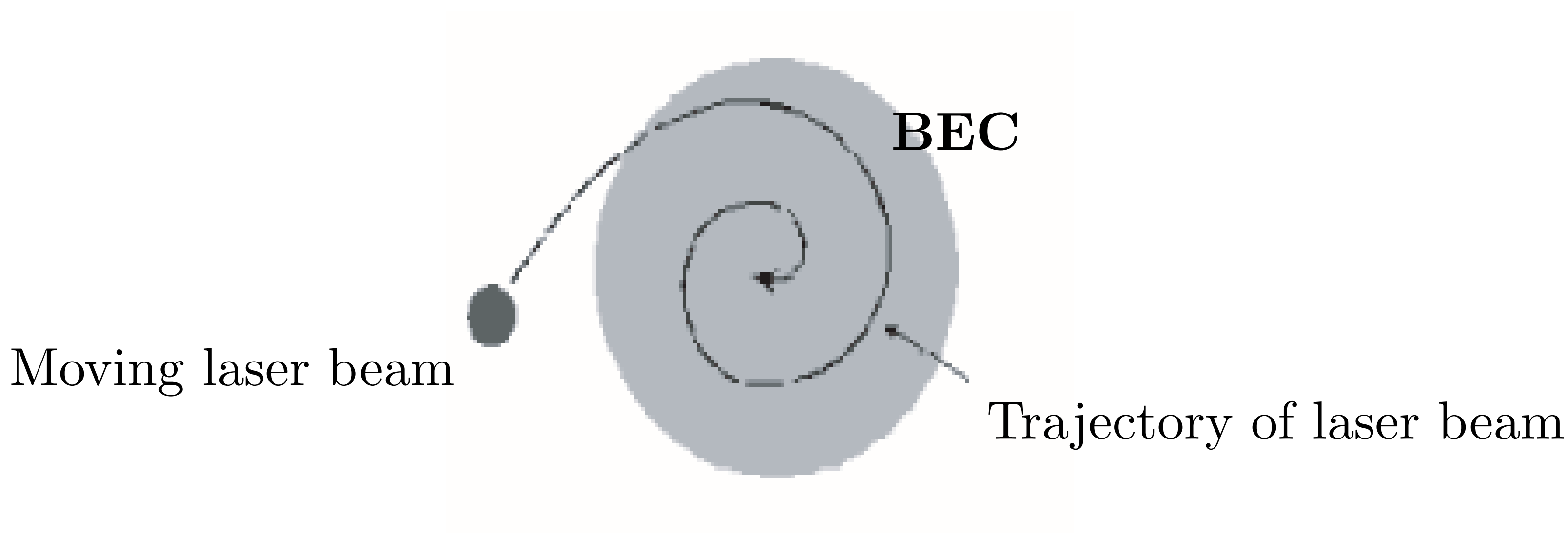
Idea of K. Staliunas

Stirred Bose-Einstein Condensates:

(1) Create one component BEC.

(2) The laser beam enters the condensate spiraling clockwise.

(3) Reaching the center of the condensate it is switched off.



Motivation

- Make a study of vortices's behavior in a two-dimensional trapped Bose-Einstein condensates.
  - PDE: time-dependent Gross-Pitaevskii equation.
  - ODE: the asymptotic motion equations of vortices.

3 Mathematical Model

- Time-dependent Gross-Pitaevskii equation

$$i\,u_t = -\Delta u + V_\epsilon(x,y)\,u + \frac{1}{\epsilon^2}(|u|^2-1)\,u, \quad t>0, \tag{3.1}$$

with the initial data $u|_{t=0} = u_0(x,y)$ and $(x,y)\in\mathbb{R}^2$.

u: a complex-valued order parameter,

$\epsilon > 0$: a small parameter,

$V_\epsilon(x,y) = \alpha_\epsilon\, x^2 + \beta_\epsilon\, y^2$: a harmonic trap potential,

$\alpha_\epsilon,\ \beta_\epsilon > 0$: depending on ϵ.

This time-dependent Gross-Pitaevskii equation was introduced as a phenomenological equation for the order parameter in superfluids.

- Dynamics of vortices in trapped BEC

  Suppose u_0 has d vortex centers at $q_j(0) = (q_{jx}(0), q_{jy}(0))^\top$. Under some specific assumptions on u_0, we obtain the asymptotic motion equations of d vortices q_j's in the following: (T. C. Lin done)

$$\begin{cases} \dot{q}_{jx} = -\sum\limits_{\substack{k=1 \\ k \neq j}}^{d} n_k \frac{q_{jy}-q_{ky}}{|q_j-q_k|^2} - \omega_1 \, q_{jy} \, , \\ \dot{q}_{jy} = \sum\limits_{\substack{k=1 \\ k \neq j}}^{d} n_k \frac{q_{jx}-q_{kx}}{|q_j-q_k|^2} + \omega_2 \, q_{jx} \, , \end{cases} \tag{3.2}$$

  where $q_j = q_j(t) = (q_{jx}(t), q_{jy}(t))$, n_j: winding numbers and $\omega_1 = -\omega + 2\beta_0$, $\omega_2 = -\omega + 2\alpha_0$. For the stability of the vortex structure in u, we require $n_j \in \{\pm 1\}, j = 1, \ldots, d$.

Results

We consider $d = 3$, then obtain

(1) the bounded and collisionless trajectories of three vortices form chaotic, quasi 2- or quasi 3-periodic orbits,

(2) a new phenomenon of $1:1$-topological synchronization is observed in the chaotic trajectories of vortices with the same sign of winding numbers..

Let d be the number of vortices.

- Aref 1979: The Kirchhoff equations (3.3) form an integrable system if $d \leq 3$. (Theoretical Proof)
- Aref 1983: The Kirchhoff equations may have chaotic motions in a bounded region if $d > 3$. (Numerical Simulation)

$$\begin{cases} \dot{q}_{jx} = -\sum\limits_{\substack{k=1 \\ k \neq j}}^{d} n_k \frac{q_{jy}-q_{ky}}{|q_j-q_k|^2}, \\ \dot{q}_{jy} = \sum\limits_{\substack{k=1 \\ k \neq j}}^{d} n_k \frac{q_{jx}-q_{kx}}{|q_j-q_k|^2}. \end{cases} \tag{3.3}$$

4 Numerical Study of Three Vortices

- Characterize the motion:
  - Lyapunov exponent,
  - Poincaré map,
  - Spectrums of waveforms.
- Indicator for ratio topologically synchronized chaotic regimes (Afraimovich et al. (1999, 2000), [1, 2]):
  - the Poincaré dimension for Poincaré recurrences.

Case $(n_1, n_2, n_3) = (1, -1, -1)$

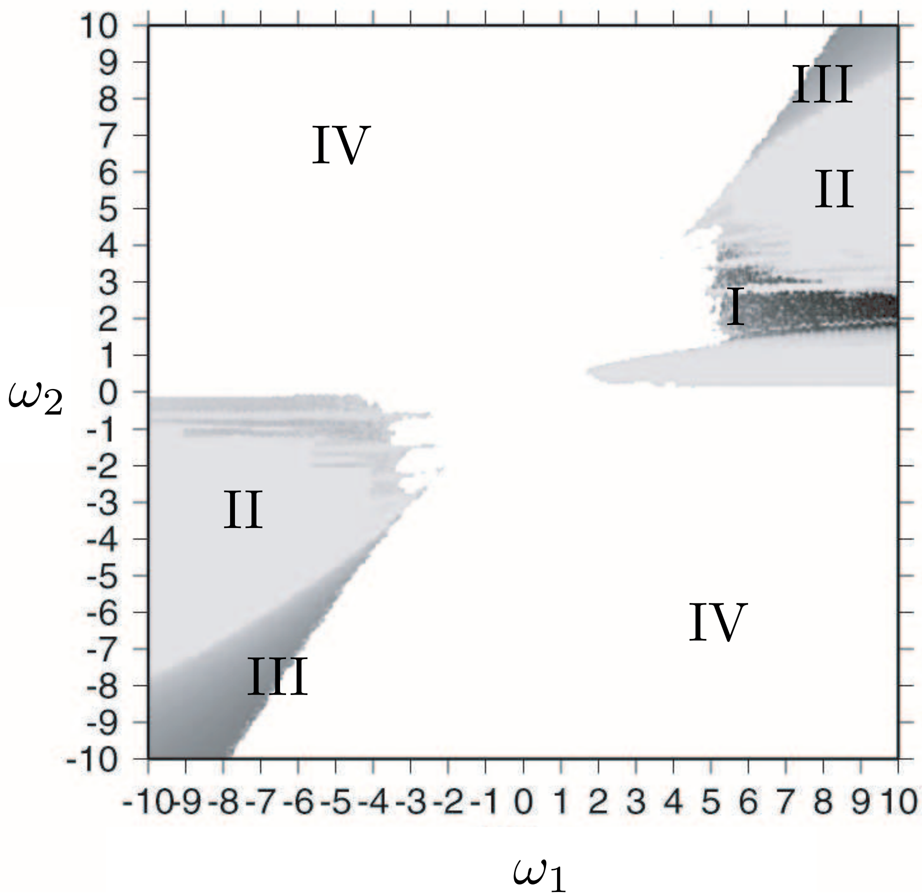


Figure 4.1: The first Lyapunov exponent.

