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Computations for Dynamical Systems

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Outline

- Dynamical System
- Computational Dynamical System
- Chaos
- Examine Chaos

Dynamical System

動態系統是要研究運動方程的解。

Computational Dynamical System

（1）離散動態系統:

$$x_{n+1} = f(x_n)$$

（2）連續動態系統:(ode45,... in MatLab)

$$y'(t) = f(t, x)$$

Computational Dynamical System

（1）離散動態系統的解：

發散(infinity) 、固定點、週期解、
擬週期(quasi-periodic)、?

（2）連續動態系統的解：

發散(infinity) 、平衡點、週期解、
極限環(limit cycle) 、
擬週期(quasi-periodic)、
?

Computational Dynamical System

- 0-D: equilibrium points
  (radial, spiral, saddle)
- 1-D: limit cycles (closed loops)
- 2-D: 2-toruses (quasiperiodic surfaces)
- *N*-D: *N*-toruses (hypersurfaces)
- Non-integer D: strange attractors (fractal)
  (attractor dimension < system dimension)

Chaos

混沌理論認為在混沌系統中，初始條件十分敏感，其微小的變化，在經過不斷放大，對未來狀態會造成極其巨大的差別。

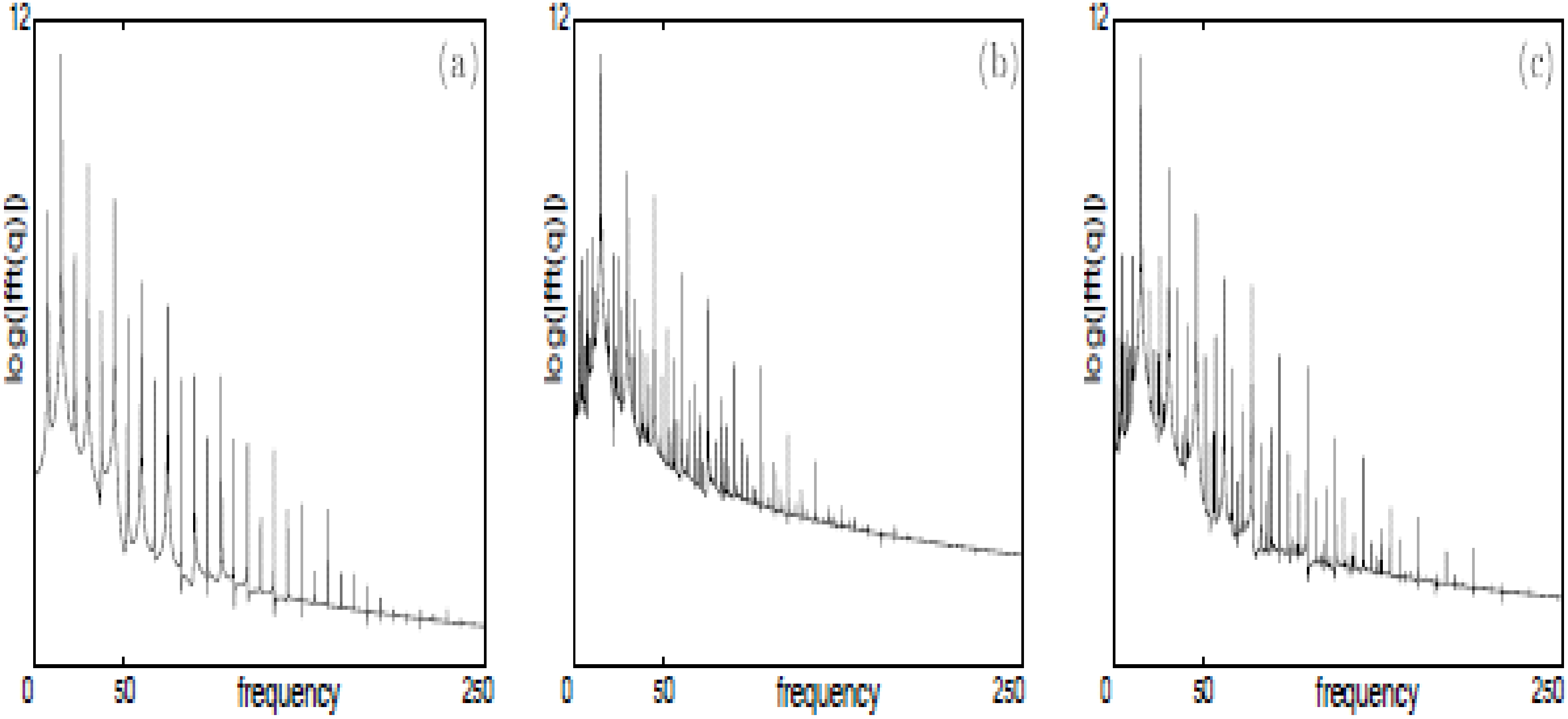
Devanvey's chaos

- 敏感性(sensitivity)：
  對初始條件非常敏感，差之毫釐失之千里。
- 傳遞性(transitivity)：
  可到處遍歷。
- 週期解稠密性(density)：
  存在任意週期。

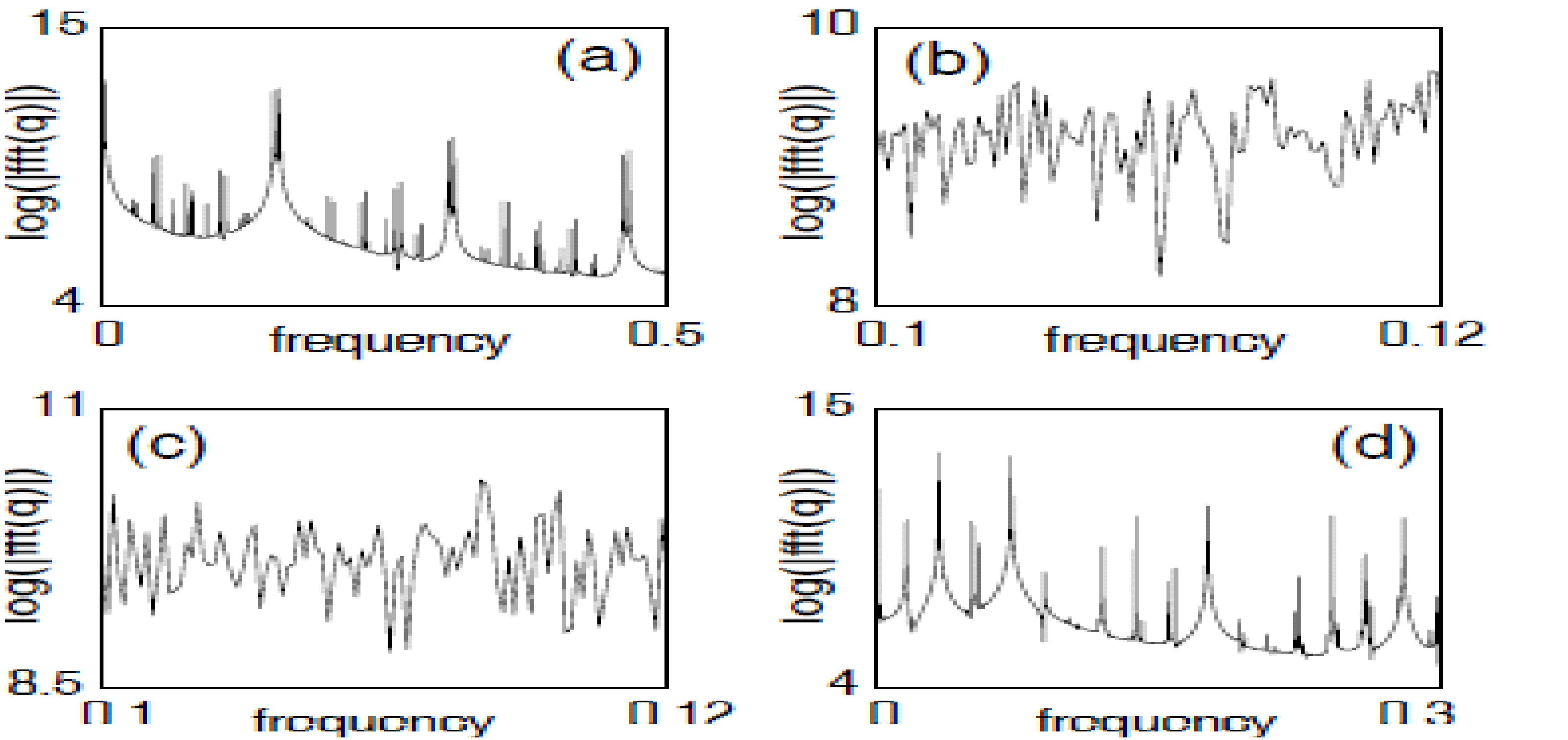
Examine Chaos

- bifurcation diagram
  (period doubling bif.: logistic map,
  intermittence: tent map)
- Feigenbaum constant
  δ= 4.669201609102990671853203820…
- spectrum analysis (fft)

FFT



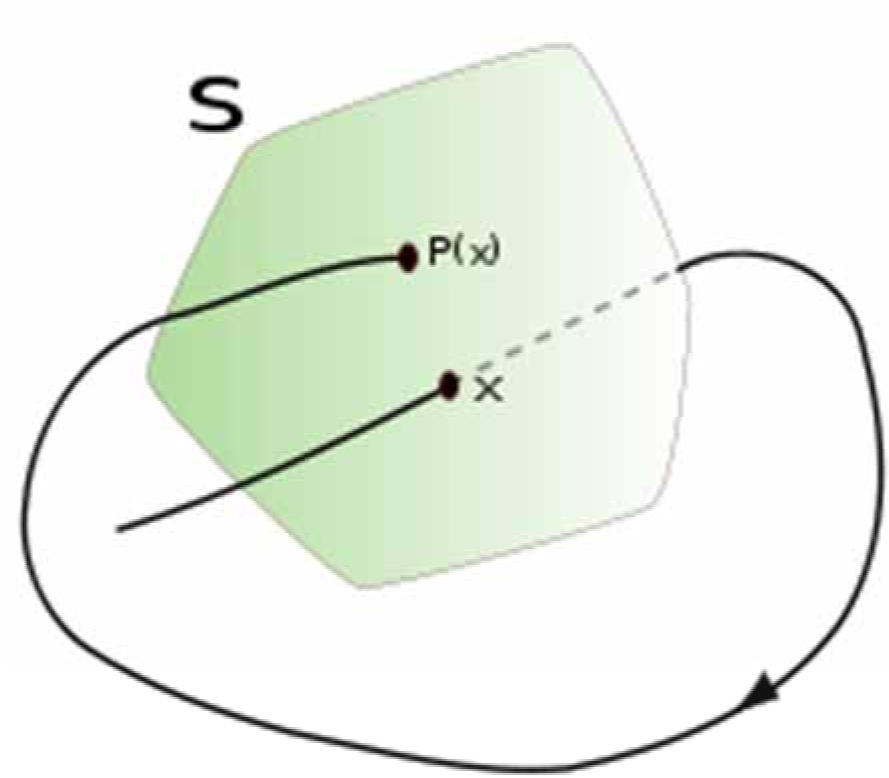
FFT


(a)
15
4
0
frequency
0.5
log(|fft(q)|)
(b)
10
8
0.1
frequency
0.12
log(|fft(q)|)
(c)
11
8.5
0.1
frequency
0.12
log(|fft(q)|)
(d)
15
4
0
frequency
0.3
log(|fft(q)|)

Examine Chaos

- Poincaré map
  (conti. D.S.)

