VERIFICATION OF MIXING PROPERTIES IN TWO-DIMENSIONAL SHIFTS OF FINITE TYPE

JUNG-CHAO BAN*, WEN-GUEI HU, SONG-SUN LIN**, AND YIN-HENG LIN

Abstract. This investigation studies topological mixing and strong specification of two-dimensional shifts of finite type. Connecting operators are introduced to reduce the order of high-order transition matrices to yield lower-order transition matrices that are useful in establishing finitely checkable conditions for the primitivity of all transition matrices. Two kinds of sufficient condition for the primitivity of transition matrices are provided; (I) invariant diagonal cycles and (II) primitive commutative cycles. After primitivity is established, the corner-extendability and crisscross-extendability are introduced to demonstrate topological mixing. In addition to these sufficient conditions for topological mixing, the hole-filling condition implies the strong specification. All mentioned conditions are finitely checkable.

1. Introduction

Multi-dimensional shift is an important and a highly active area of ongoing research in dynamical system. It is also closely related to lattice models in the scientific modeling of spatial structure. Relevant investigations have been performed on phase transitions and chemical reactions [4, 6, 7, 11, 18, 22, 26, 27, 28, 29, 30, 36, 37, 38, 39, 40, 50, 51, 52], biology [8, 9] and image processing and pattern recognition [17, 19, 20, 23, 24, 25, 33]. Lattice models can be better understood if multi-dimensional shifts of finite type are understood. The most interesting properties of shifts include entropy and various mixing properties, such as topological mixing and strong specification (or strong irreducibility). These properties enjoy many of the important properties of dynamical systems [12, 13, 14, 15, 16, 24, 25, 30, 35, 41, 42, 44, 45, 47, 49, 54]. However, determining whether a given system exhibits topological mixing or strong specification in multi-dimensions is not easy. The intrinsic difficulty is related to undecidability of multi-dimensional coloring problem [10, 21, 23, 32, 34, 46, 48, 53]. Nevertheless, this study provides some easily checked sufficient conditions for topological mixing and strong specification of two-dimensional shifts of finite type.

Let $\mathbb{Z}^2$ be the two-dimensional planar lattice. Vertex (or corner) coloring is considered first. For any $m, n \geq 1$ and $(i, j) \in \mathbb{Z}^2$, the $m \times n$ rectangular lattice with the left-bottom vertex (i, j) is denoted by

*The author would like to thank the National Science Council, R.O.C. (Contract No. NSC 99-2115-M-259-003) and the National Center for Theoretical Sciences for partially supporting this research.

**The author would like to thank the National Science Council, R.O.C. (Contract No. NSC 98-2115-M-009) for partially supporting this research.

$$\mathbb{Z}_{m\times n}((i,j)) = \{(i+n_1, j+n_2) \mid 0 \le n_1 \le m-1, 0 \le n_2 \le n-1\}.$$

In particular,

$$\mathbb{Z}_{m\times n} = \mathbb{Z}_{m\times n}((0,0)).$$

Let $\mathcal{S}_p$ be a set of p (≥ 2) colors or symbols. For $m, n \ge 1$, $\Sigma_{m\times n}(p) = \mathcal{S}_p^{\mathbb{Z}_{m\times n}}$ is the set of $m \times n$ local patterns.

Let $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ be a basic set of allowable local patterns. For any lattice $R \subset \mathbb{Z}^2$, the set of all $\mathcal{B}$-admissible patterns on R is defined as

$$\Sigma_R(\mathcal{B}) = \left\{U \in \mathcal{S}_p^R : U\mid_{\mathbb{Z}_{2\times 2}((i,j))} \in \mathcal{B} \text{ if } \mathbb{Z}_{2\times 2}((i,j)) \subset R\right\}.$$

Denote $\Sigma_{m\times n}(\mathcal{B}) = \Sigma_{\mathbb{Z}_{m\times n}}(\mathcal{B})$ for $m, n \ge 2$. $\Sigma(\mathcal{B}) = \Sigma_{\mathbb{Z}^2}(\mathcal{B})$ is the set of all global patterns that can be constructed from the local patterns in $\mathcal{B}$. Notably, any $\mathbb{Z}^2$-shift of finite type can be represented by some $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ for some $p \ge 2$ [43]. Hence, only the case of $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, $p \ge 2$, is considered here.