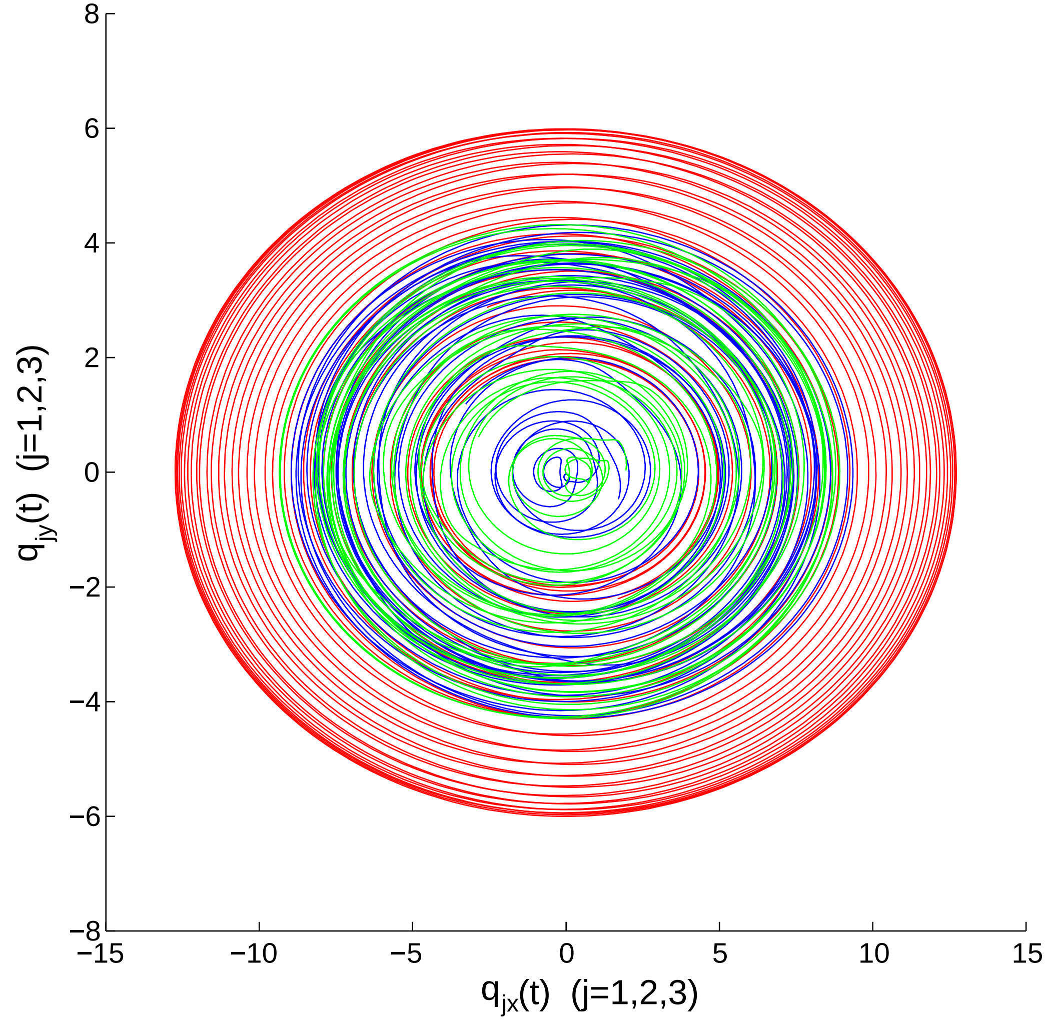


Figure 4.2: Chaotic trajectories: $(\omega_1, \omega_2) = (9.88, 2.24), t = 25,050 \sim 25,100$ sec.

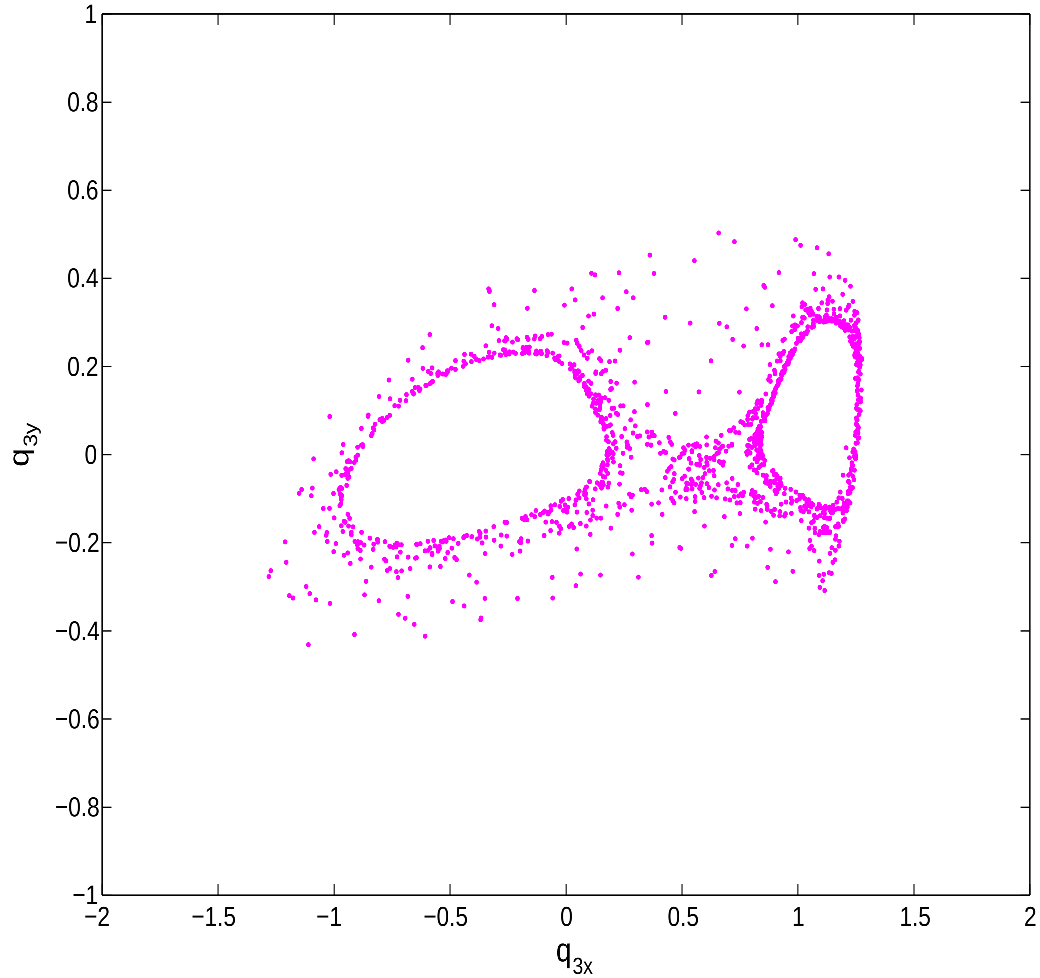


Figure 4.3: Chaotic second-Poincaré maps (4 dim.), $t = 1,393 \sim 5,000,000$ sec.