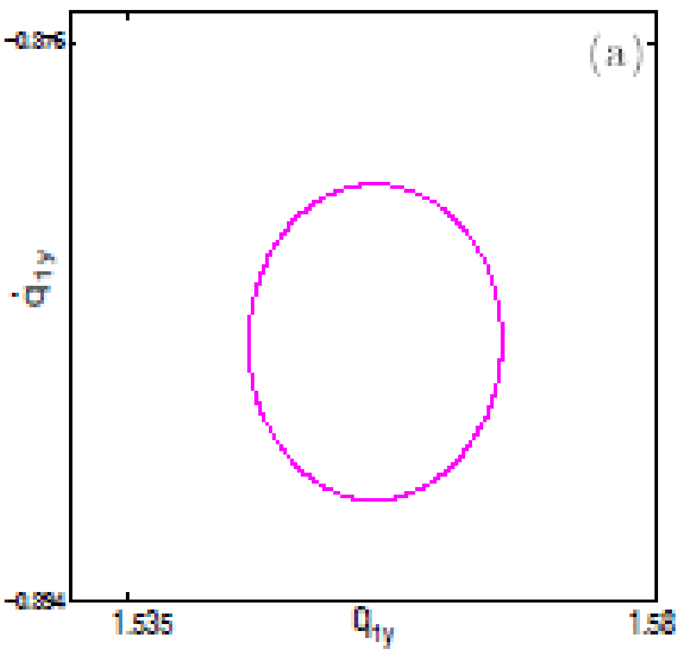
Poincaré map

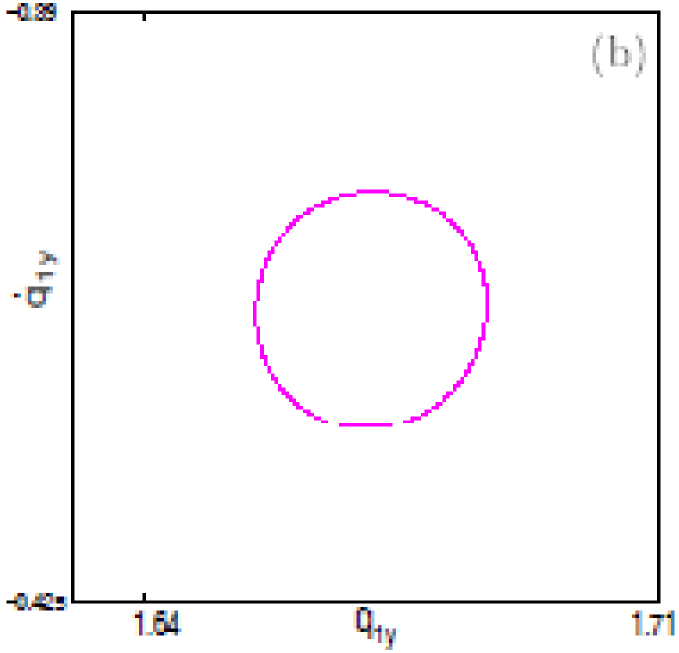


Poincaré map

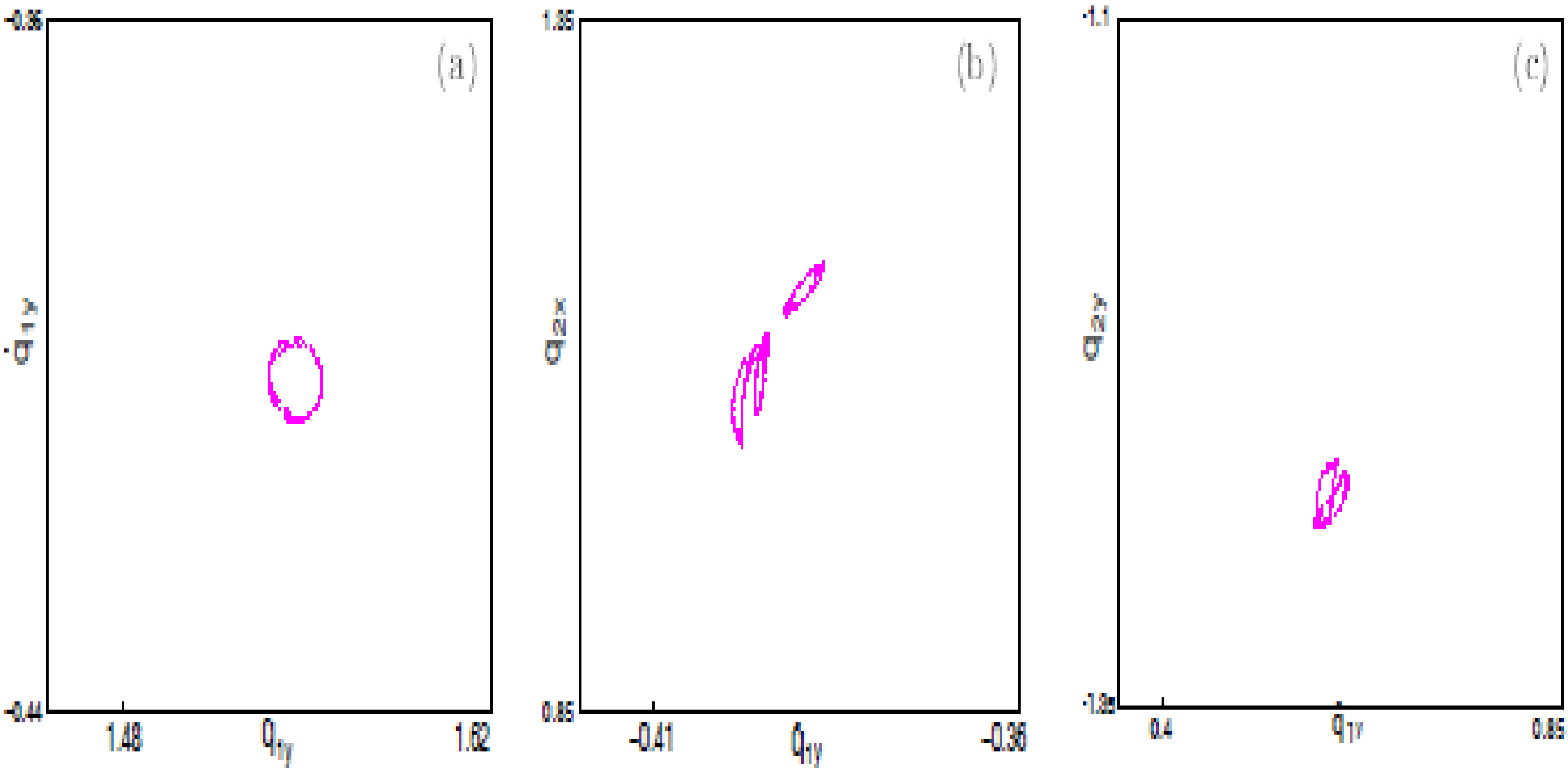
- 發散(infinity)、平衡點
- 週期解
- 極限環(limit cycle)
- 擬週期(quasi-periodic)
- chaos

Poincaré map

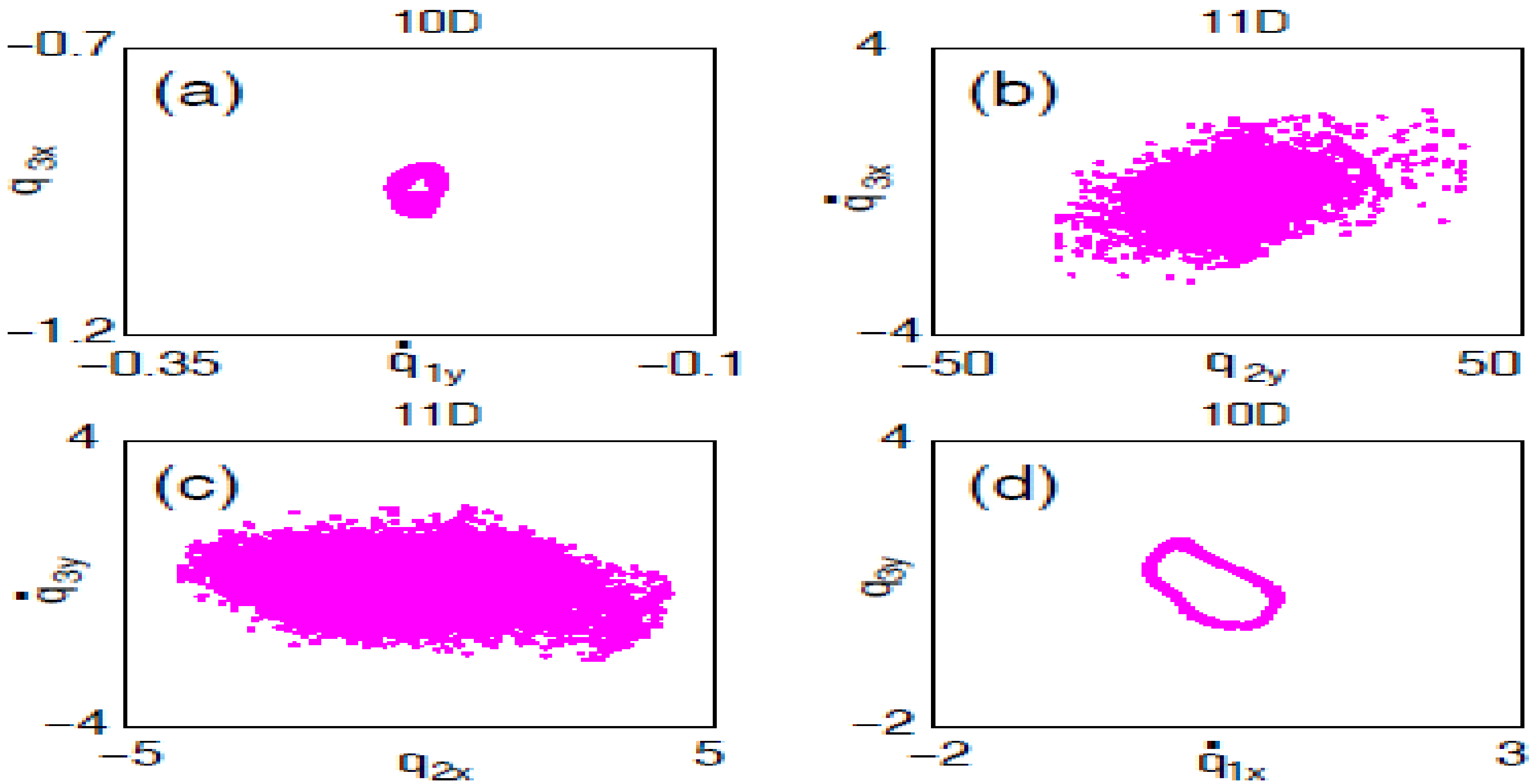




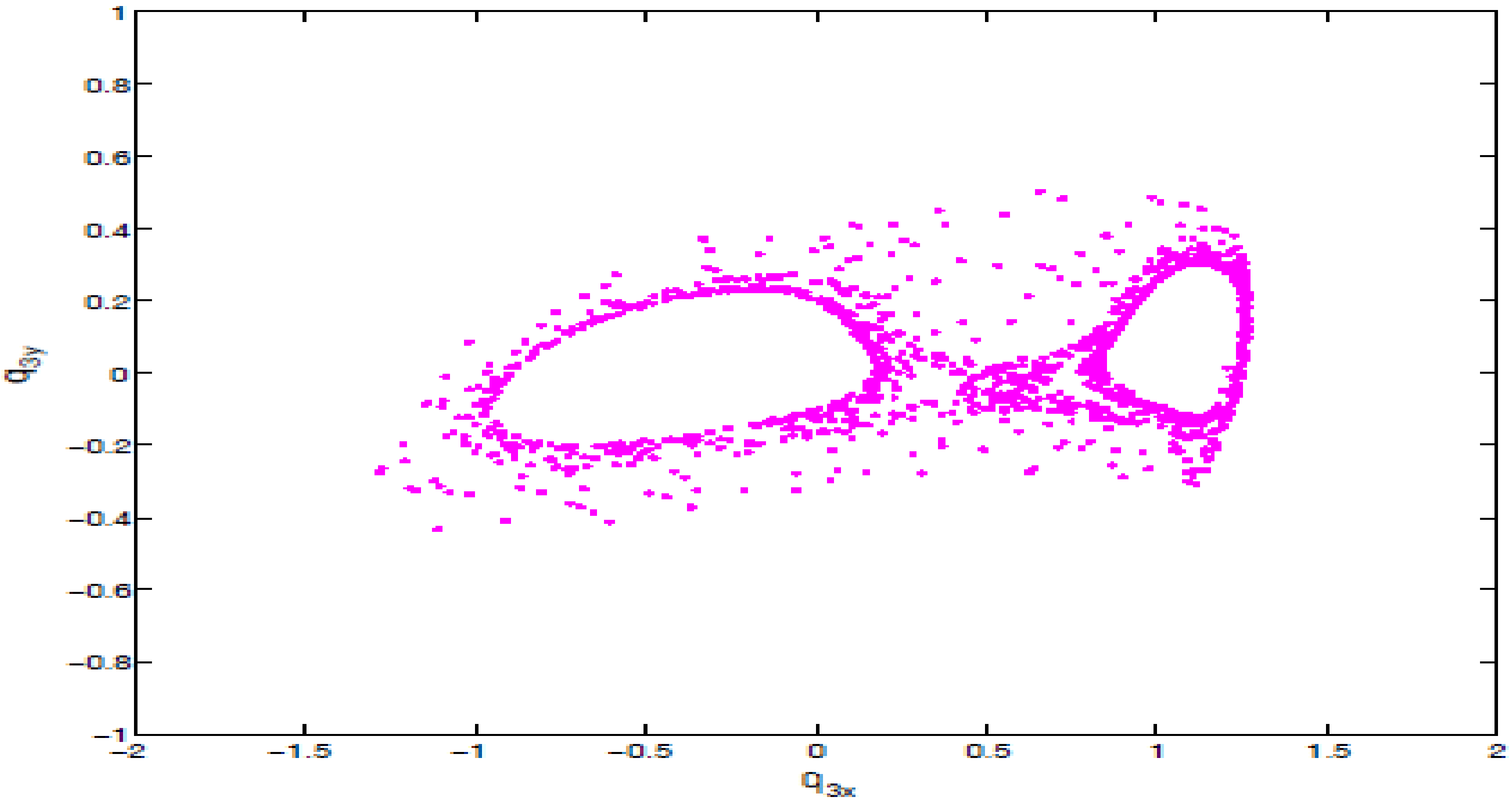
Poincaré map



Poincaré map



Poincaré map



Examine Chaos

- Lyapunov exponent (Lyapunov characteristic exponent)
- Poincaré recurrence
- homoclinic orbit (snapback repellor)

Lyapunov exponent

$$|\delta \mathbf{Z}(t)| \approx e^{\lambda t} |\delta \mathbf{Z}_0|$$

where λ is the Lyapunov exponent.

The maximal Lyapunov exponent can be defined as follows:

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}.$$

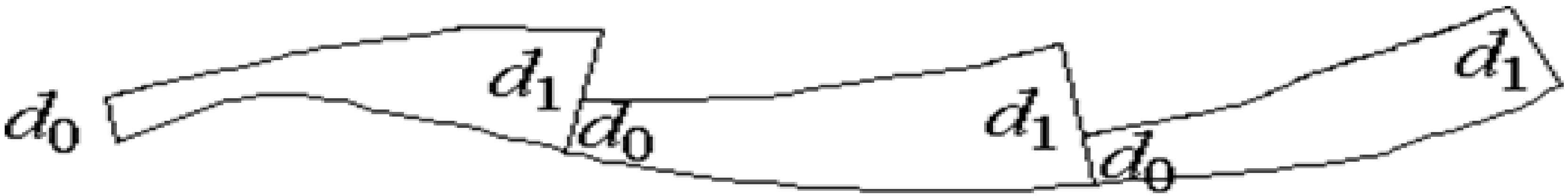
Lyapunov exponent

- [Definition] global Lyapunov exponent
- [Computation] local Lyapunov exponent (average the phase-space volume expansion along trajectory)

Local Lyapunov exponent

Numerical Calculation of Largest Lyapunov Exponent

1. Start with any initial condition in the basin of attraction
2. Iterate until the orbit is on the attractor
3. Select (almost any) nearby point (separated by d_0)
4. Advance both orbits one iteration and calculate new separation d_1
5. Evaluate log $|d_1/d_0|$ in any convenient base
6. Readjust one orbit so its separation is d_0 **in same direction as $\boldsymbol{d_1}$**
7. Repeat steps 4-6 many times and calculate average of step 5
8. The largest Lyapunov exponent is $\boldsymbol{\lambda_1 = \langle \log |d_1/d_0| \rangle}$
9. If map approximates an ODE, then $\boldsymbol{\lambda_1 = \langle \log |d_1/d_0| \rangle / h}$
10. A positive value of λ_1 indicates chaos