First, topological mixing is introduced. For any shift Σ and any subset $R \subset \mathbb{Z}^2$, $\Pi_R(\Sigma) : \Sigma \to \mathcal{S}_p^R$ is the restriction map. Denote by d the Euclidean metric on $\mathbb{Z}^2$. A $\mathbb{Z}^2$ shift Σ is topologically mixing (mixing, for short) if for any finite subsets R_1 and R_2 of $\mathbb{Z}^2$, a constant $M(R_1, R_2)$ exists such that for all $\mathbf{v} \in \mathbb{Z}^2$ with $d(R_1, R_2+\mathbf{v}) \ge M$, and for any two allowable patterns $U_1 \in \Pi_{R_1}(\Sigma)$ and $U_2 \in \Pi_{R_2+\mathbf{v}}(\Sigma)$, there exists a global pattern $W \in \Sigma$ with $\Pi_{R_1}(W) = U_1$ and $\Pi_{R_2+\mathbf{v}}(W) = U_2$; see [54].

On the other hand, Σ has strong specification if a number $M(\Sigma) \ge 1$ exists such that for any two allowable patterns $U_1 \in \Pi_{R_1}(\Sigma)$ and $U_2 \in \Pi_{R_2}(\Sigma)$ with $d(R_1, R_2) \ge M$, where R_1, R_2 are subsets of $\mathbb{Z}^2$, there exists a global pattern $W \in \Sigma$ with $\Pi_{R_1}(W) = U_1$ and $\Pi_{R_2}(W) = U_2$; see [54]. Clearly, strong specification implies topological mixing.

Very few results that verify that $\Sigma(\mathcal{B})$ is mixing or has strong specification are known [3, 44]. Previously, in studying pattern generation problems [2], the authors introduced connecting operators to study the entropy of $\Sigma(\mathcal{B})$. In this paper, connecting operators are also used to provide sufficient conditions for the mixing of or strong specification of $\Sigma(\mathcal{B})$.

First, topological mixing is considered. Given two patterns U_1 and U_2 defined on R_1 and $R_2+\mathbf{v}$ respectively, in general, R_1 and $R_2+\mathbf{v}$ are not located horizontally or vertically. Typically, the gluing process is decomposed into three steps, as presented in Fig. 1.1. For clarity, in Fig 1.1, the patterns U's are presented and the underlying lattices R's are omitted.

Step (1): Extend U_2 to $\widetilde{U}_2$ such that U_1 can connect $\widetilde{U}_2$ horizontally. The combined pattern becomes an L-shaped pattern $U_1 \bigcup \widetilde{U}_1 \bigcup U_2 \bigcup \widetilde{U}_2$.
Step (2): Extend the L-shaped pattern to a rectangular pattern.
Step (3): Extend the rectangular pattern to a global pattern on $\mathbb{Z}^2$.

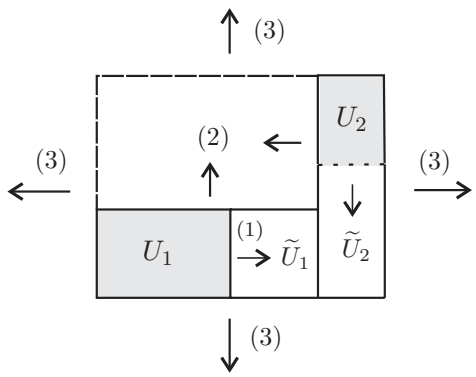


Figure 1.1.

To ensure that all processes are executable, the following sufficient conditions are proposed in each step:

(i) The primitivity of horizontal transition matrices $\mathbb{H}_n$ and vertical transition matrices $\mathbb{V}_n$, for each $n \geq 2$.
(ii) The corner-extendability for L-shaped lattices.
(iii) The rectangle-extendability to the $\mathbb{Z}^2$-plane.