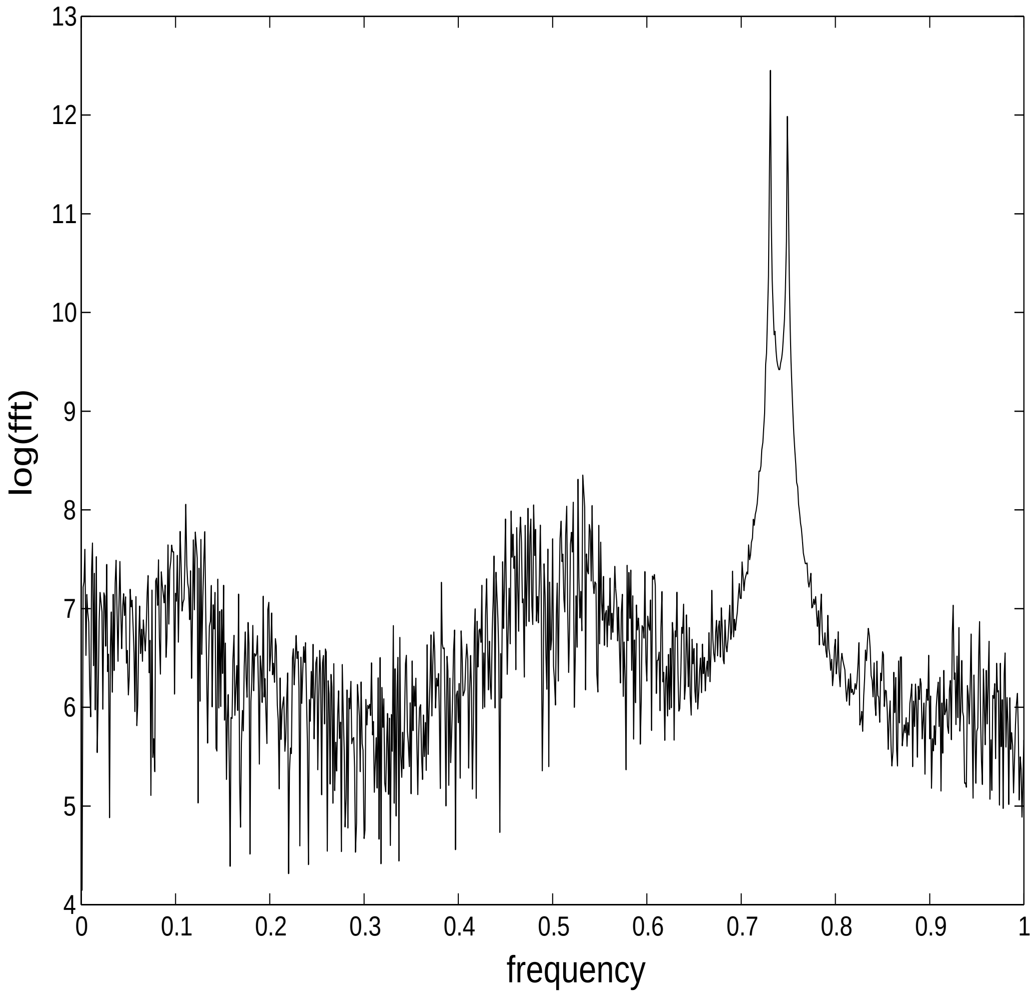


Figure 4.4: Chaotic spectrum of waveforms, $t = 1,050 \sim 25,500$ sec.

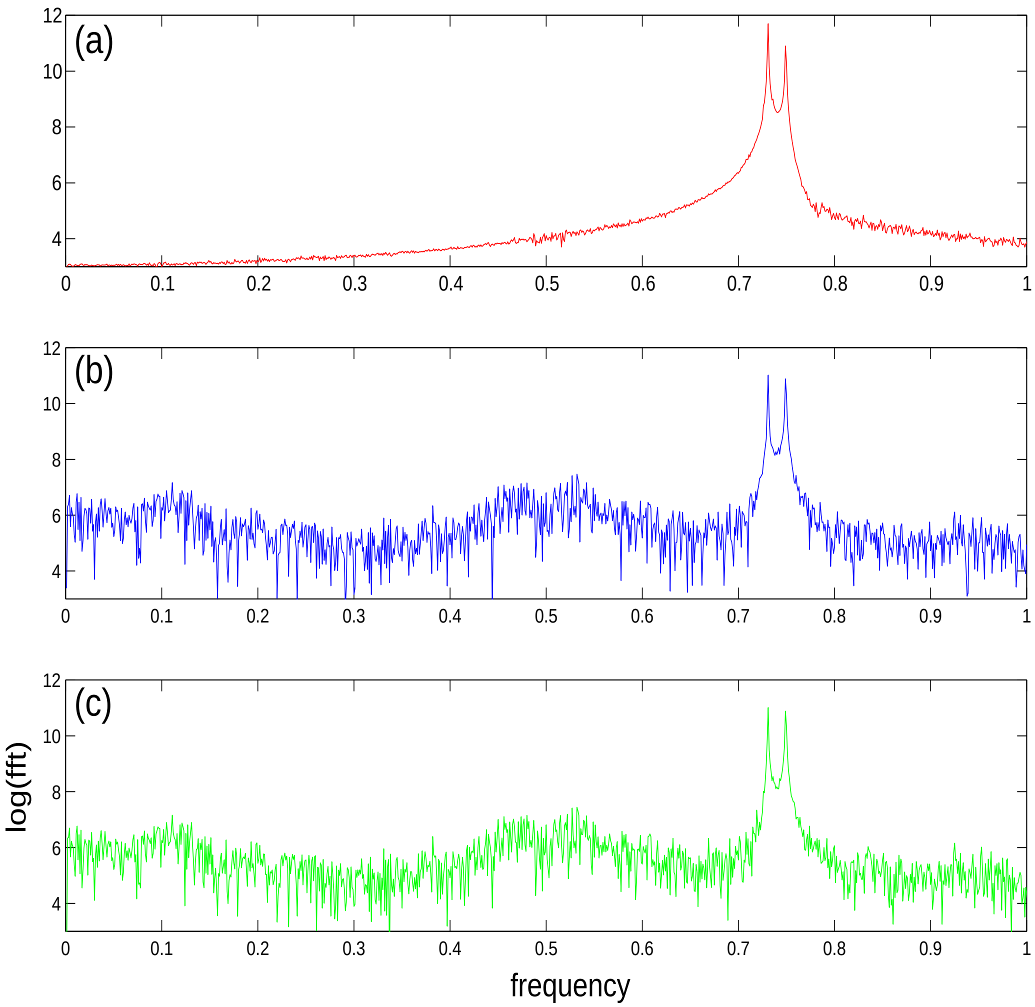


Figure 4.5: Chaotic individual spectrum, $t = 1,000 \sim 25,500$ sec.