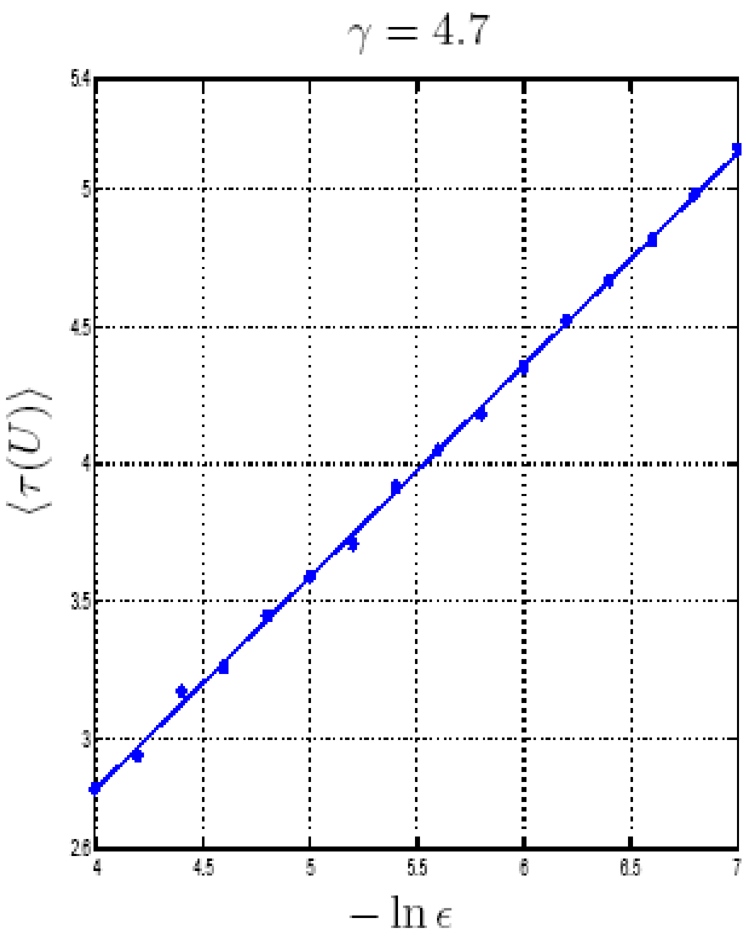
Local Lyapunov exponent

- Choose $T > 0$, an initial state x_0, and an integer K.
  Let $x^{(0)} := x_0$ and $x^{(k)} := \phi_T(x^{(k-1)})$ for $k = 1, \ldots, K$.
  Then

$$\Phi_{KT}(x_0) = \Phi_T(x^{(K-1)}) \cdots \Phi_T(x^{(0)}).$$

- If T is not too big, then $\Phi_T(x^{(k)})$ can be integrated accurately.
- Difficulty: how to compute the eigenvalues of $\Phi_{KT}(x)$.

Poincaré recurrence



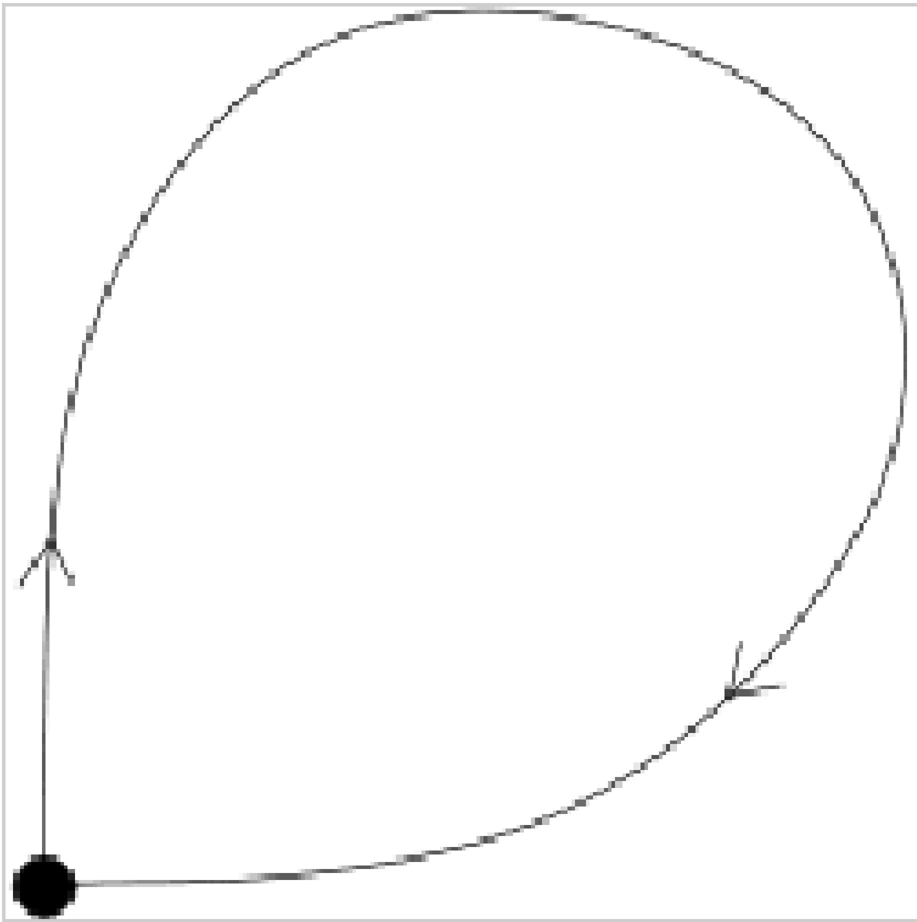
$$\langle \tau(U) \rangle \sim \frac{b}{q_0}(-\ln \epsilon)$$

positive topological entropy

$$\lambda_\mu \geq \left(\lim_{\epsilon \to 0} \frac{\tau(x, U)}{-\ln \epsilon} \right)^{-1},$$

where λ_μ is the Lyapunov exponent

Homoclinic orbit



MLM: modified logistic map

For $\gamma > 0$, Modified Logistic Map (MLM)

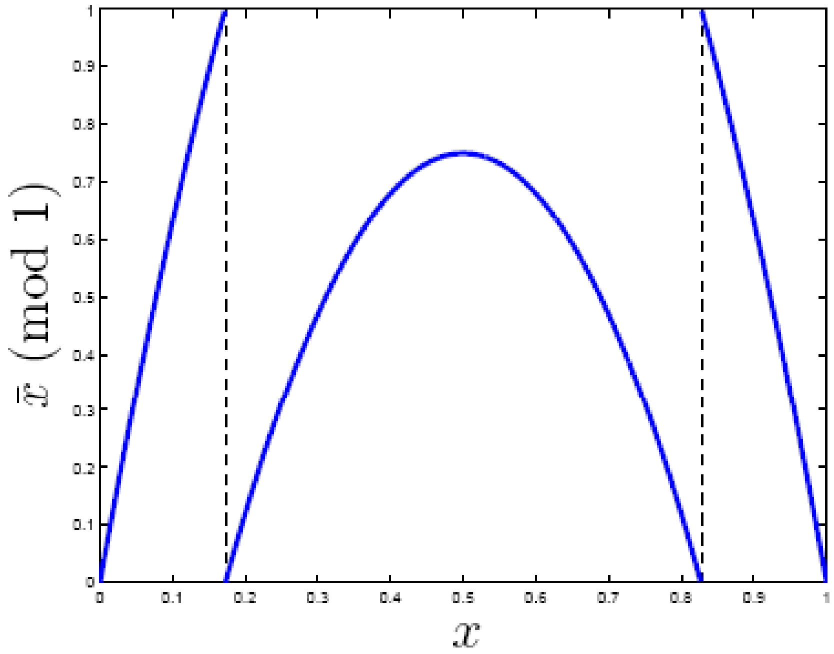
$$f_\gamma(x) : [0,1] \to [0,1],$$

$$f_\gamma(x) = \begin{cases} \gamma x(1-x) \pmod 1, & \text{if } x \in [0,1] \setminus (\eta_1, \eta_2), \\ \frac{\gamma x(1-x) \pmod 1}{\frac{\gamma}{4} \pmod 1}, & \text{if } x \in (\eta_1, \eta_2), \end{cases}$$

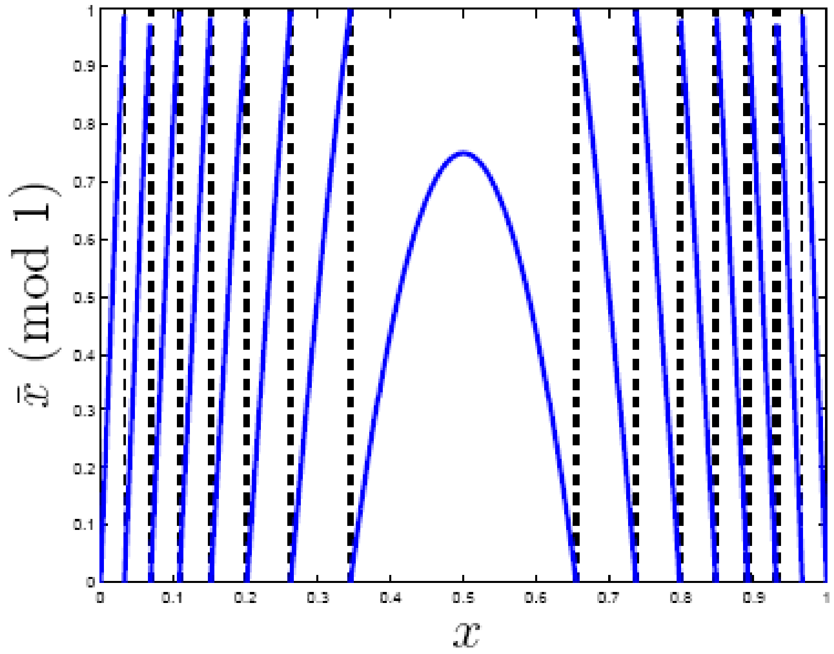
where $\eta_1 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{[\frac{\gamma}{4}]}{\gamma}}$, $\eta_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{[\frac{\gamma}{4}]}{\gamma}}$,

$[z]$: the greatest integer less than or equal to z.