Notably, a matrix A is primitive (or n_0-primitive) if there exists $n_0 \geq 1$ such that $A^n > 0$ for all $n \geq n_0$; here, $A^n > 0$ means that each entry of A^n is positive except in positions of A where a zero row or zero column is present. To find finitely checkable sufficient conditions of (i), two kinds of sufficient conditions for the primitivity of $\mathbb{H}_n$ and $\mathbb{V}_n$ are introduced.

(I) invariant diagonal cycles,
(II) primitive commutative cycles.

Both of these conditions are applied to construct the primitive diagonal submatrices of $\mathbb{H}_n^{M(n)}$ for some $M(n)$, $n \geq 2$; then, they are used to show that $\mathbb{H}_n$ is primitive. Furthermore, when either condition applies, only finitely many $\mathbb{H}_n$ have to be checked to ensure that $\mathbb{H}_n$ is primitive for all $n \geq 2$.

After the primitivity of $\mathbb{H}_n$ and $\mathbb{V}_n$ is established, the corner-extendable conditions $C(1) \sim C(4)$ and crisscross-extendability are introduced (Definitions 3.2 and 3.11) to extend the L-shaped pattern and the rectangular pattern into a global pattern, and then to establish that $\Sigma(\mathcal{B})$ is mixing. The main theorem is given by Theorem 3.14.

Theorem 1.1. *If*

(i) $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ *is crisscross-extendable,*
(ii) $\mathcal{B}$ *satisfies three of corner-extendable conditions* $C(i)$, $1 \leq i \leq 4$,

then $\mathbb{H}_n(\mathcal{B})$ *and* $\mathbb{V}_n(\mathcal{B})$ *are primitive for all* $n \geq 2$ *if and only if* $\Sigma(\mathcal{B})$ *is mixing.*

Next, strong specification is considered. Since strong specification is stronger than topological mixing. Apart from the processes in Fig. 1.1 which concern the situation in which regions R_1 and $R_2 + \mathbf{v}$ are far away, the case in which one pattern is enclosed in another pattern, as in Fig. 1.2, must be studied.

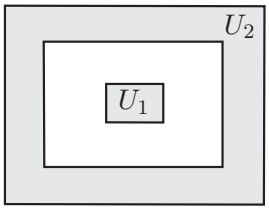


Figure 1.2.

Notably, in studying topological mixing, Fig. 1.2 can be reduced to Fig. 1.1. However, in studying strong specification, U_1 and U_2 cannot be removed since the relative positions of R_1 and R_2 are fixed. Now, the gluing of U_1 and U_2 can be completed by the following two processes.

Step (4): Extend U_1 horizontally and vertically to form a crisscross pattern that touches U_2, as presented in Fig. 1.3.

Step (5): Fill the holes that are surrounded by the rectangularly annular lattice to form a rectangular pattern, as presented in Fig. 1.4.

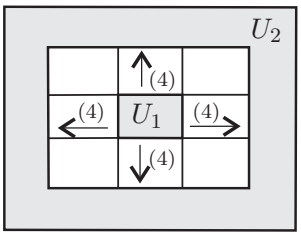


Figure 1.3.

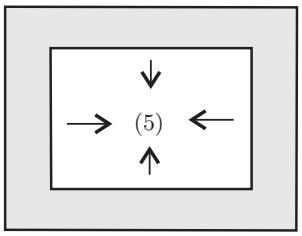


Figure 1.4.

Then, repeat Step (3) and extend the rectangular pattern to a global pattern on $\mathbb{Z}^2$.

The hole-filling condition (HFC) in Step (5) is finitely checkable: any hole of size (M, N) that is surrounded by any admissible annular lattice with width $L \geq 2$ can be filled by admissible local patterns and forms an $(M + 2L) \times (N + 2L)$ pattern. Moreover, k hole-filling condition $(\mathrm{HFC})_k$, $k \geq 2$, with $(\mathrm{HFC})_2 = \mathrm{HFC}$ is introduced (see Definition 5.1), which is weaker than HFC and is also finitely checkable. $(\mathrm{HFC})_k$ is closely related to the extension property called square filling [41, 42]. The main theorem for strong specification is given by Theorem 5.4.