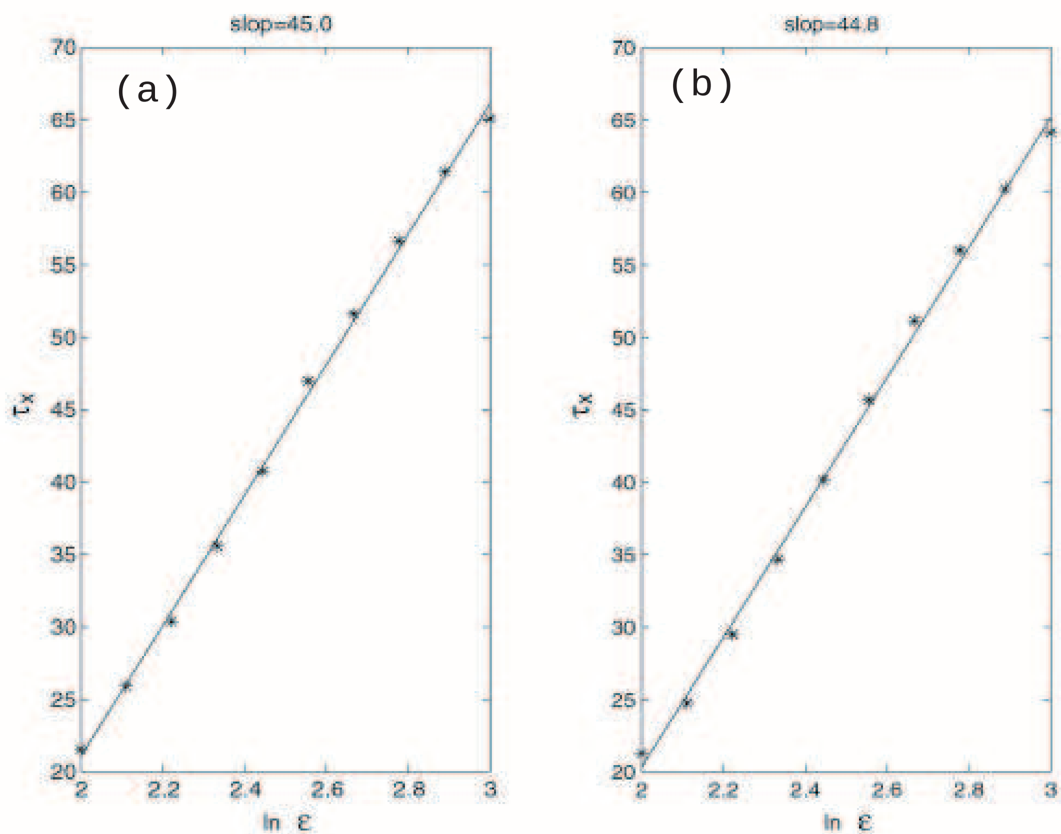


Figure 4.6: The ratio of slopes $= 45.0/44.8 \approx 1.006$

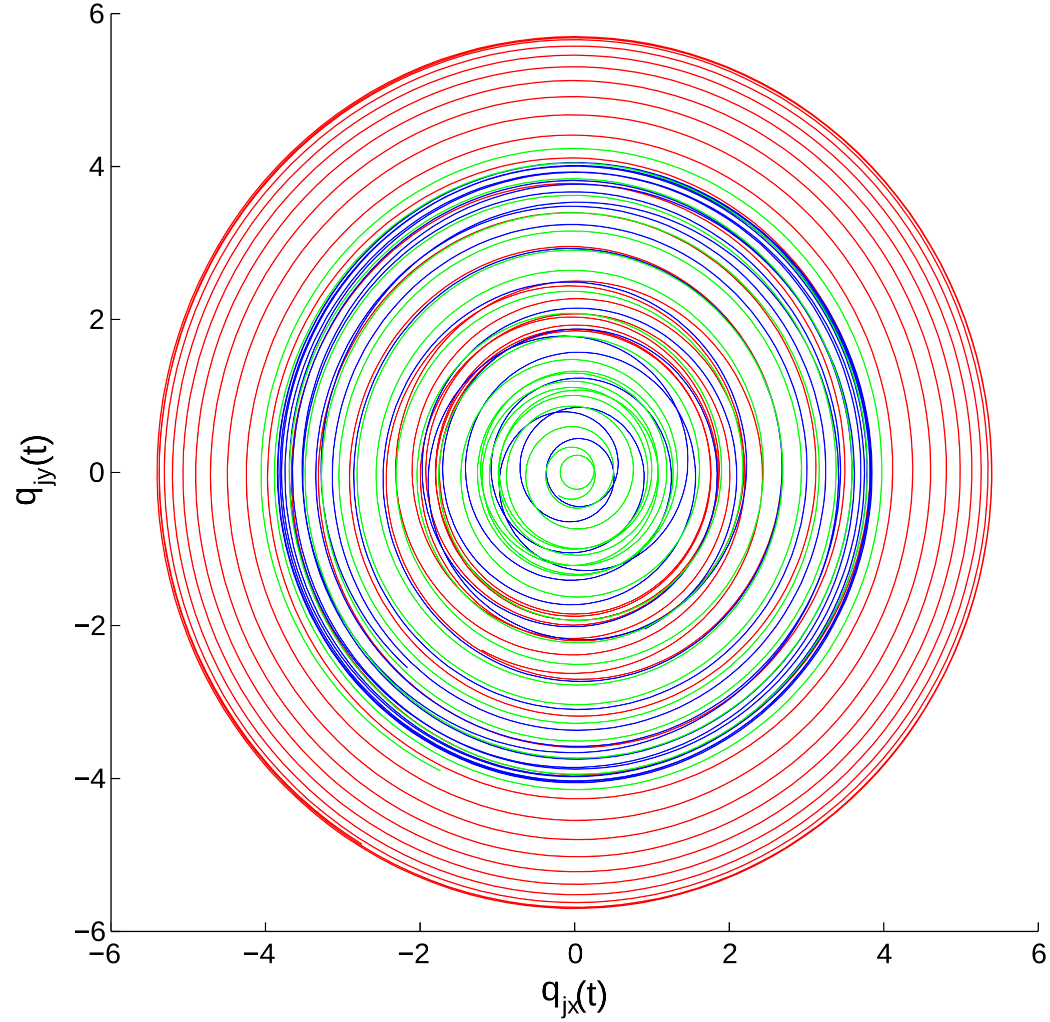


Figure 4.7: Quasi 3-periodic trajectories: $(\omega_1, \omega_2) = (9, 10), t = 25,080 \sim 25,095$ sec.

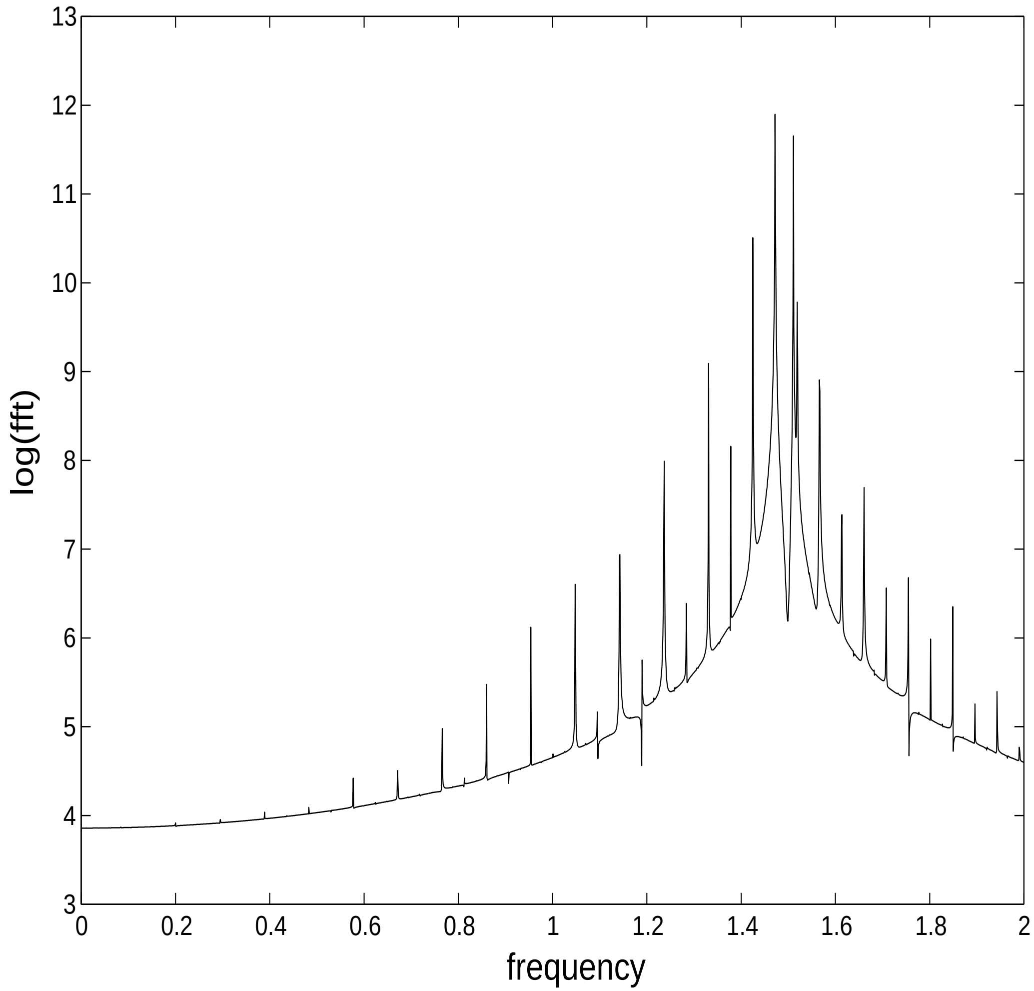


Figure 4.8: Quasi 3-periodic spectrum, $t = 1,000 \sim 25,500$ sec.