Logistic map

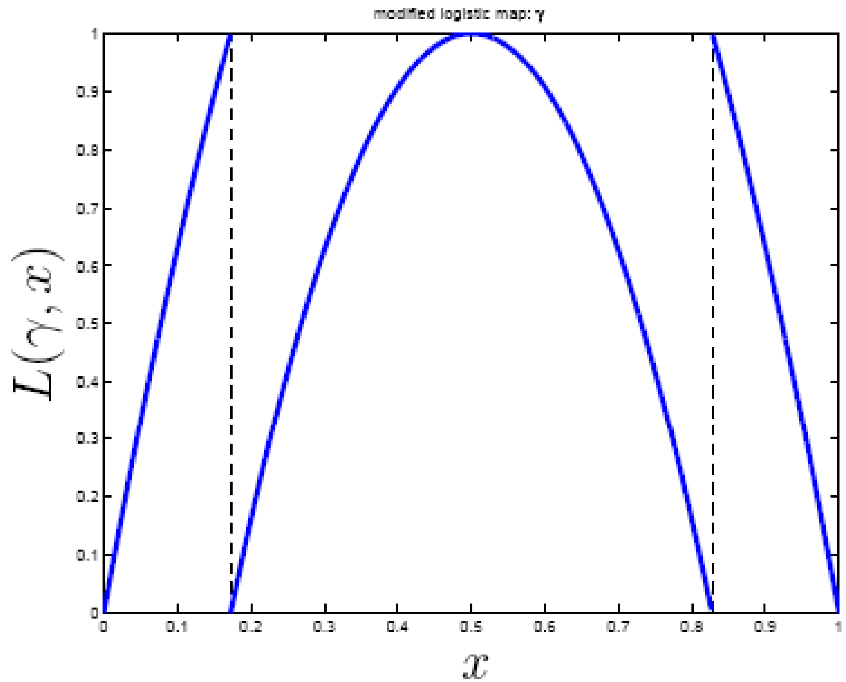


(a) $\gamma = 7$

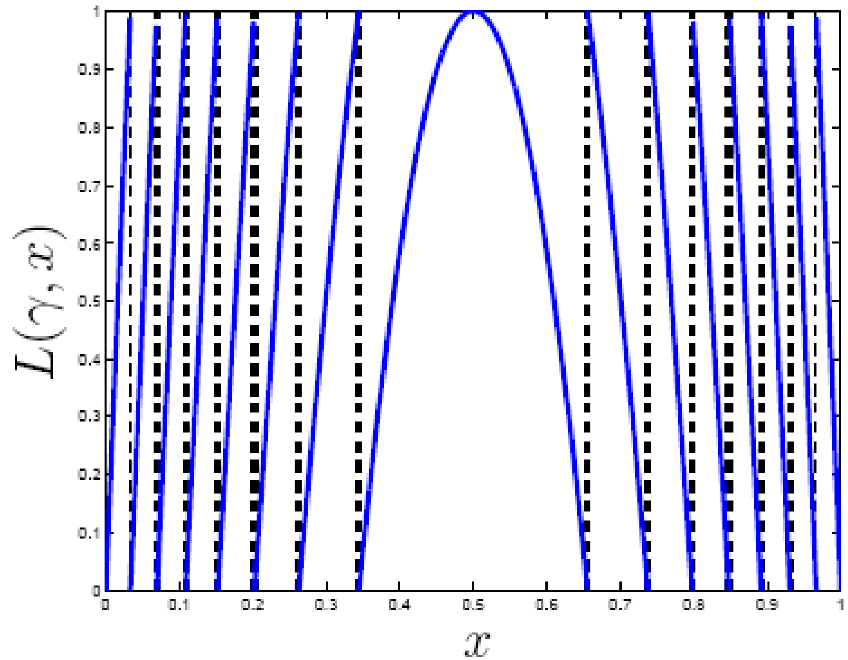


(b) $\gamma = 31$

Modified Logistic map



(a) $\gamma = 7$



(b) $\gamma = 31$

Properties of MLM

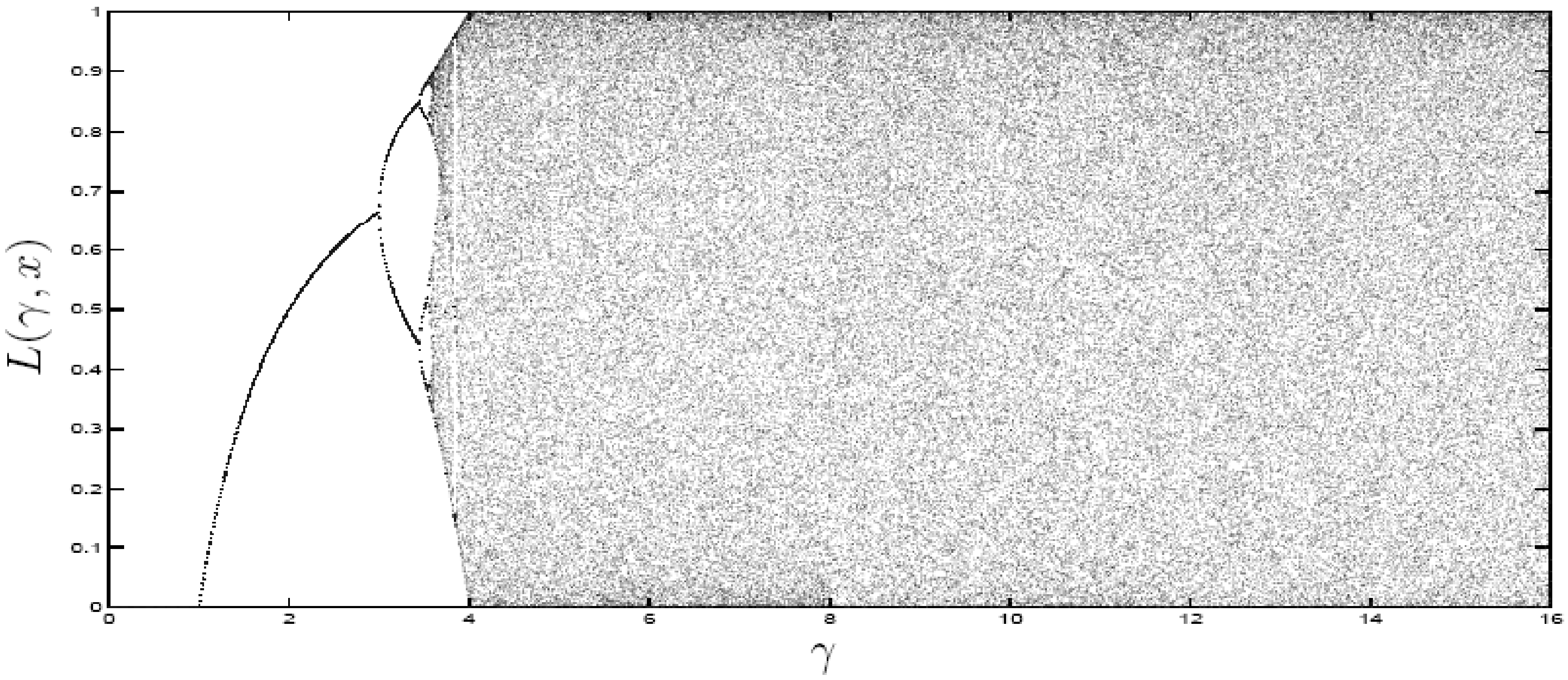
- Chaotic map
- No windows
- Uniform distribution
- Equivalent
- Pseudorandom

MLM: chaotic map

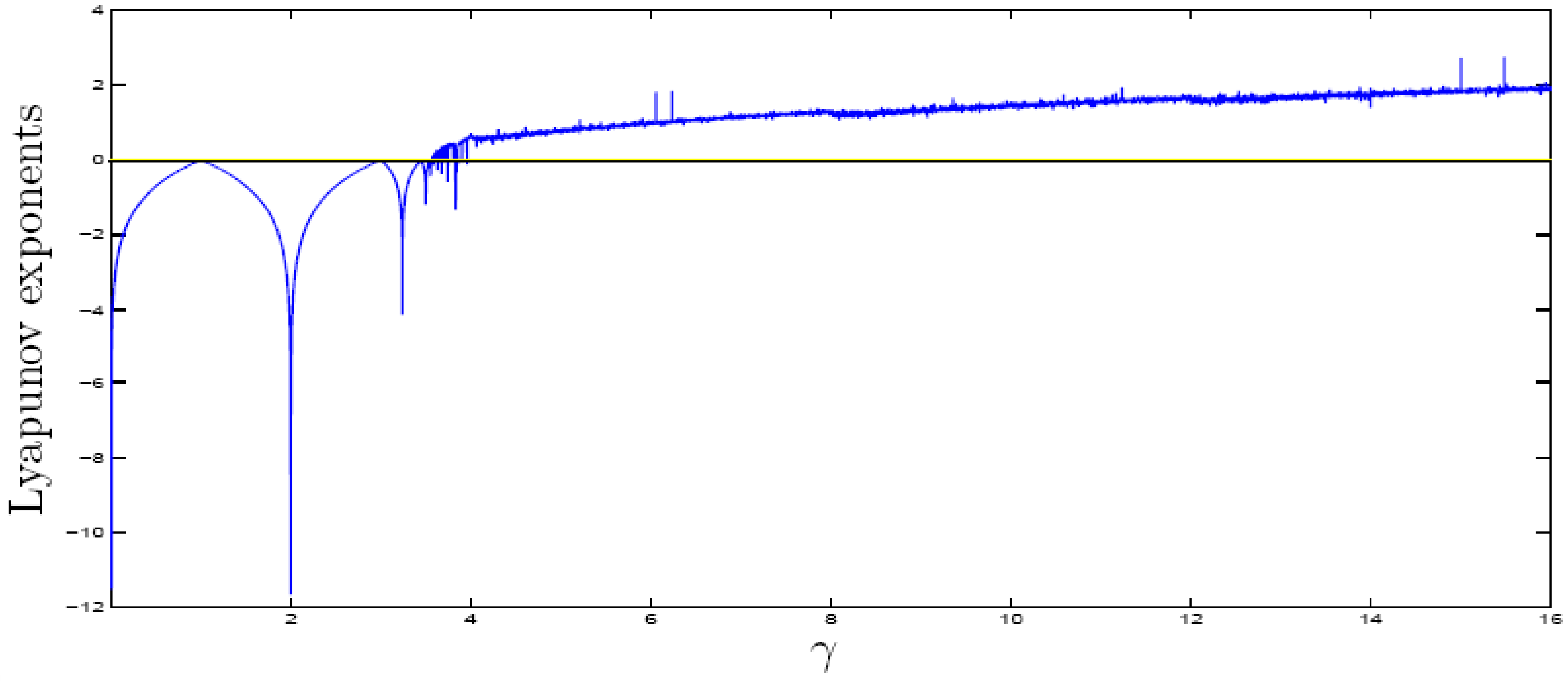
Definition 1. *Let $f : I \to I$ be a map, where I is a closed interval. We say that f exhibits* Devanvey's chaos *on I if the following conditions are satisfied:*

1. *the set of periodic points is dense in I;*
2. *the map f is* topologically transitive, *i.e., for any given pair of nonempty open sets U and V in I, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$; and*
3. *the map f has* sensitive dependence on initial conditions; *i.e., there exists $\alpha > 0$ such that for any $x \in I$ and any $\epsilon > 0$, there are $y \in I$ and $n \in \mathbb{N}$ such that $|x - y| < \epsilon$ and $|f^n(x) - f^n(y)| > \alpha$.*