Theorem 1.2. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, if there exists $k \geq 2$ such that*

(i) *$\mathcal{B}$ is k crisscross-extendable,*
(ii) *$\mathcal{B}$ satisfies $(HFC)_k$ with size (M, N) for some $M, N \geq 2k - 3$,*
(iii) *$\mathbb{H}_k$ is $(M - 2k + 5)$-primitive and $\mathbb{V}_k$ is $(N - 2k + 5)$-primitive,*

then $\Sigma(\mathcal{B})$ has strong specification.

Theorems 1.1 and 1.2 are very powerful in verifying mixing properties. The most known results for strong specification and topological mixing in the literature can be checked successfully by them. Their significance in classification of mixing properties is discussed as follows. Previously, Boyle et al. [12] discussed various mixing properties, including strong irreducibility, uniform filling property, corner gluing, block gluing and topological mixing (see Definition 3.17). The range of

$(\mathrm{HFC})_k$ and these mixing properties for $\mathbb{Z}^2$ shifts of finite type are listed in Figure 1.5. For brevity, we use the following notations.

(a) : $\mathcal{B}$ satisfies $(\mathrm{HFC})_k$ (i.e., condition (ii) of Theorem 1.2) and conditions (i) and (iii) of Theorem 1.2,
(b) : $\Sigma(\mathcal{B})$ has strong specification,
(c) : $\Sigma(\mathcal{B})$ has the UFP,
(d) : $\Sigma(\mathcal{B})$ is corner gluing,
(e) : $\Sigma(\mathcal{B})$ is block gluing,
(f) : $\Sigma(\mathcal{B})$ is topologically mixing.

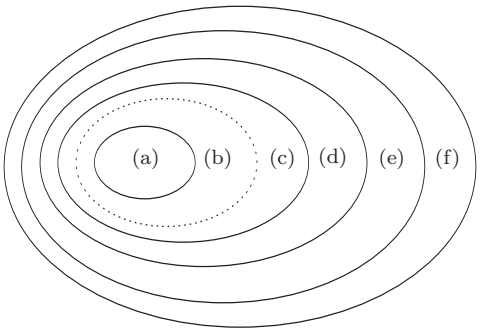


Figure 1.5.

Notably, the solid line indicates that the inner property is strictly stronger than the outer one; the dotted line indicates that inner property is stronger than or equivalent to the outer one. Example 6.8 in this paper is an example for $(b) \not\Rightarrow (a)$, and the examples for $(d) \not\Rightarrow (c)$ and $(e) \not\Rightarrow (d)$ were given by Boyle et al. [12]. It is still unsolved whether or not $(b) \Leftarrow (c)$; see [12]. Note that strong specification, periodic specification and strong irreducibility are all equivalent; see Remark 3.18.

In viewing Fig. 1.5, Theorem 1.1 and primitive results in Section 4 ensure the weakest case–topological mixing–holds. Theorem 1.2 ensures the strongest case–strong specification–holds.

In studying both topological mixing and strong specification, the transition matrices $\mathbb{H}_n$ and $\mathbb{V}_n$ and the connecting operator $\mathbb{S}_m$ or $\mathbb{C}_m$, introduced in Section 2, are extensively used. Indeed, invariant diagonal cycles, primitive commutative cycles and $(\mathrm{HFC})_k$ can be expressed in terms of transition matrices and connecting operators as the finitely checkable sufficient conditions. All cases with certain extendability conditions can be verified by using transition matrices and connecting operators except strong specification. In conclusion, the mixing properties in Fig. 1.5 with certain extendable conditions can be expressed in terms of transition matrices and connecting operators except strong specification. The related theorems are listed in Table 1.1.

On he other hand, the extendability problem is known to be undecidable [10, 21, 23, 32, 34, 46, 48, 53]. Presumably, the gluing problem is also undecidable. Therefore, it is not possible to obtain a necessary and sufficient condition which are finitely checkable for topological mixing and strong specification. However, the gluing problems in this paper are decomposed into primitivity problems of transition matrices of (1) and (4), and the extending problems of (2), (3) and (5). Although

extending problem (3) is generally undecidable, the proposed finitely checkable sufficient conditions ensure that is answered affirmatively under our situation.