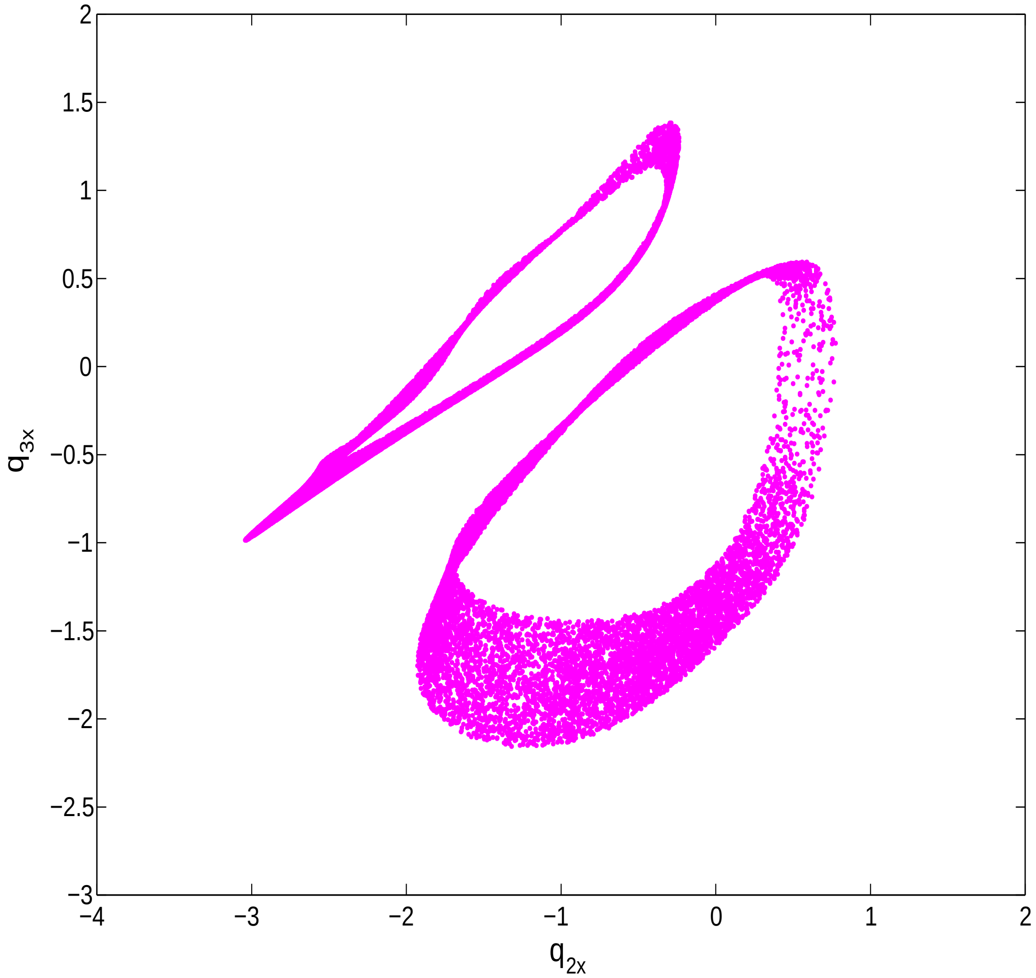


Figure 4.9: Quasi 3-periodic second-order Poincaré maps (4 dim.), $t = 41,179 \sim 4,000,000$ sec.

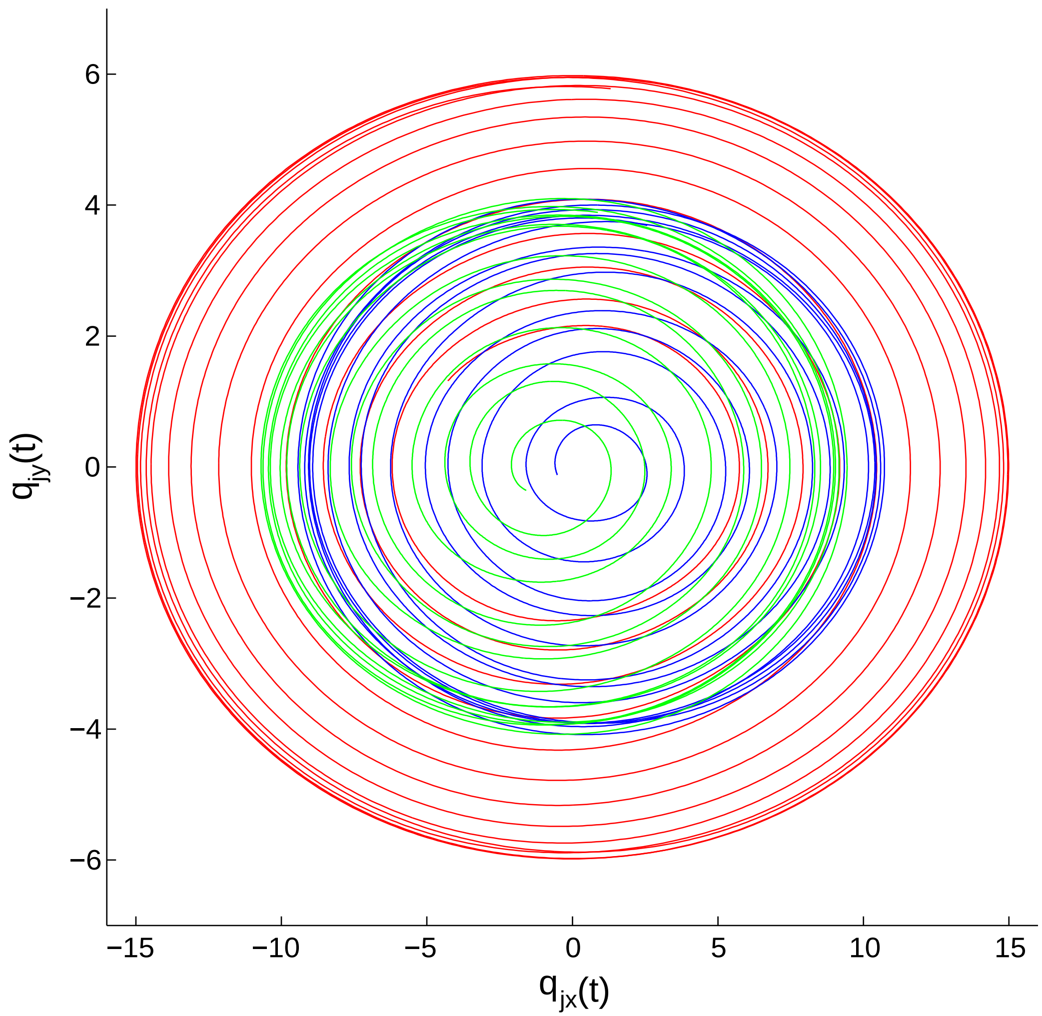


Figure 4.10: Quasi 2-periodic trajectories: $(\omega_1, \omega_2) = (6, 1), t = 25,155 \sim 25,190$ sec.