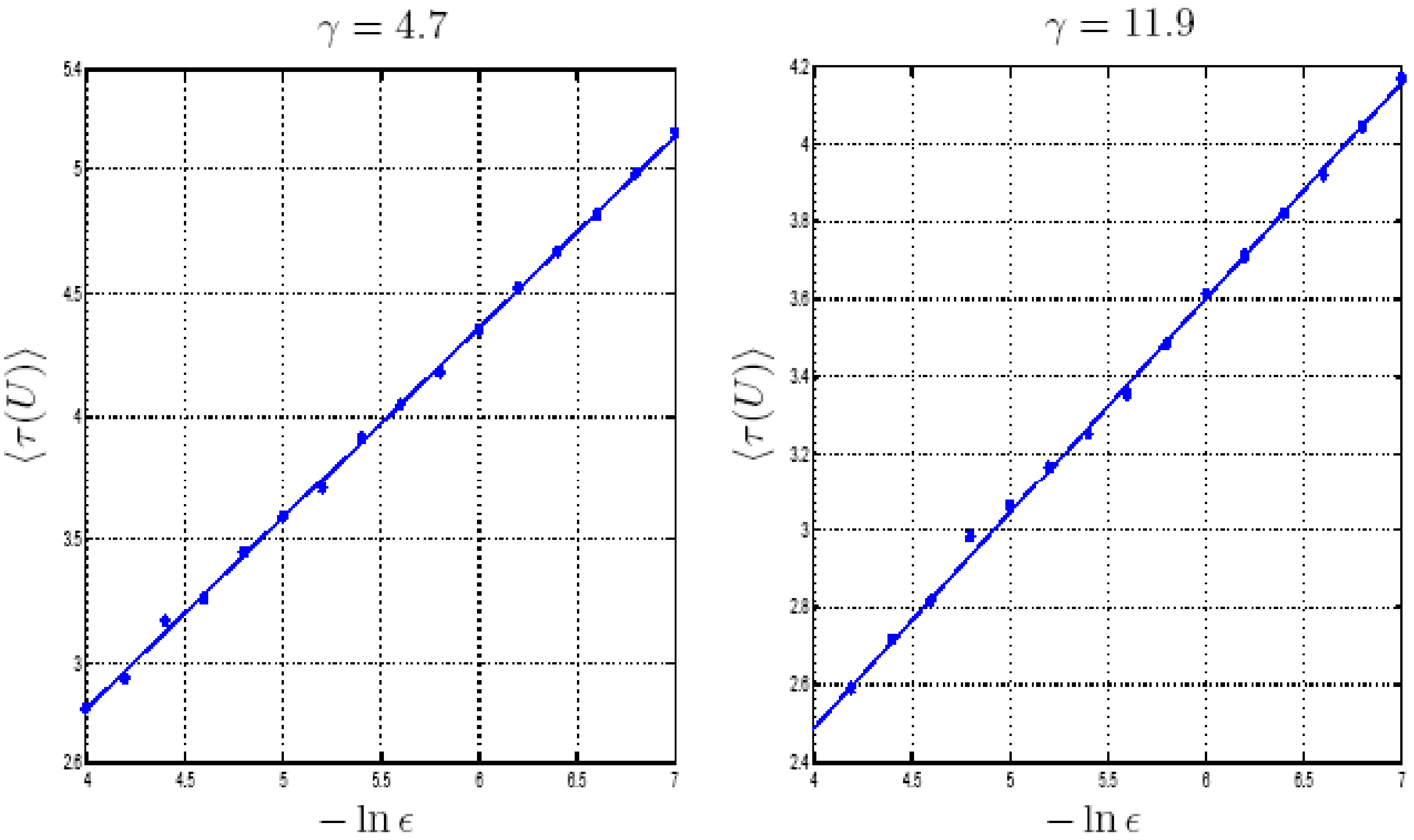
MLM: no windows



MLM: no windows

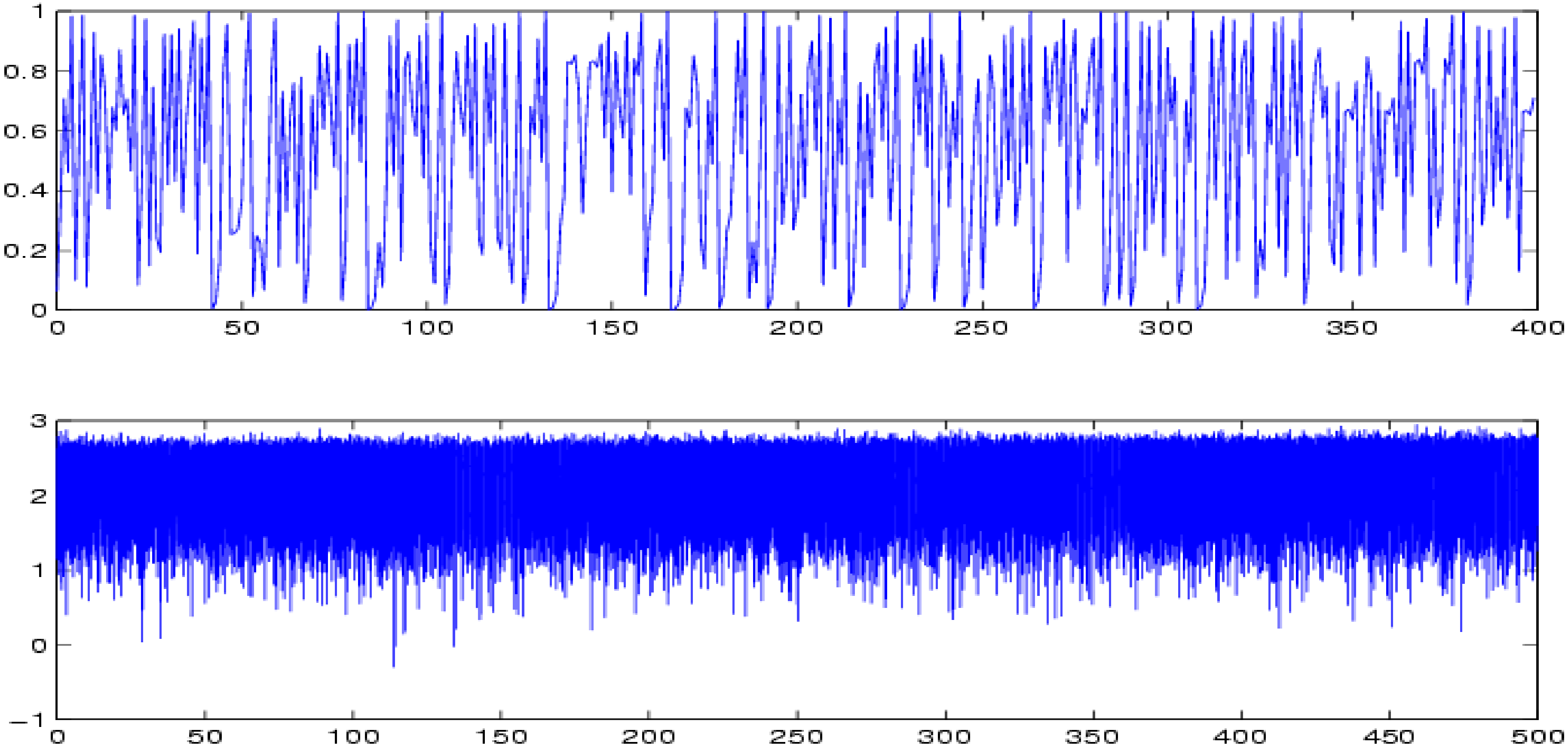


MLM: Poincaré recurrence

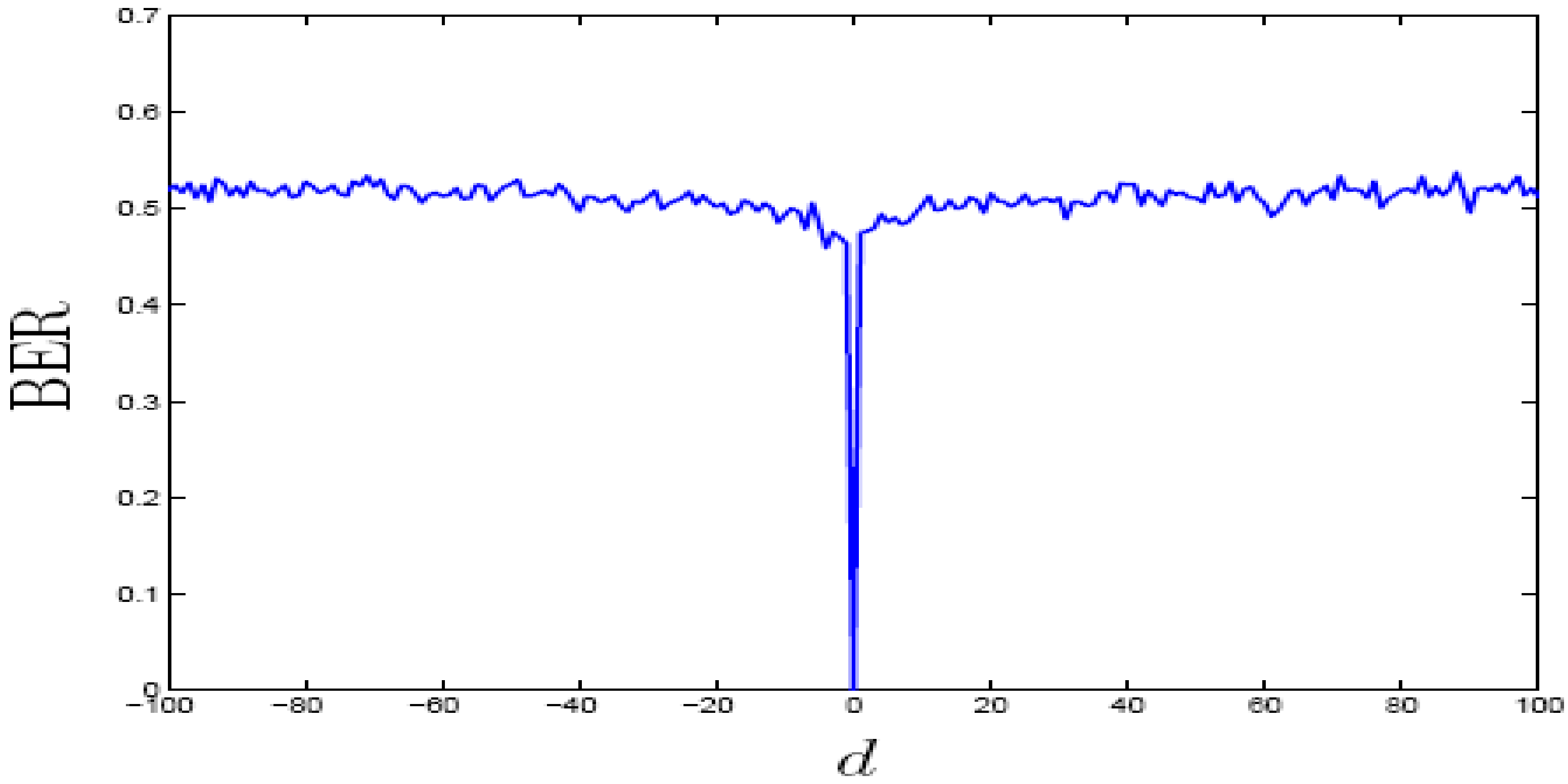


MLM: uniform distribution (FFT)

r = 5.9



MLM: equivalent (bits error rate analysis)



MLM: pseudorandom

Rukhin, A., Soto, J., Nechvatal, J., Smid, M., Barker, E., Leigh, S., Levenson, M., Vangel, M., Banks, D., Heckert, A., Dray, J., & Vo, S. [2001] "A statistical test suite for random and pseudorandom number generators for cryptographic applications," *Technical Report NIST Special Publication 800-22* (National Inst. of Standards and Technology, Gaithersburg, MD).

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MLM: pseudorandom (SP 800-22)

Frequency	0.99	0.98	1.00	1.00	0.99	1.00
Block Frequence	1.00	0.98	0.99	0.99	0.98	0.99
Cumulative-sums	0.99	0.98	1.00	1.00	0.99	1.00
Run	0.99	0.98	0.99	0.99	0.99	0.99
Long Runs of Ones	1.00	1.00	0.99	1.00	1.00	1.00
Rank	1.00	0.98	0.98	1.00	1.00	1.00
Spectral DFT	0.99	0.98	0.98	1.00	0.99	0.97
Non-overlapping Template	0.99	0.99	0.99	0.99	0.99	0.99
Overlapping Templates	0.99	0.99	0.97	0.99	0.99	0.98
Universal	0.99	1.00	0.99	0.98	0.98	0.99
Approximate Entropy	0.98	1.00	1.00	0.99	0.99	1.00
Random Excursions	0.99	0.98	0.98	0.99	0.98	0.98
Random Excursions Variant	0.99	0.99	0.99	0.99	0.99	1.00
Lempel Ziv Complexity	1.00	0.97	1.00	1.00	0.97	0.98
Serial	0.99	0.98	0.98	0.99	1.00	1.00
Num. of "< 0.97"	0	0	0	0	0	0

Random vs. Chaos

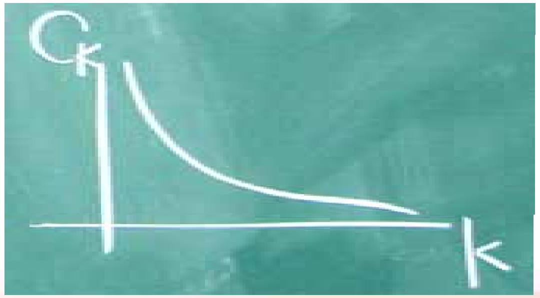
Random numbers $(x_0, x_1, \ldots, x_n, \ldots)$

Chaotic signals $(y_0, y_1, \ldots, y_n, \ldots)$

Identity：

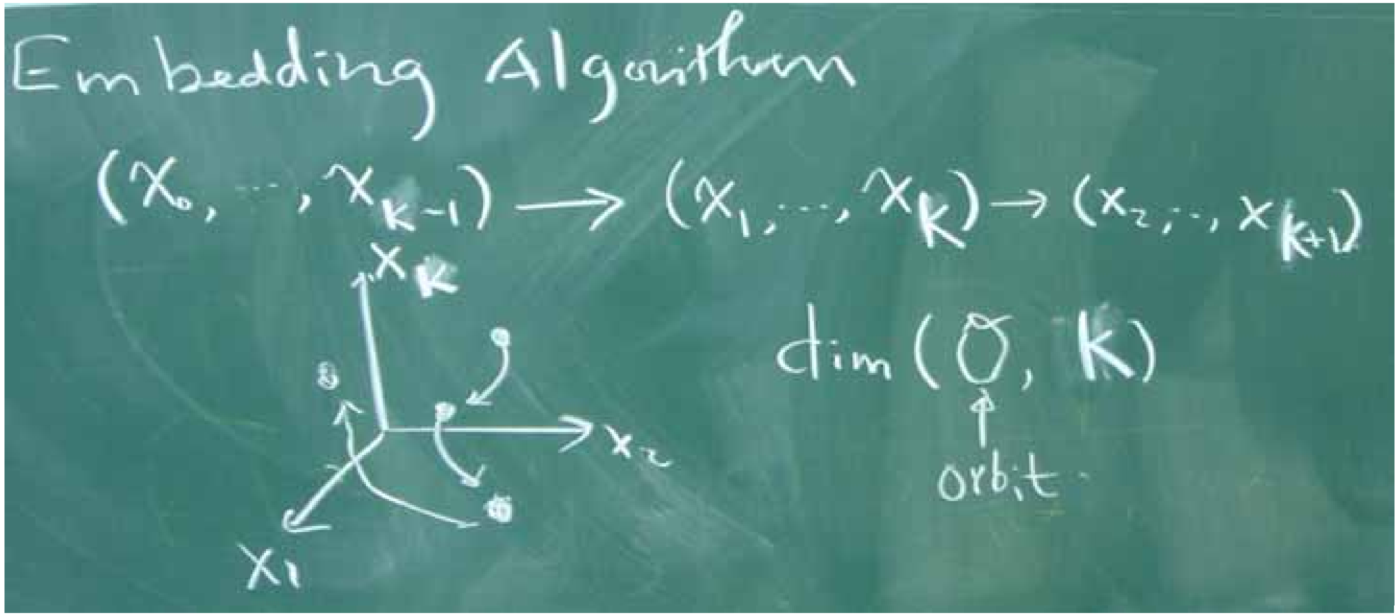
1. Continuous Spectrum
2. Correlation Function：

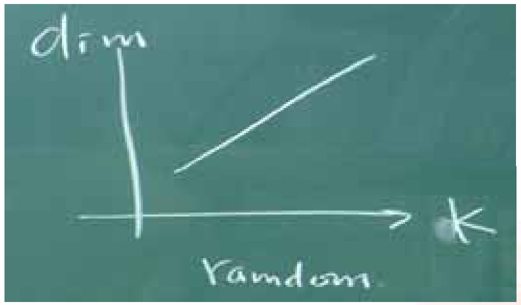
$$C_k = \sum_{n=1}^{\infty} x_{n+k} x_n$$