Results / Mixing properties	Expressions in $\mathbb{H}$ and $\mathbb{S}$	Finite checkable sufficient conditions
Strong specification		Theorem 1.2
Uniform filling property	Theorem A.2	
Corner gluing	Theorem A.3	
Block gluing	Theorem 3.19	
Topological mixing	Theorem 1.1	Theorem 4.28

Table 1.1.

In many physical problems, edge coloring is very common. The results of vertex coloring can easily be extended to edge coloring. In particular, the six-vertex and eight-vertex ice models in statistical physics can be shown to exhibit topological mixing and strong specification, respectively. The other related cases can be treated analogously [6, 7].

The rest of this paper is organized as follows. Section 2 introduces ordering matrices of local patterns, transition matrices and connecting operators. Section 3 introduces corner-extendable conditions and crisscross-extendability to study rectangle-extendability and mixing. Section 4 introduces invariant diagonal cycles and primitive commutative cycles to establish sufficient conditions for the primitivity of $\mathbb{H}_n$ or $\mathbb{V}_n$. Section 5 introduces k hole-filling condition to develop finitely checkable conditions for strong specification. Section 6 presents the theory of edge coloring. In Appendix, we present the expressions of $(\mathrm{HFC})_k$, uniform filling property and corner gluing with rectangle-extendability by $\mathbb{H}_n$ and $\mathbb{S}_m$.

2. Preliminary

This section reviews the essential aspects of the ordering matrices of local patterns and their associated transition matrices [1]. Then, connecting operators are introduced. This study depends on more precise properties of connecting operators which were only outlined in the previous paper [2]. Most of the proofs can be obtained by the arguments similar to those in [1, 2] and are omitted.

As presented elsewhere [1], with $p \geq 2$ fixed, the ordering matrices $\mathbf{X}_n$ and $\mathbf{Y}_n$ are introduced to arrange systematically all local patterns in $\Sigma_{2\times n}(p)$ and $\Sigma_{n\times 2}(p)$, respectively.

Since the vertex coloring and face coloring are equivalent on $\mathbb{Z}^2$, in the following, the colors of patterns are drawn on faces instead of on vertices.

For an n-sequence $\overline{U}_n = (u_1, u_2, \cdots, u_n)$ with $u_k \in \mathcal{S}_p$, $1 \leq k \leq n$, $\overline{U}_n$ is assigned the number by the n-th order counting function $\psi \equiv \psi_n$:

$$\psi(\overline{U}_n) = \psi(u_1, u_2, \cdots, u_n) = 1 + \sum_{k=1}^{n} u_k p^{(n-k)}. \tag{2.1}$$

The horizontal and vertical ordering matrices $\mathbf{X}_2 = [x_{i_1,j_1}]_{p^2\times p^2}$ and $\mathbf{Y}_2 = [y_{i_2,j_2}]_{p^2\times p^2}$ are defined by

$$x_{i_1,j_1} = \begin{array}{|c|c|}\hline u_{0,1} & u_{1,1} \\ \hline u_{0,0} & u_{1,0} \\ \hline \end{array} \quad \text{and} \quad y_{i_2,j_2} = \begin{array}{|c|c|}\hline u'_{0,1} & u'_{1,1} \\ \hline u'_{0,0} & u'_{1,0} \\ \hline \end{array}, \tag{2.2}$$

where $u_{s,t}, u'_{s,t} \in \mathcal{S}_p$, $0 \le s,t \le 1$, with

$$\begin{cases} i_1 = \psi(u_{0,0}, u_{0,1}) \\ j_1 = \psi(u_{1,0}, u_{1,1}) \end{cases} \quad \text{and} \quad \begin{cases} i_2 = \psi(u'_{0,0}, u'_{1,0}) \\ j_2 = \psi(u'_{0,1}, u'_{1,1}). \end{cases}$$