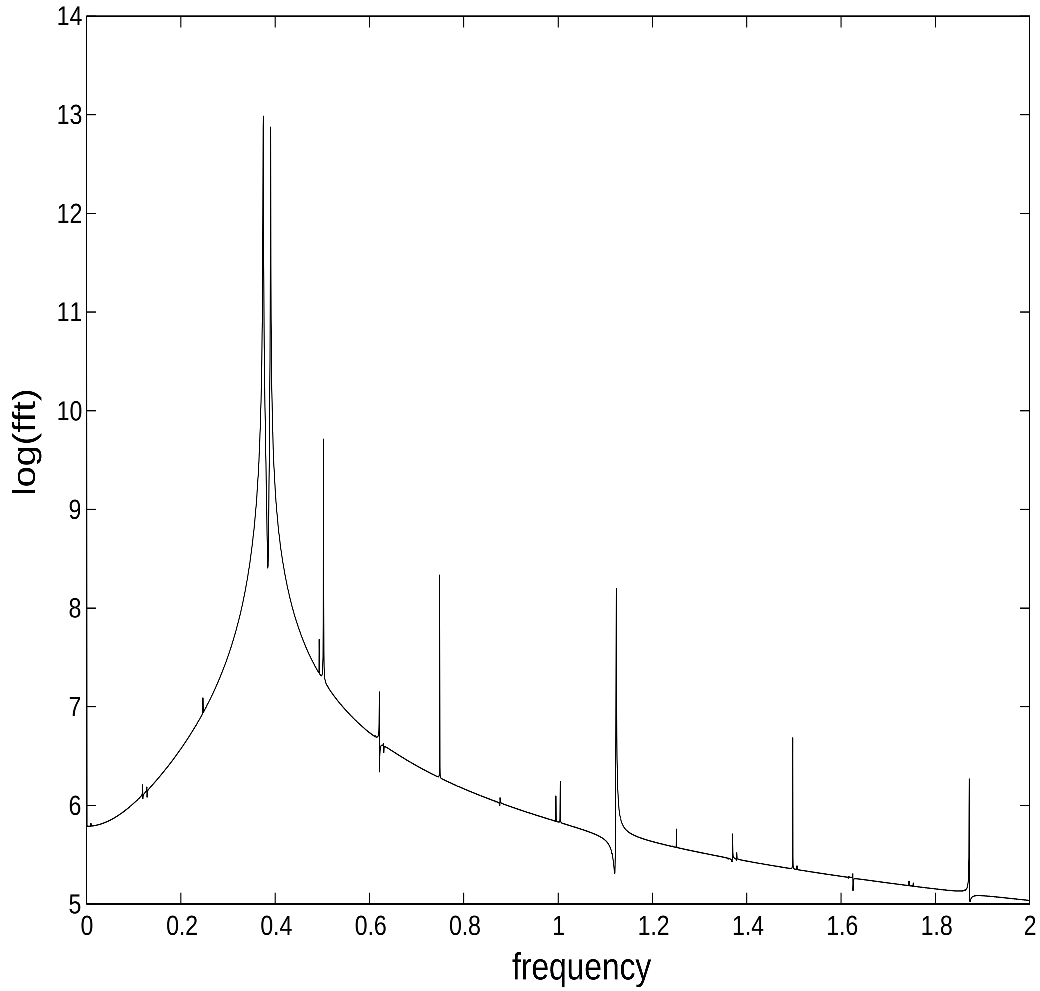


Figure 4.11: Quasi 2-periodic spectrum, $t = 2,000 \sim 25,500$ sec.

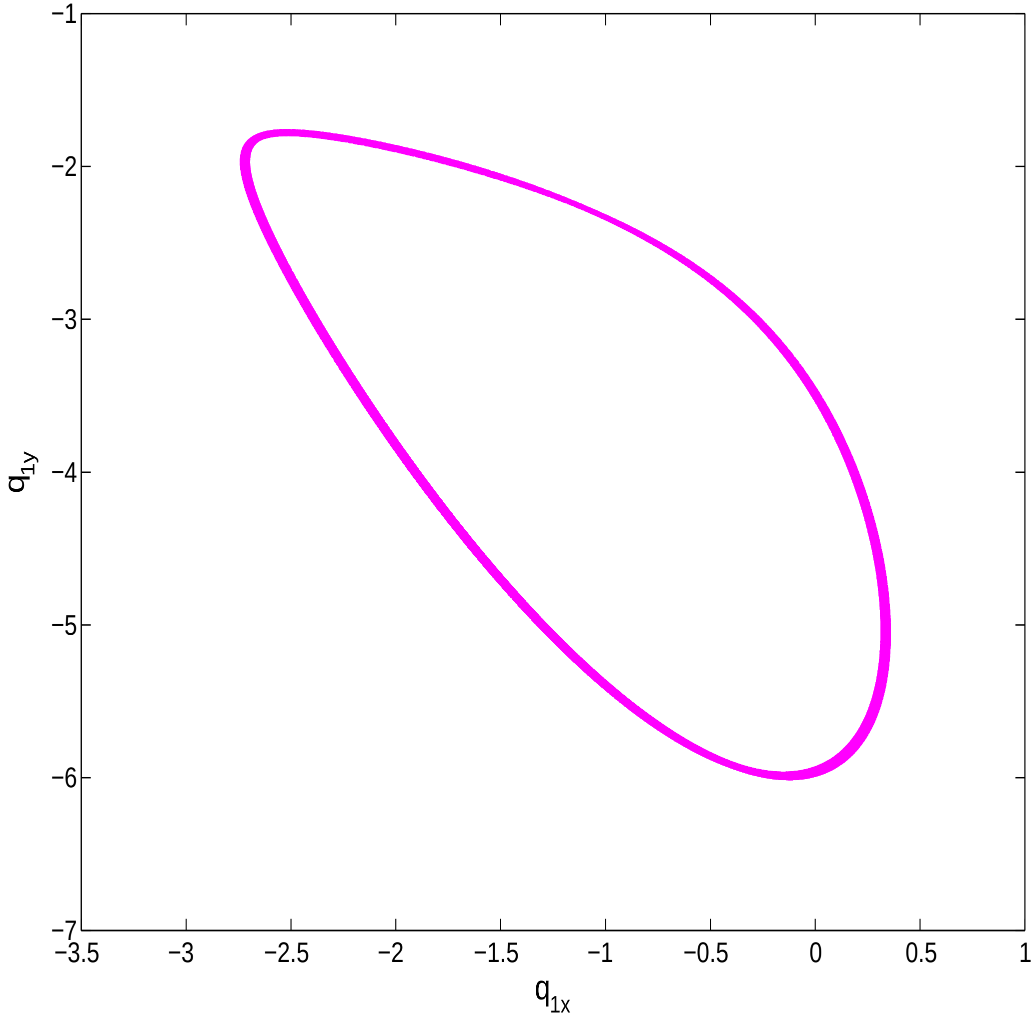


Figure 4.12: Quasi 2-periodic first-order Poincaré maps (5 dim.), $t = 37,193 \sim 1,000,000$ sec.

Case $(n_1, n_2, n_3) = (1, 1, 1)$

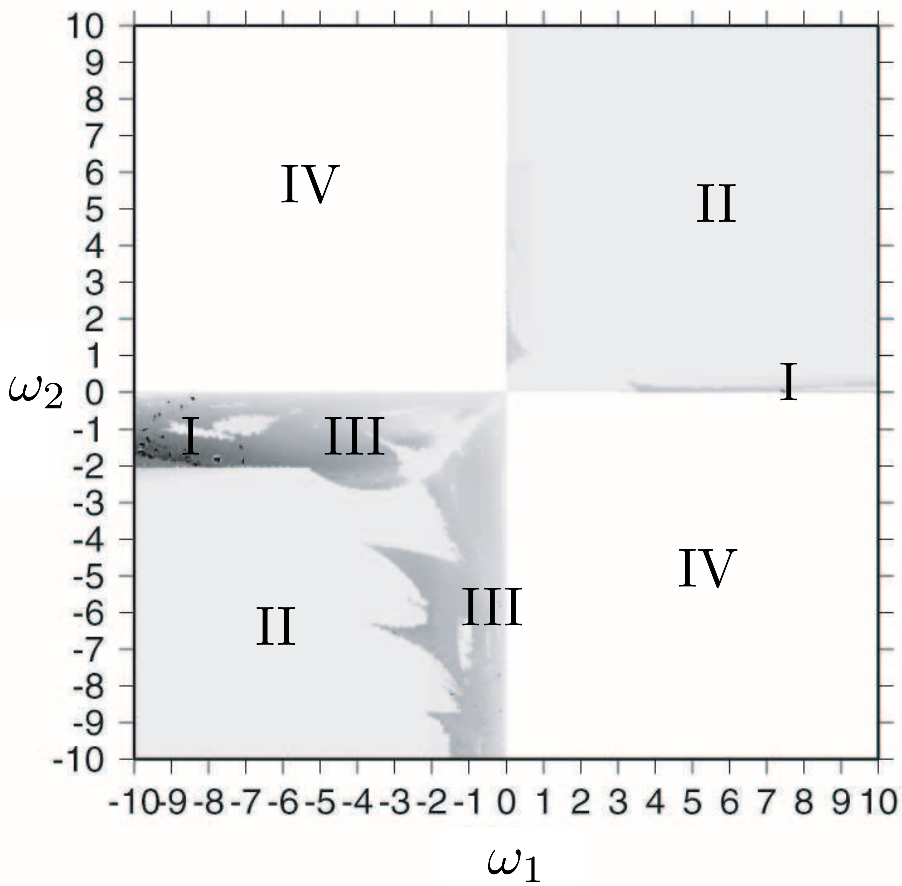


Figure 4.13: The first Lyapunov exponent.