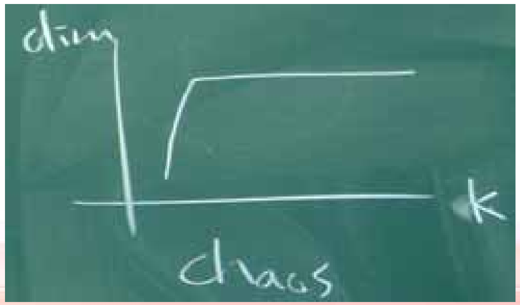


Random vs. Chaos

Distinction：



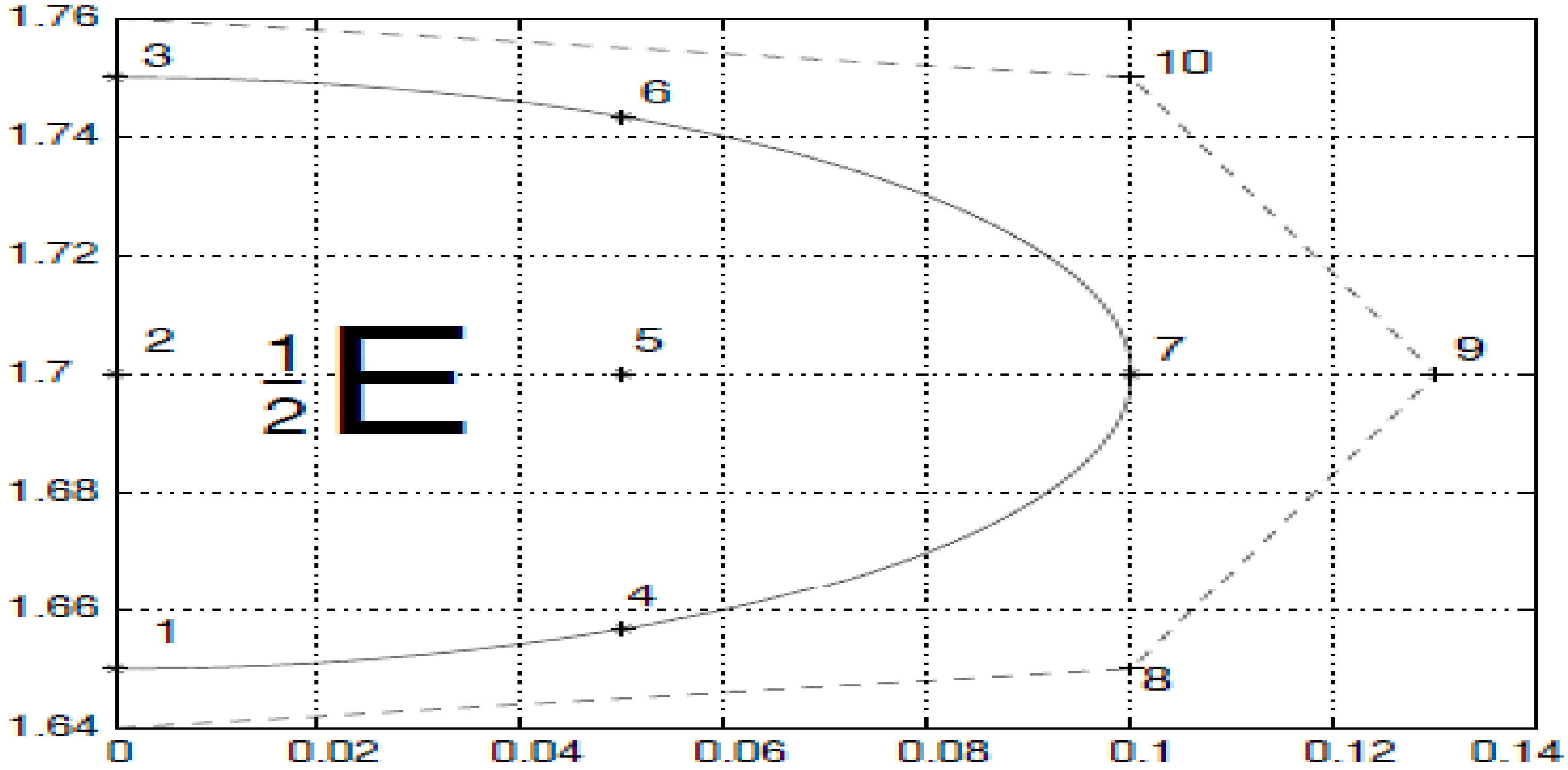




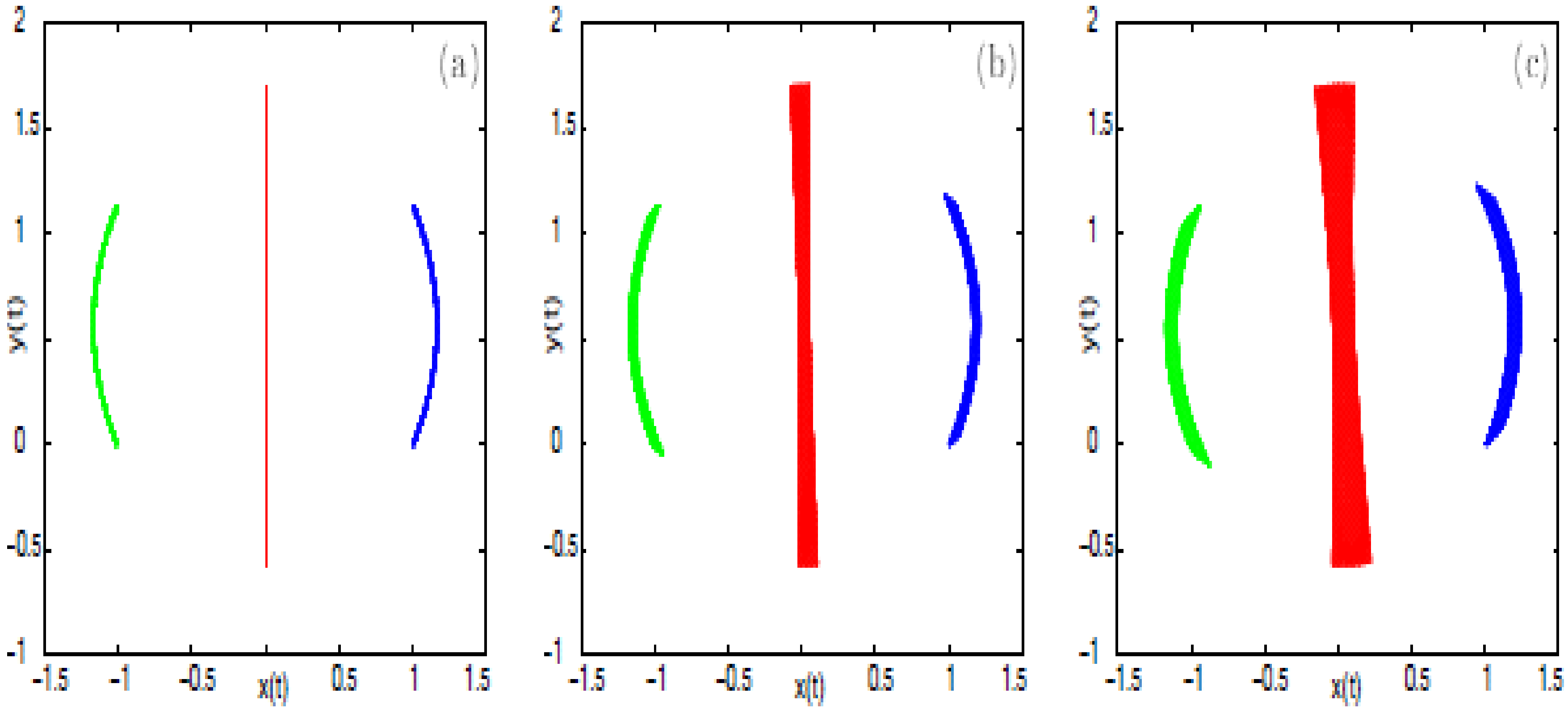
3 2D charged particles

$$\begin{cases} \ddot{q}_1 & = n_1 \, n_2 \, \dfrac{q_1 - q_2}{|q_1 - q_2|^2} + n_1 \, n_3 \, \dfrac{q_1 - q_3}{|q_1 - q_3|^2} + f_0 \, \dot{q}_1^{\perp} \\ \ddot{q}_2 & = n_2 \, n_1 \, \dfrac{q_2 - q_1}{|q_2 - q_1|^2} + n_2 \, n_3 \, \dfrac{q_2 - q_3}{|q_2 - q_3|^2} + f_0 \, \dot{q}_2^{\perp} \\ \ddot{q}_3 & = n_3 \, n_1 \, \dfrac{q_3 - q_1}{|q_3 - q_1|^2} + n_3 \, n_2 \, \dfrac{q_3 - q_2}{|q_3 - q_2|^2} + f_0 \, \dot{q}_3^{\perp} \end{cases}$$