For instance, when $p = 2$,

(2.3)

$$\mathbf{X}_2 = \begin{array}{c|c} & \begin{array}{cccc} \begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 0\\\hline\end{array} & \begin{array}{|c|}\hline 0\\\hline 1\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array} \end{array} \\ \begin{array}{c} \begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array} \\ \begin{array}{|c|}\hline 1\\\hline 0\\\hline\end{array} \\ \begin{array}{|c|}\hline 0\\\hline 1\\\hline\end{array} \\ \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array} \end{array} & \left[\begin{array}{cccc} \begin{array}{|c|c|}\hline 0&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&1\\\hline\end{array} \\ \begin{array}{|c|c|}\hline 1&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&1\\\hline\end{array} \\ \begin{array}{|c|c|}\hline 0&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&1\\\hline\end{array} \\ \begin{array}{|c|c|}\hline 1&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array} \end{array}\right] \end{array} \quad \text{and} \quad \mathbf{Y}_2 = \begin{array}{c|c} & \begin{array}{cccc} \begin{array}{|c|c|}\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \end{array} \\ \begin{array}{c} \begin{array}{|c|c|}\hline 0&0\\\hline\end{array} \\ \begin{array}{|c|c|}\hline 0&1\\\hline\end{array} \\ \begin{array}{|c|c|}\hline 1&0\\\hline\end{array} \\ \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \end{array} & \left[\begin{array}{cccc} \begin{array}{|c|c|}\hline 0&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&0\\\hline\end{array} \\ \begin{array}{|c|c|}\hline 0&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&1\\\hline\end{array} \\ \begin{array}{|c|c|}\hline 0&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&0\\\hline\end{array} \\ \begin{array}{|c|c|}\hline 0&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array} \end{array}\right] \end{array}.$$

Now, $\mathbf{X}_2$ and $\mathbf{Y}_2$ are closely related to each other as follows.

$$\mathbf{X}_2 = \begin{bmatrix} X_{2;1} & X_{2;2} & \cdots & X_{2;p} \\ X_{2;p+1} & X_{2;p+2} & \cdots & X_{2;2p} \\ \vdots & \vdots & \ddots & \vdots \\ X_{2;p(p-1)+1} & X_{2;p(p-1)+2} & \cdots & X_{2;p^2} \end{bmatrix}, \tag{2.4}$$

where

$$X_{2;\alpha} = \begin{bmatrix} y_{\alpha,1} & y_{\alpha,2} & \cdots & y_{\alpha,p} \\ y_{\alpha,p+1} & y_{\alpha,p+2} & \cdots & y_{\alpha,2p} \\ \vdots & \vdots & \ddots & \vdots \\ y_{\alpha,p(p-1)+1} & y_{\alpha,p(p-1)+2} & \cdots & y_{\alpha,p^2} \end{bmatrix}; \tag{2.5}$$

the case for $\mathbf{Y}_2$ is similar.

The higher-order ordering matrices $\mathbf{X}_n = [x_{n;i,j}]_{p^n\times p^n}$ of $\Sigma_{2\times n}(p)$, $n \ge 3$, are defined recursively as

$$\mathbf{X}_n = \begin{bmatrix} X_{n;1} & X_{n;2} & \cdots & X_{n;p} \\ X_{n;p+1} & X_{n;p+2} & \cdots & X_{n;2p} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n;p(p-1)+1} & X_{n;p(p-1)+2} & \cdots & X_{n;p^2} \end{bmatrix}, \tag{2.6}$$

where

(2.7)
$$X_{n;\alpha} = \begin{bmatrix} y_{\alpha,1}X_{n-1;1} & y_{\alpha,2}X_{n-1;2} & \cdots & y_{\alpha,p}X_{n-1;p} \\ y_{\alpha,p+1}X_{n-1;p+1} & y_{\alpha,p+2}X_{n-1;p+2} & \cdots & y_{\alpha,2p}X_{n-1;2p} \\ \vdots & \vdots & \ddots & \vdots \\ y_{\alpha,p(p-1)+1}X_{n-1;p(p-1)+1} & y_{\alpha,p(p-1)+2}X_{n-1;p(p-1)+2} & \cdots & y_{\alpha,p^2}X_{n-1;p^2} \end{bmatrix}$$
is a $p^{n-1} \times p^{n-1}$ matrix. Notably, the entry $x_{n;i,j}$ is the $2 \times n$ local pattern $U_{2\times n} = (u_{s,t})_{0\le s\le 1, 0\le t\le n-1}$ with

$$i = \psi(u_{0,0}, u_{0,1}, \cdots, u_{0,n-1}) \quad \text{and} \quad j = \psi(u_{1,0}, u_{1,1}, \cdots, u_{1,n-1}). \tag{2.8}$$

Similarly, the higher-order ordering matrix $\mathbf{Y}_n$ can be defined recursively, as above.