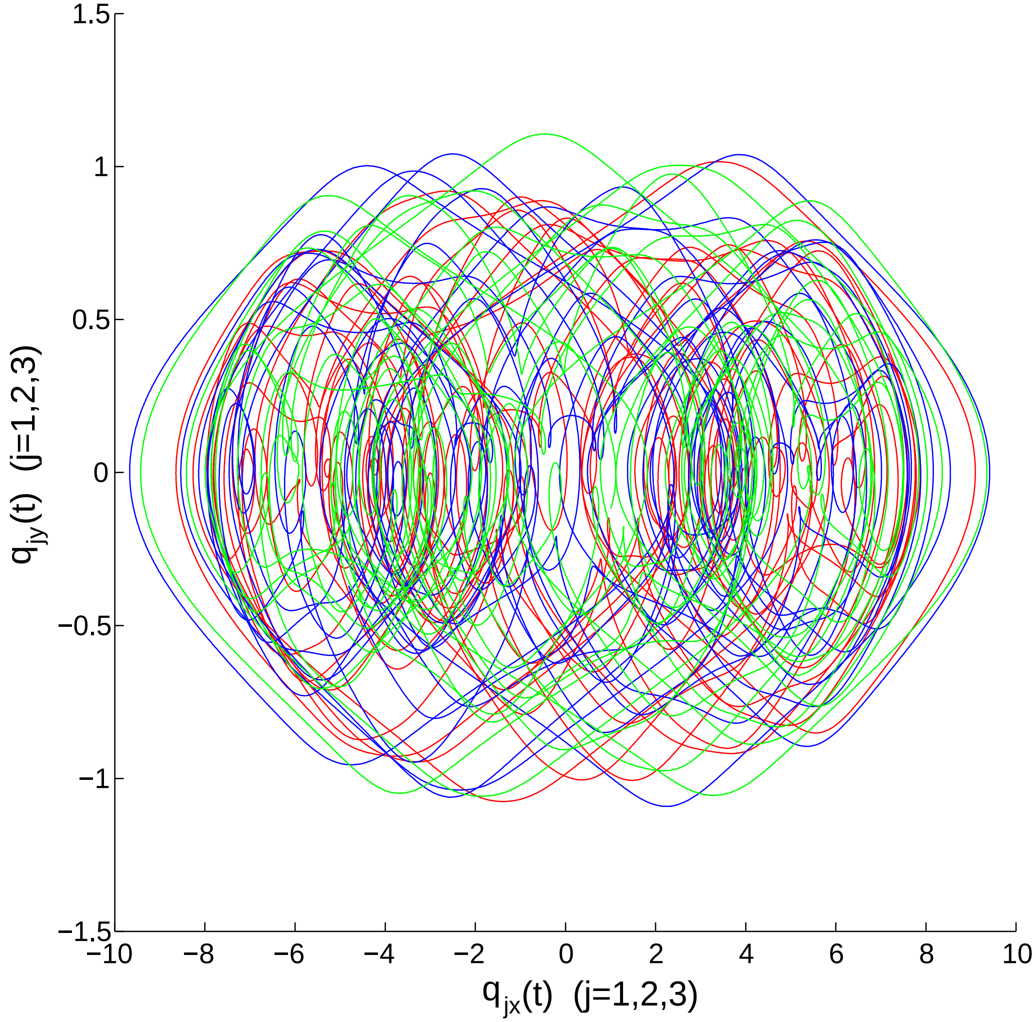


Figure 4.14: Chaotic trajectories: $(\omega_1, \omega_2) = (7.4, 0.025), t = 25,050 \sim 25,070$ sec.

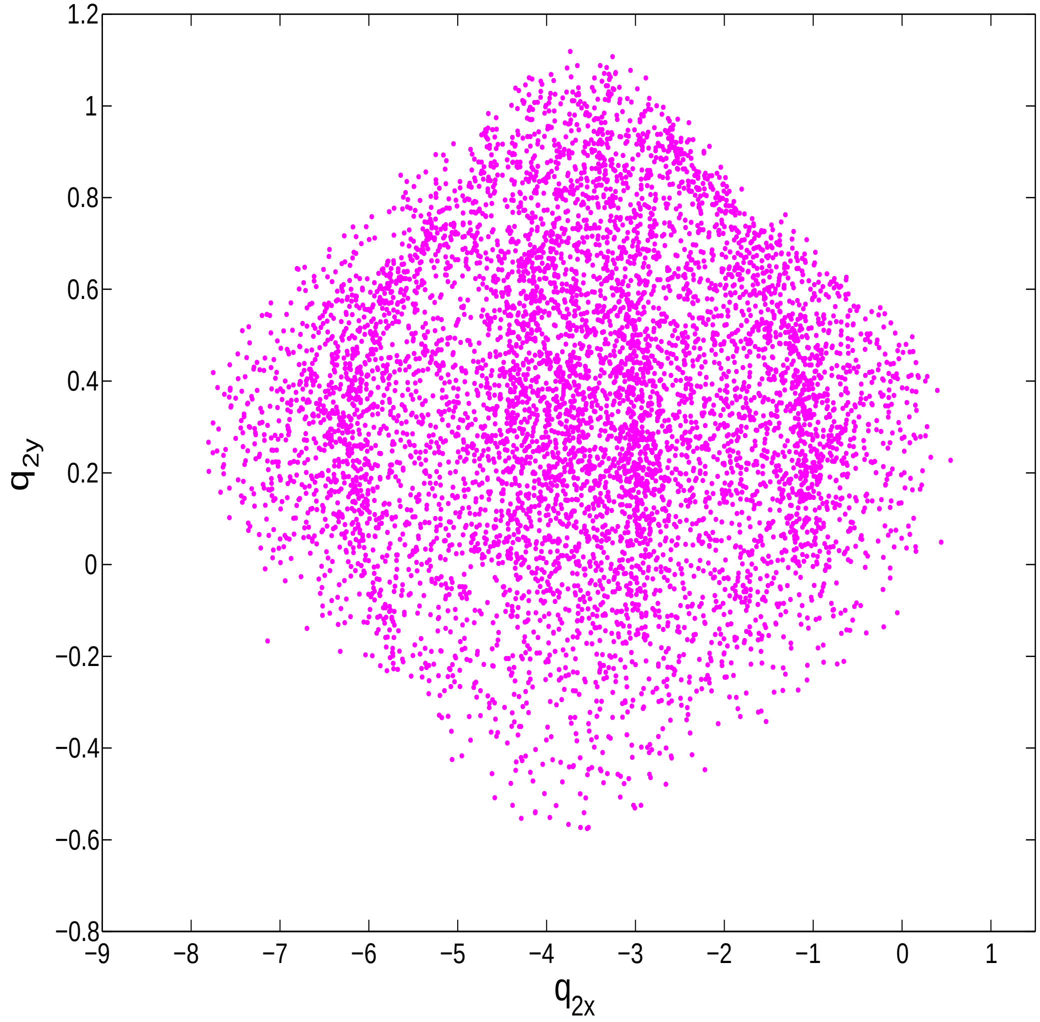


Figure 4.15: Chaotic first-order Poincaré maps (5 dim.), $t = 1,000 \sim 100,000$ sec.