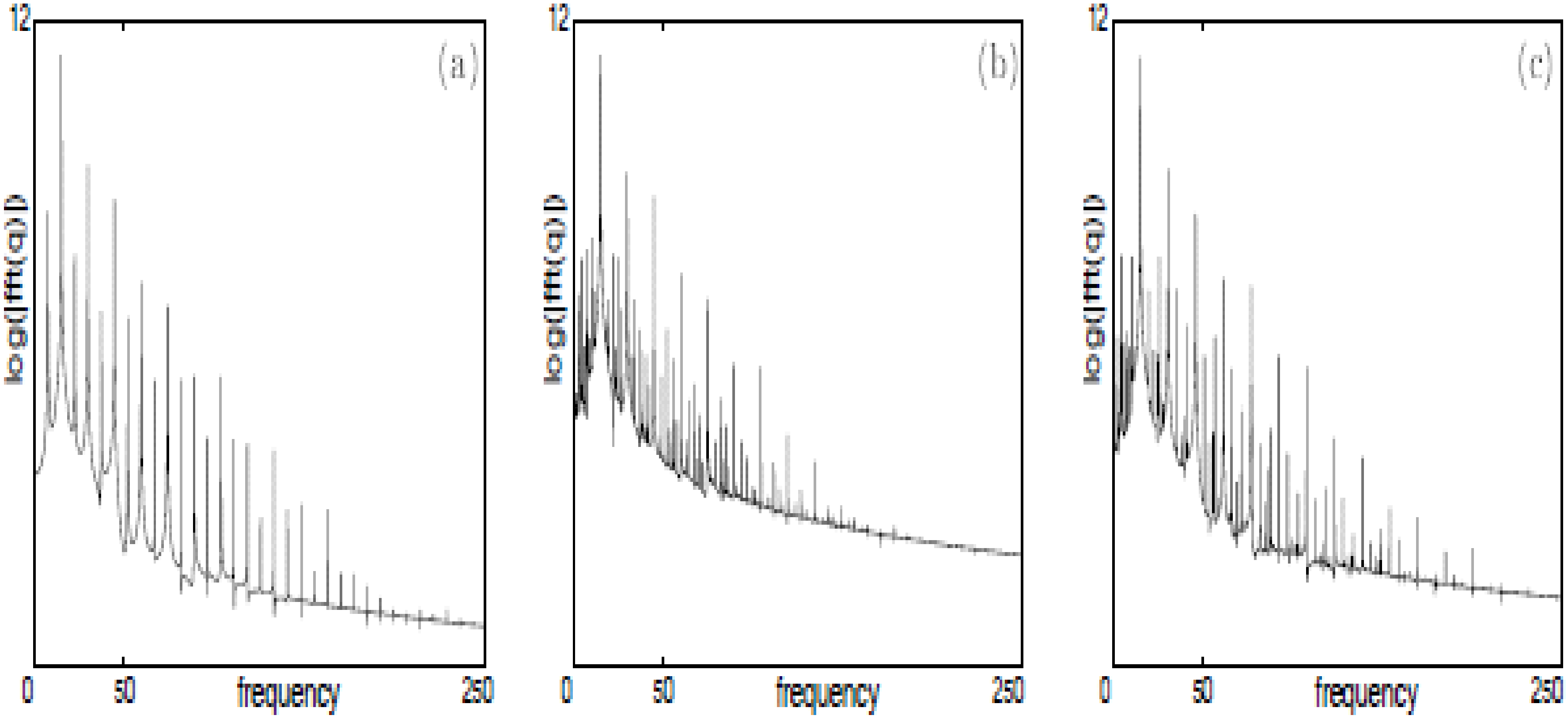
3 2D charged particles



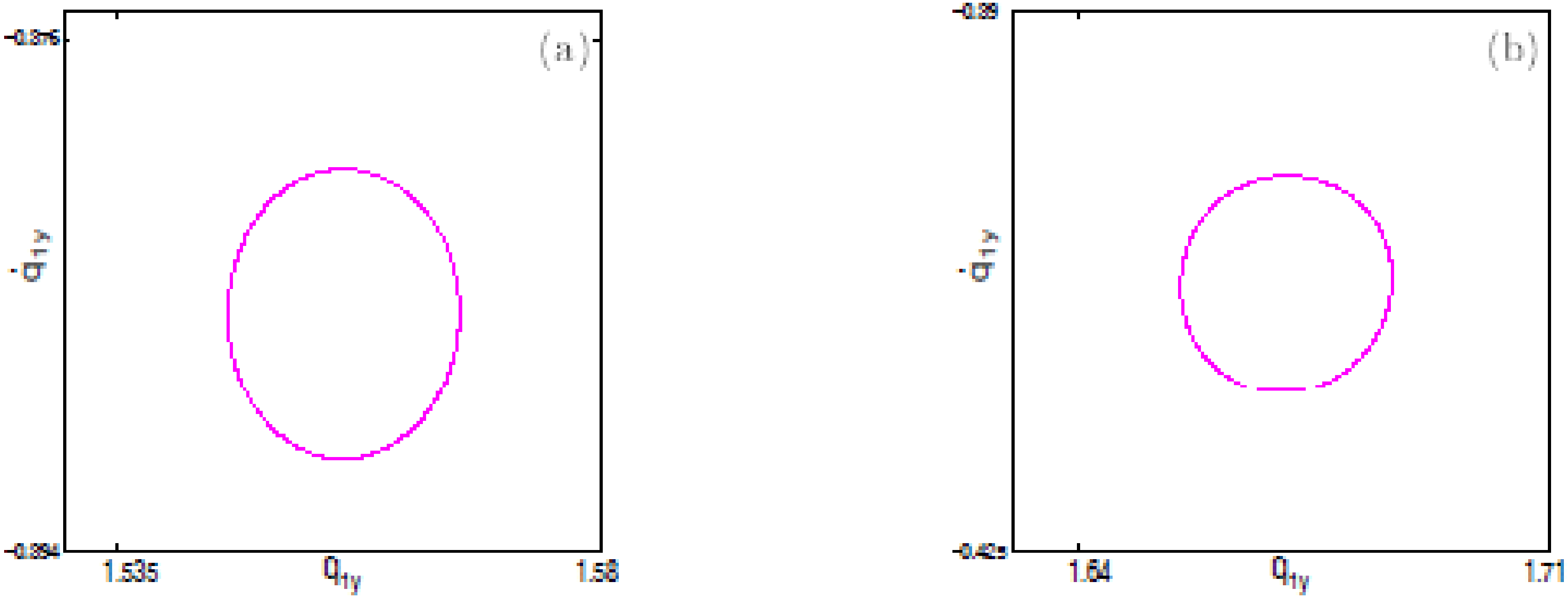
3 2D charged particles



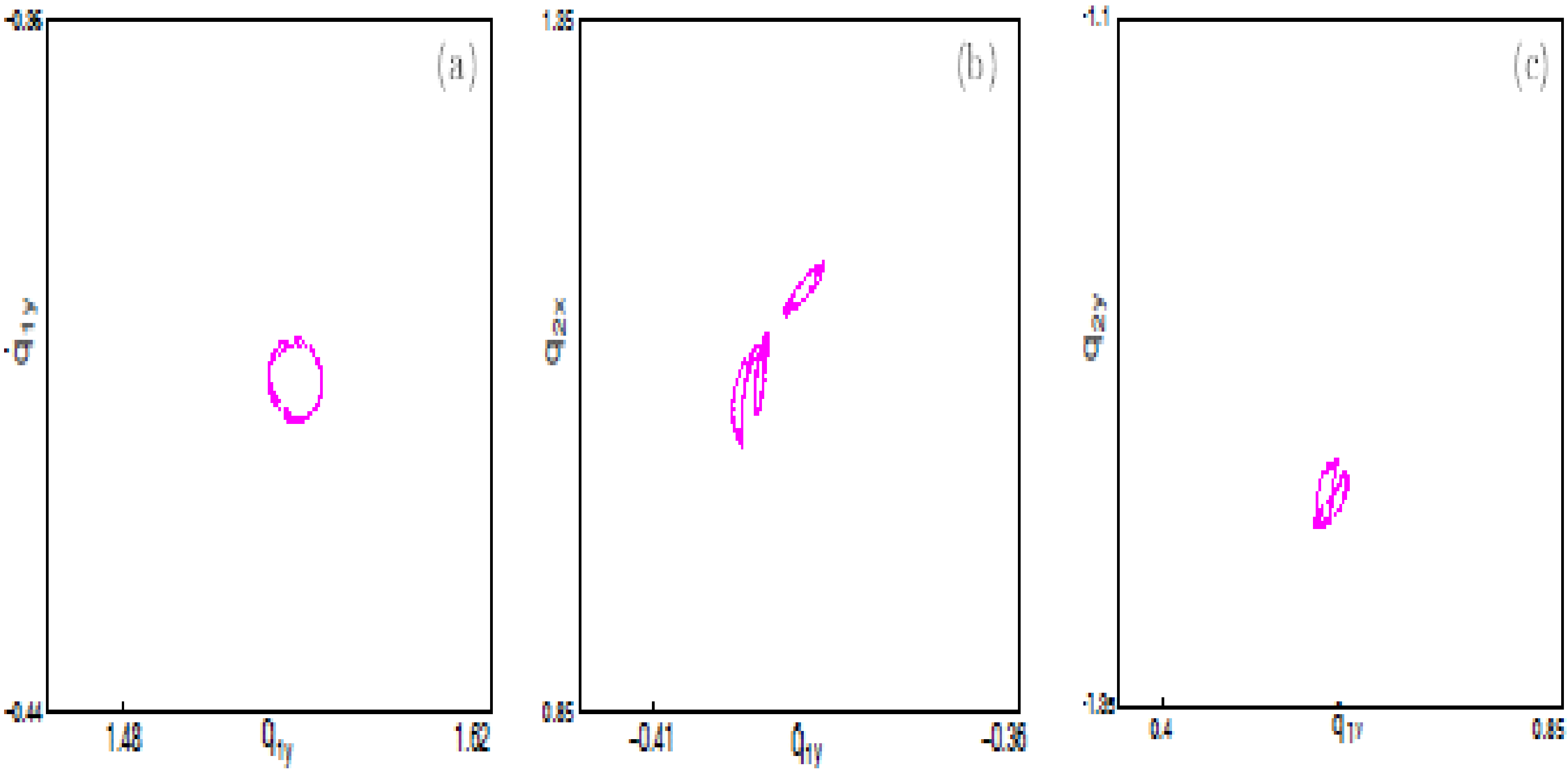
3 2D charged particles



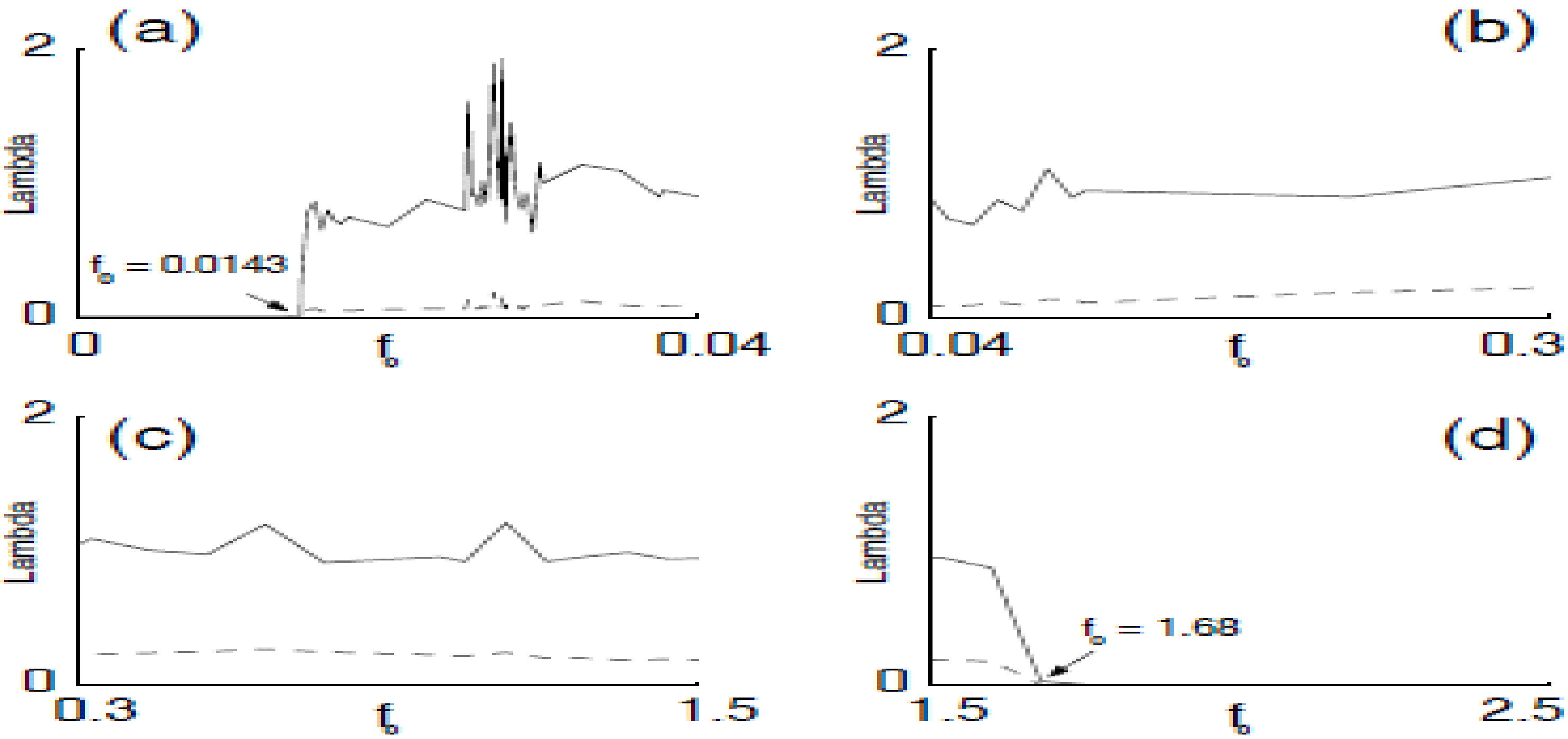
3 2D charged particles



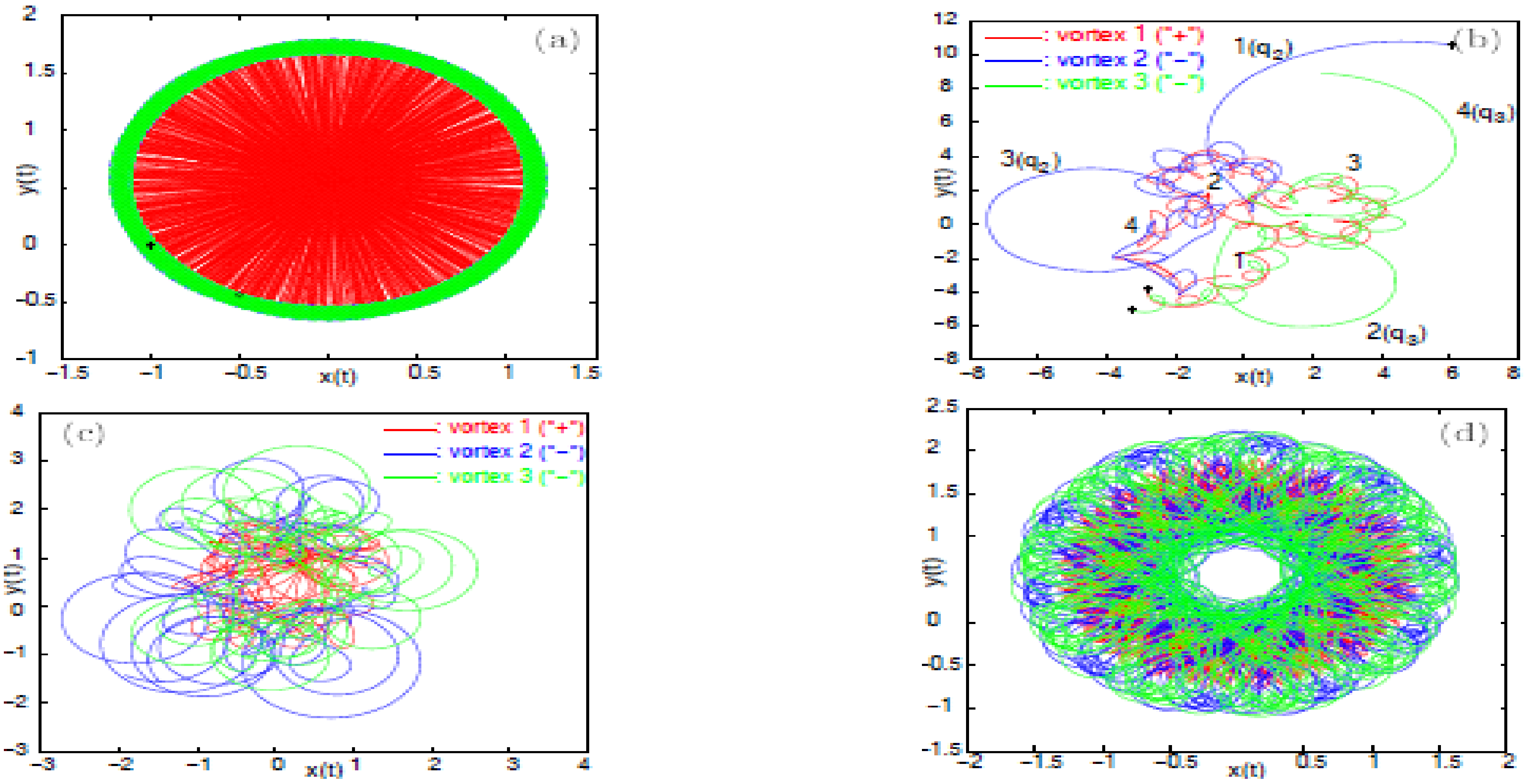
3 2D charged particles



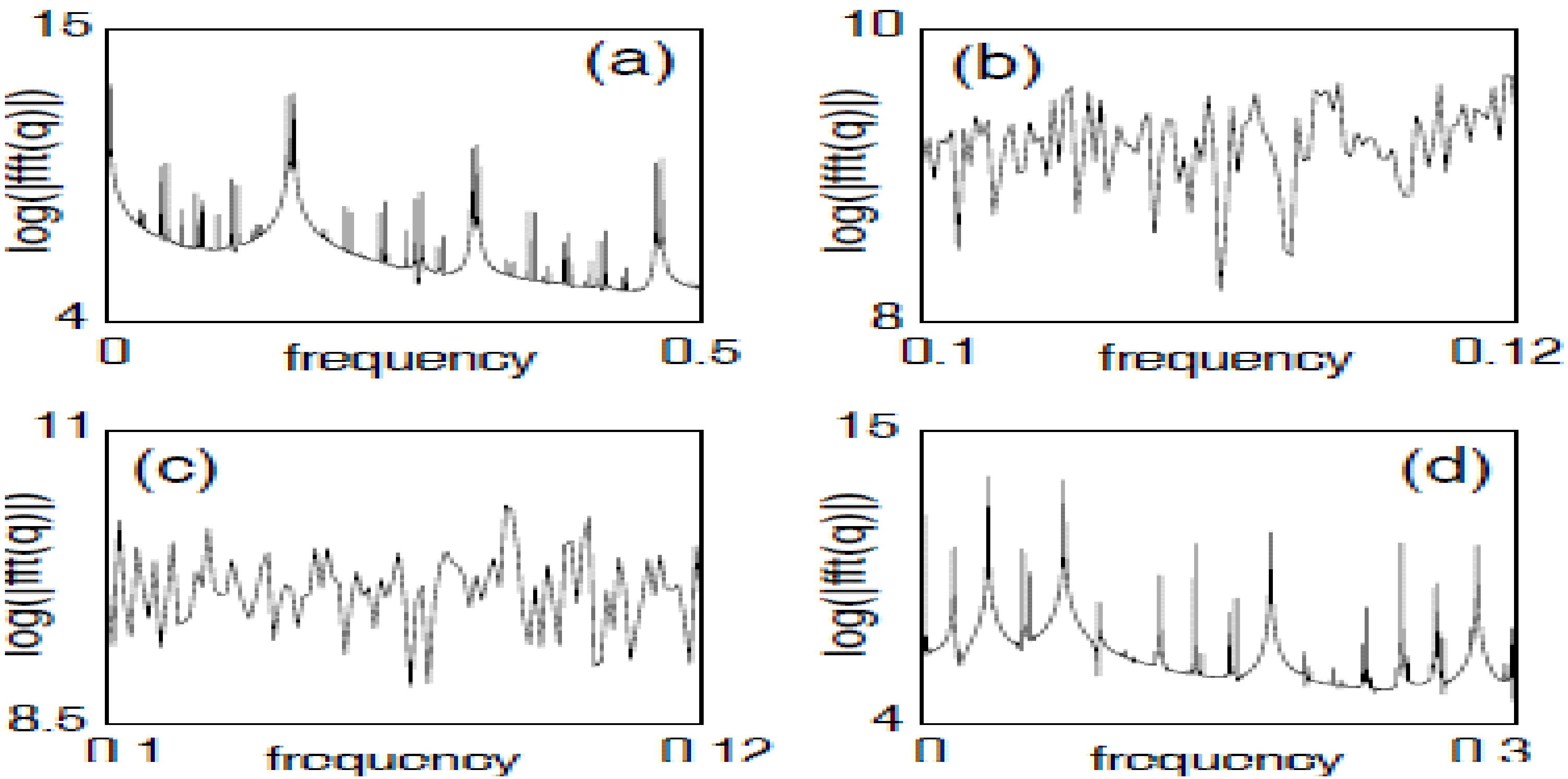
3 2D charged particles



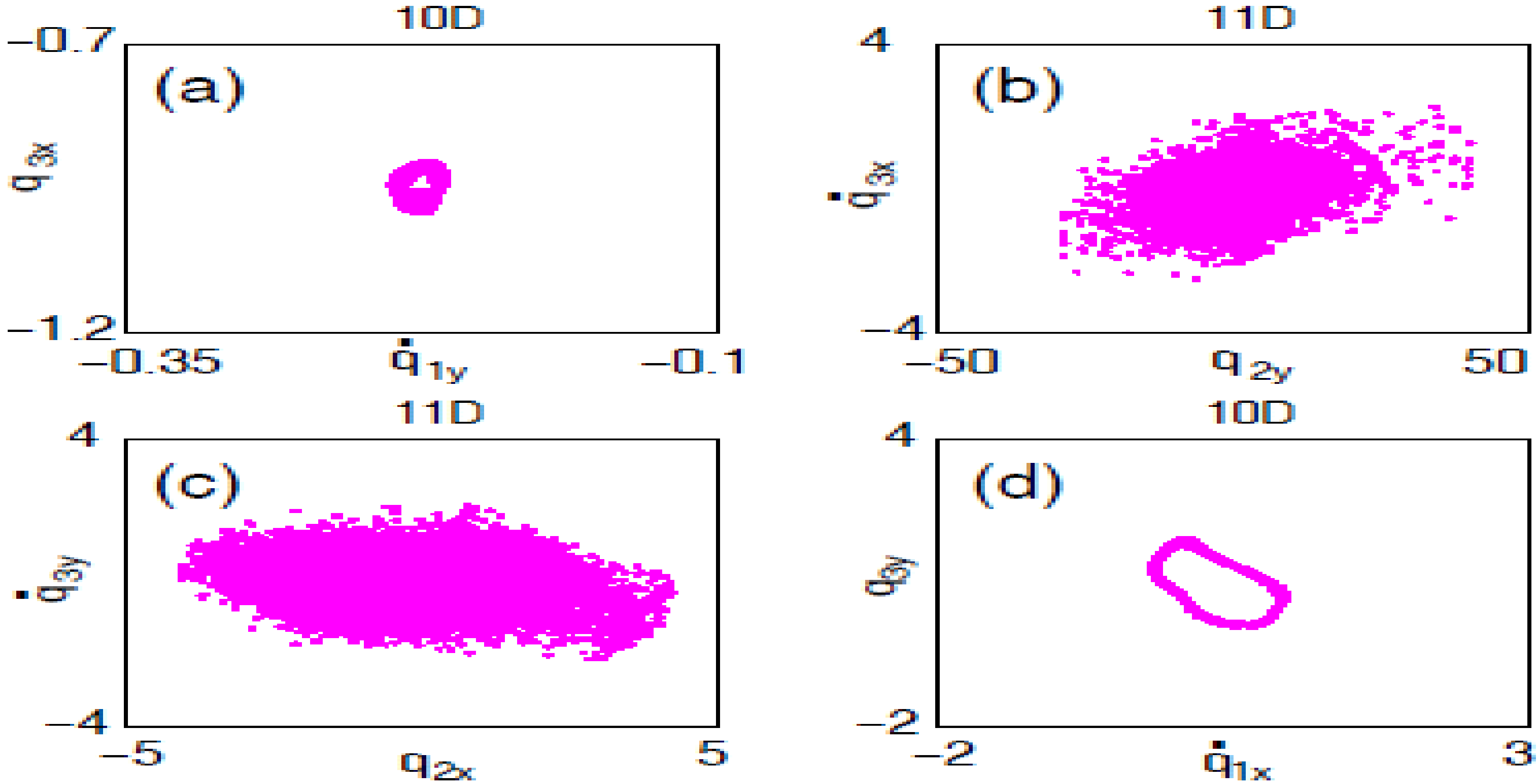
3 2D charged particles



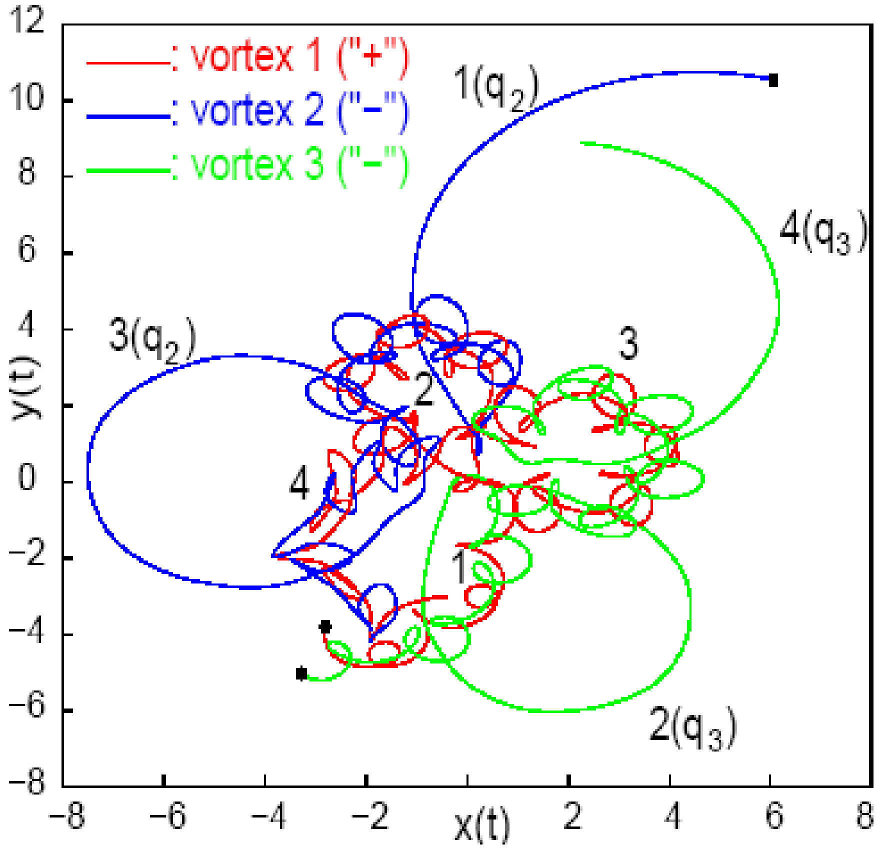
3 2D charged particles

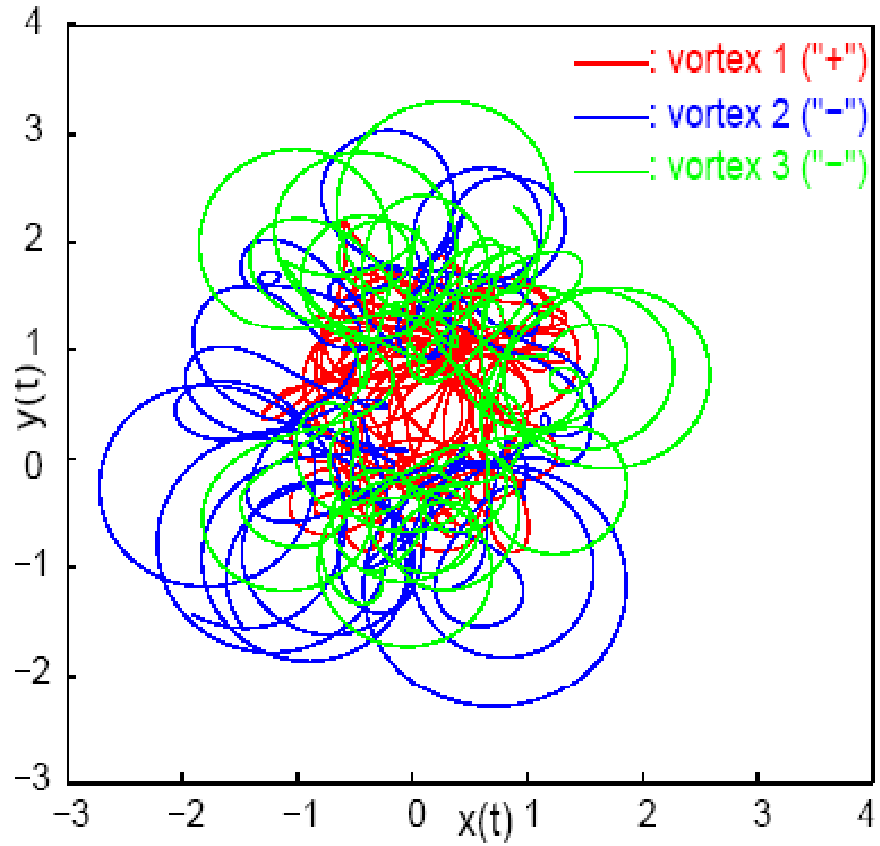


3 2D charged particles

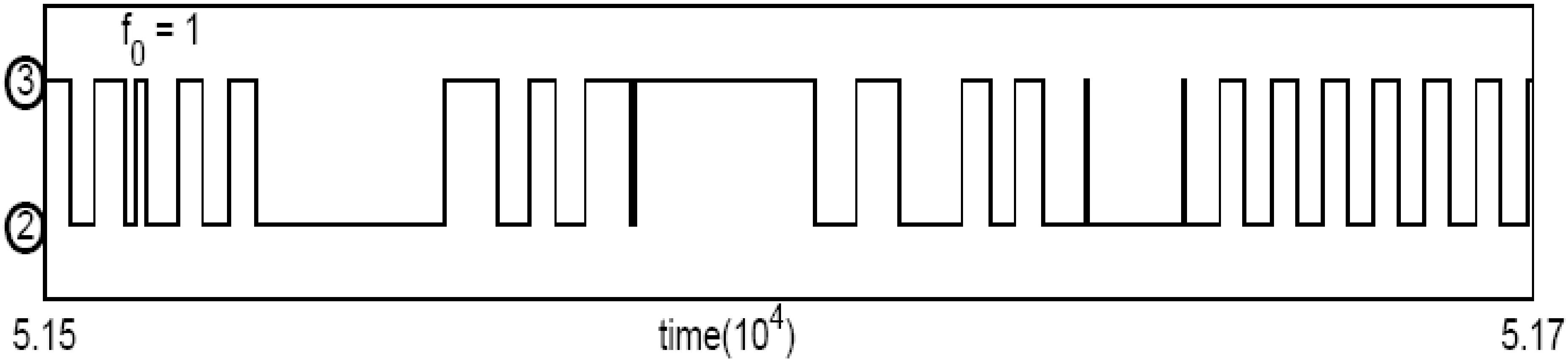


3 2D charged particles





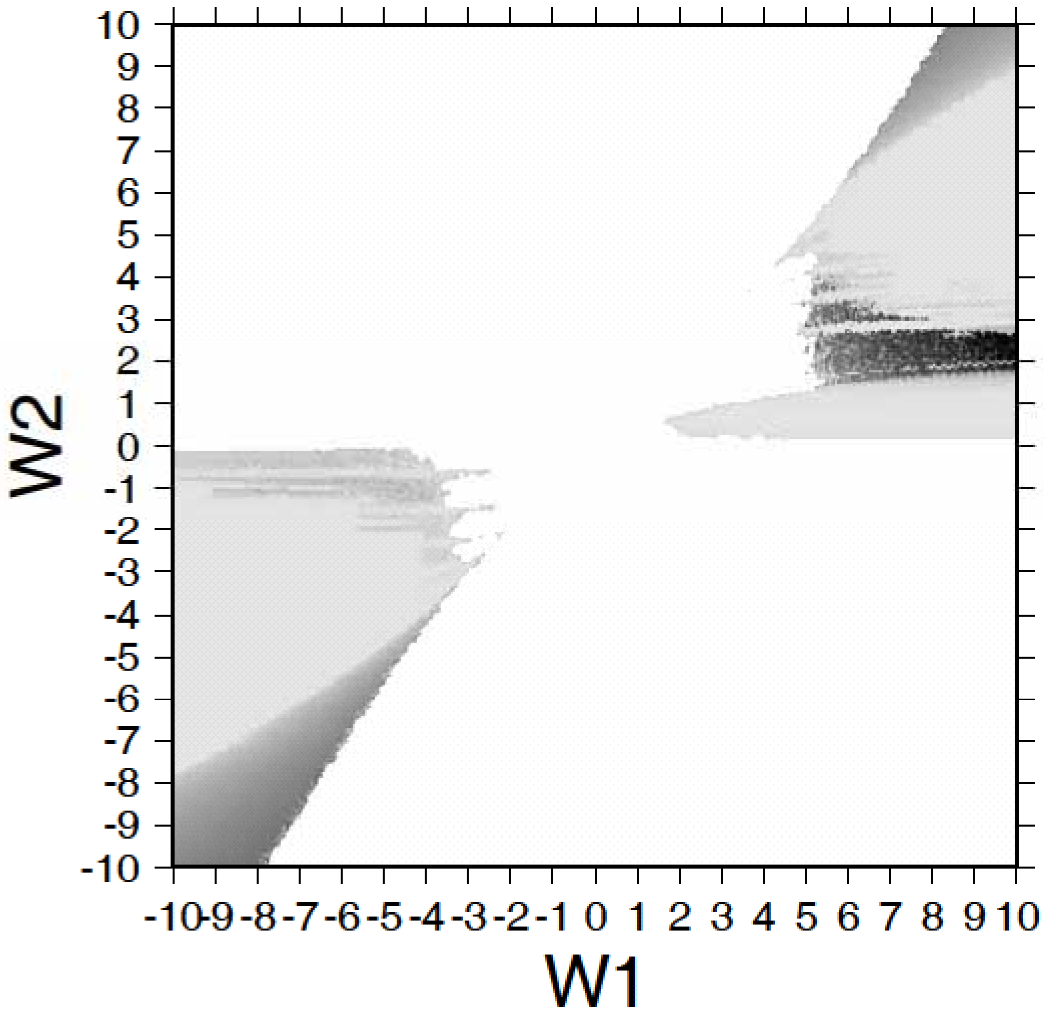
3 2D charged particles



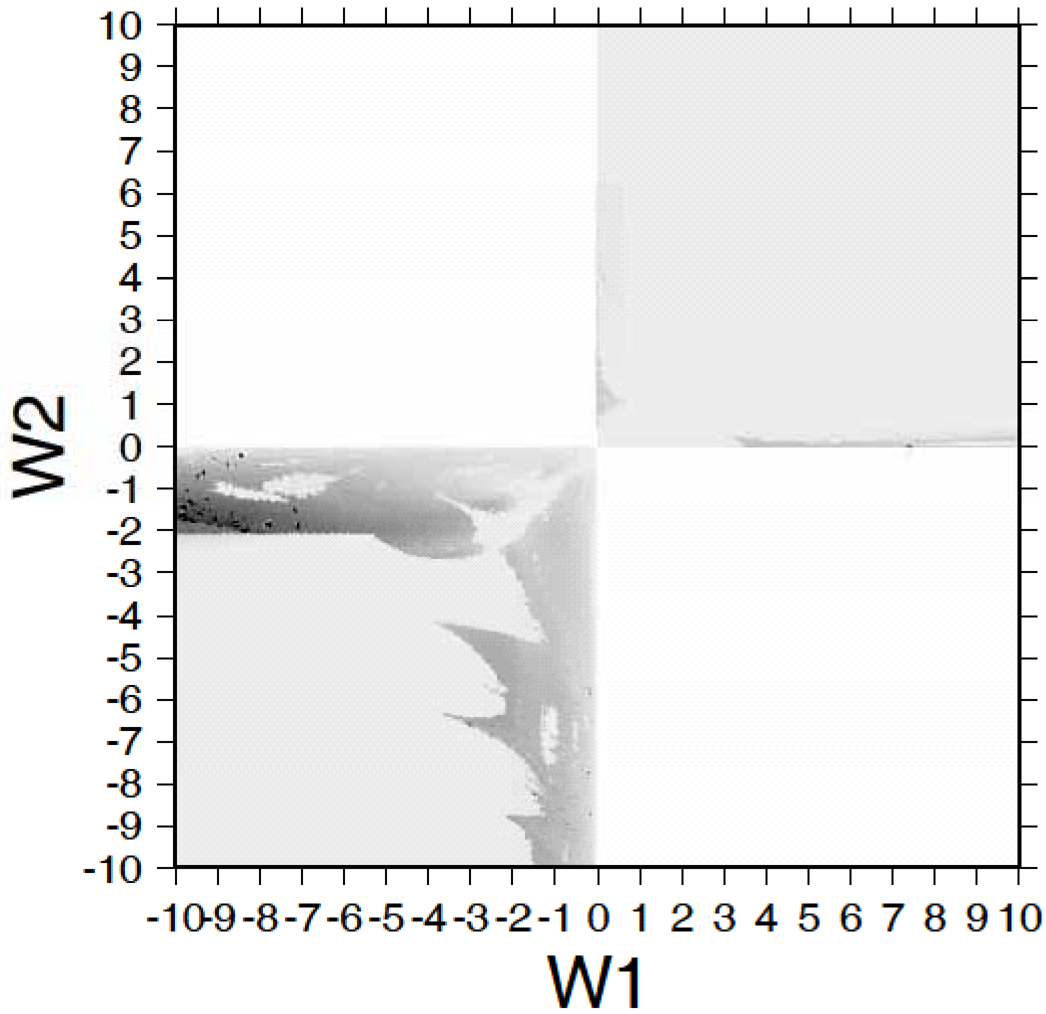
3 vortices system

$$\begin{cases} \dot{q}_{jx} = -\sum_{\substack{k=1 \\ k \neq j}}^{3} n_k \frac{q_{jy} - q_{ky}}{|q_j - q_k|^2} - \omega_1 q_{jy}, \\ \dot{q}_{jy} = \sum_{\substack{k=1 \\ k \neq j}}^{3} n_k \frac{q_{jx} - q_{kx}}{|q_j - q_k|^2} + \omega_2 q_{jx} \end{cases}$$

3 vortices system



3 vortices system



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Thank you for your attention!