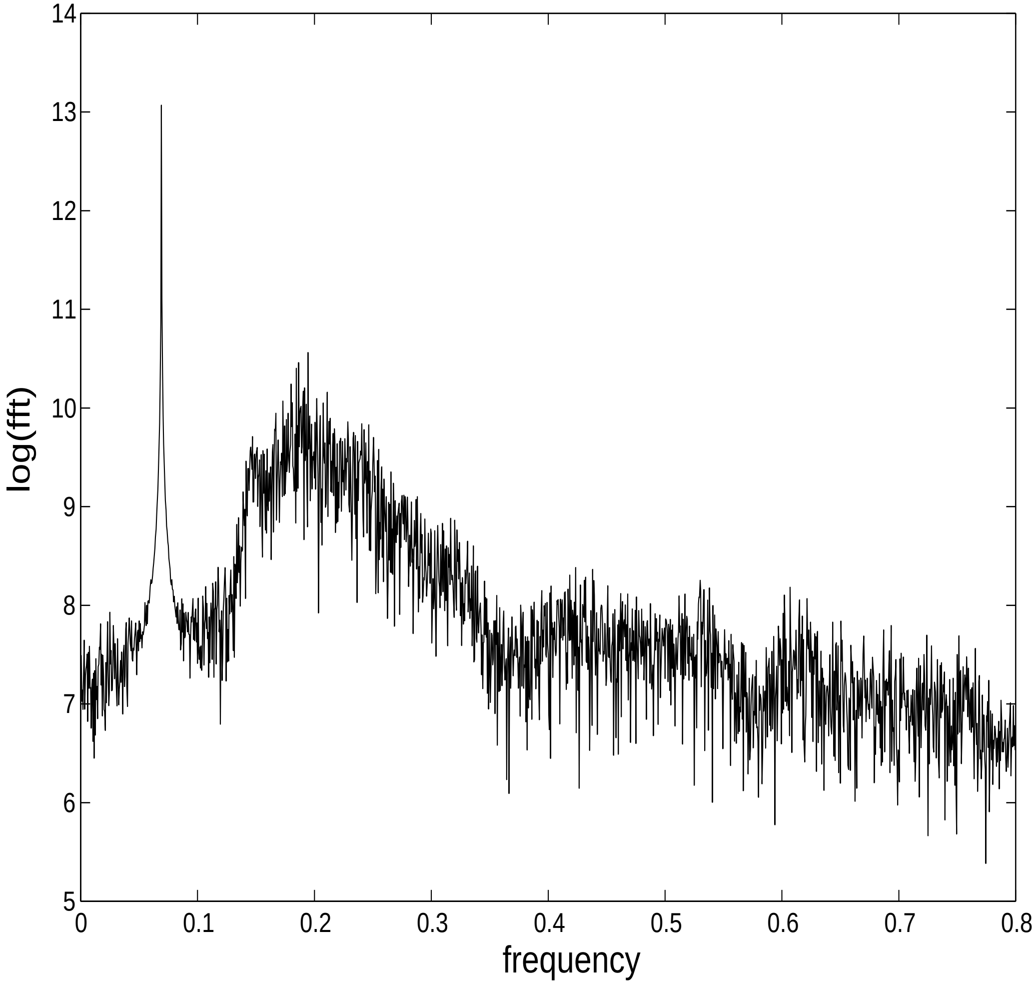


Figure 4.16: Chaotic spectrum, $t = 2,000 \sim 25,500$ sec.

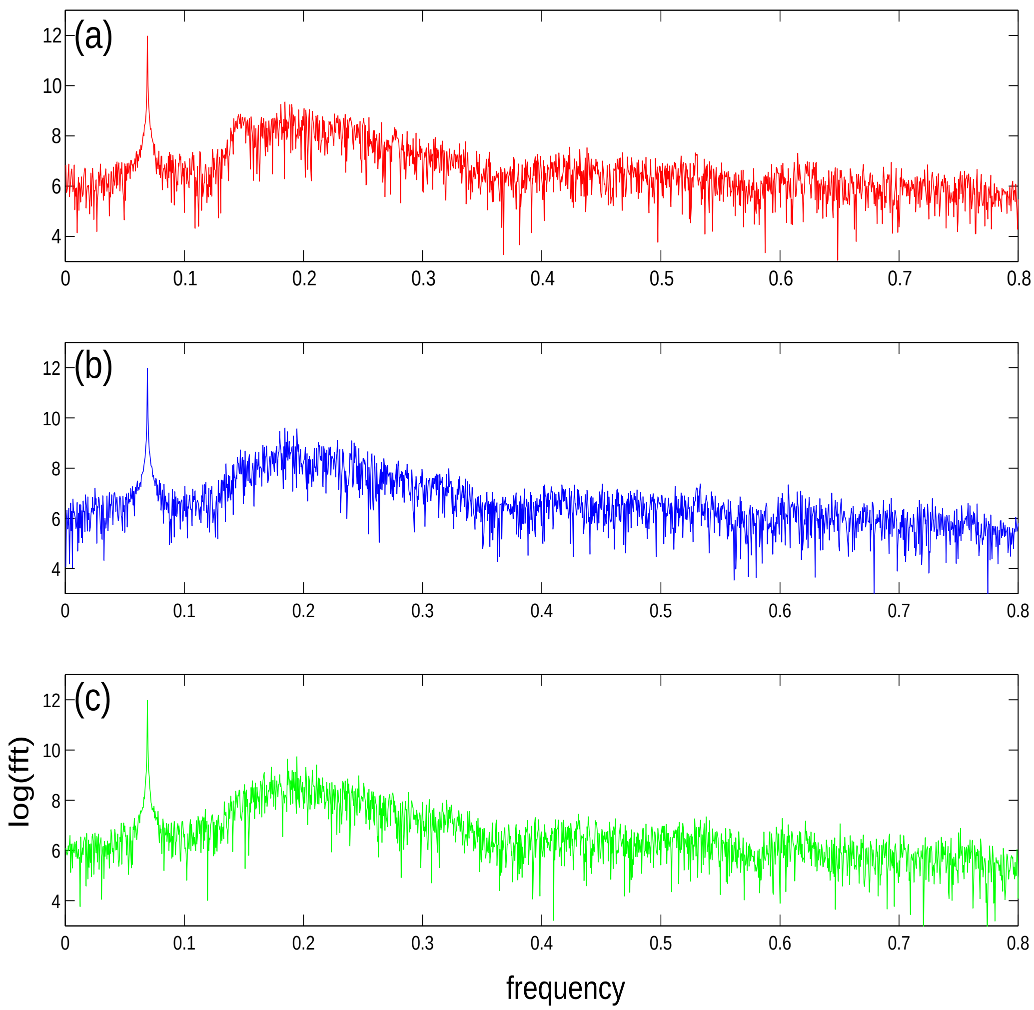


Figure 4.17: Chaotic individual spectrum, $t = 2,000 \sim 25,500$ sec.

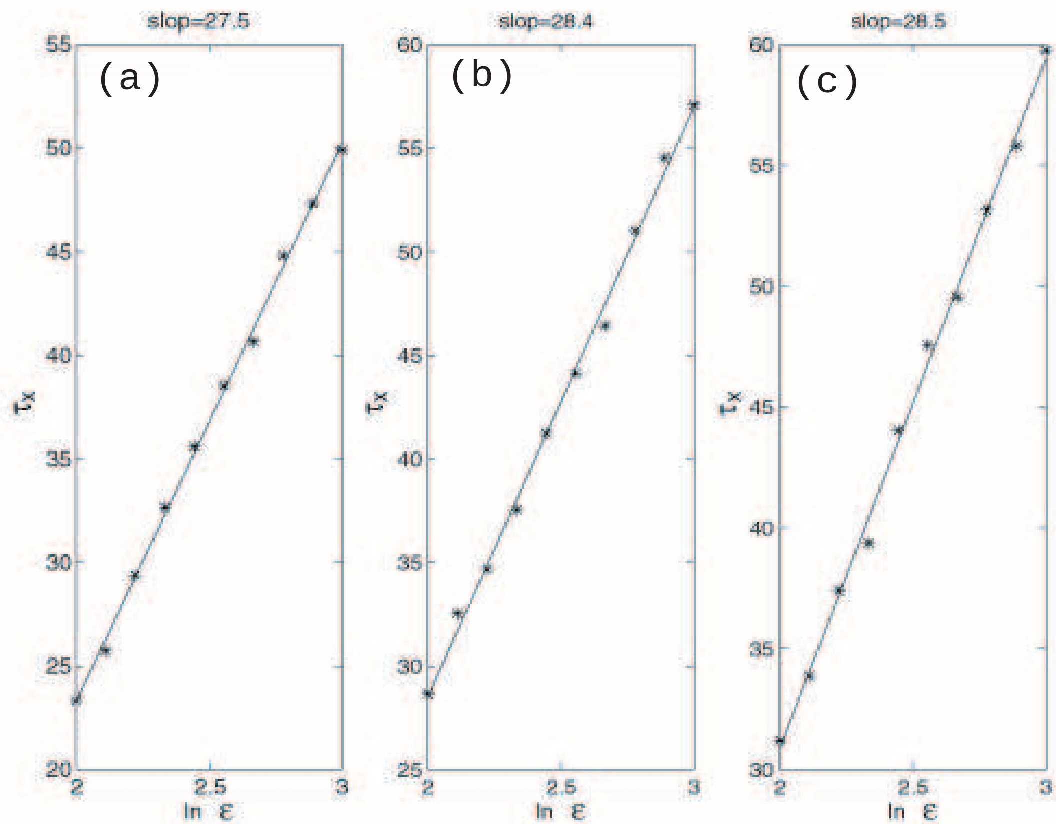


Figure 4.18: The ratio of slopes = 27.5 : 28.4 : 28.5 ≈ 0.97 : 0.996 : 1.