Given a basic set $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, the horizontal and vertical transition matrices $\mathbb{H}_2 = \mathbb{H}_2(B) = [h_{i,j}]_{p^2\times p^2}$ and $\mathbb{V}_2 = \mathbb{V}_2(B) = [v_{i,j}]_{p^2\times p^2}$ are given by

$$\begin{cases} h_{i,j} = 1 & \text{if } x_{i,j} \in \mathcal{B}, \\ h_{i,j} = 0 & \text{if } x_{i,j} \notin \mathcal{B}, \end{cases} \quad \text{and} \quad \begin{cases} v_{i,j} = 1 & \text{if } y_{i,j} \in \mathcal{B}, \\ v_{i,j} = 0 & \text{if } y_{i,j} \notin \mathcal{B}. \end{cases} \tag{2.9}$$

According to (2.6) and (2.7), the higher-order transition matrices $\mathbb{H}_n$, $n \ge 3$, can be defined as

$$\mathbb{H}_n = \begin{bmatrix} H_{n;1} & H_{n;2} & \cdots & H_{n;p} \\ H_{n;p+1} & H_{n;p+2} & \cdots & H_{n;2p} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n;p(p-1)+1} & H_{n;p(p-1)+2} & \cdots & H_{n;p^2} \end{bmatrix}, \tag{2.10}$$

where

(2.11)
$$H_{n;\alpha} = \begin{bmatrix} v_{\alpha,1}H_{n-1;1} & v_{\alpha,2}H_{n-1;2} & \cdots & v_{\alpha,p}H_{n-1;p} \\ v_{\alpha,p+1}H_{n-1;p+1} & v_{\alpha,p+2}H_{n-1;p+2} & \cdots & v_{\alpha,2p}H_{n-1;2p} \\ \vdots & \vdots & \ddots & \vdots \\ v_{\alpha,p(p-1)+1}H_{n-1;p(p-1)+1} & v_{\alpha,p(p-1)+2}H_{n-1;p(p-1)+2} & \cdots & v_{\alpha,p^2}H_{n-1;p^2} \end{bmatrix}$$
is a $p^{n-1} \times p^{n-1}$ zero-one matrix.

Furthermore, for any $n \ge 2$ and $q \ge 1$, $\mathbb{H}_{n+q}$ are decomposed by applying (2.10) $q+1$ times, as follows. For any $q \ge 1$ and $0 \le r \le q-1$, define

$$\begin{aligned} & H_{n+q;\beta_1;\beta_2;\cdots;\beta_{r+1}} \\ = & \begin{bmatrix} H_{n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};1} & H_{n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};2} & \cdots & H_{n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};p} \\ H_{n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};p+1} & H_{n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};p+2} & \cdots & H_{n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};2p} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};p(p-1)+1} & H_{n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};p(p-1)+2} & \cdots & H_{n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};p^2} \end{bmatrix}. \end{aligned}$$

Therefore, for any $q \ge 0$, $\mathbb{H}_{n+q}$ can be represented as a $p^{q+1} \times p^{q+1}$ matrix

$$\mathbb{H}_{n+q} \equiv [H_{n+q;i,j}]_{p^{q+1}\times p^{q+1}} = \left[H_{n+q;\beta_1;\beta_2;\cdots;\beta_{q+1}}\right]. \tag{2.12}$$

In particular, when $q = 0$,

$$\mathbb{H}_n = \begin{bmatrix} H_{n;1,1} & H_{n;1,2} & \cdots & H_{n;1,p} \\ H_{n;2,2} & H_{n;2,2} & \cdots & H_{n;2,p} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n;p,1} & H_{n;p,2} & \cdots & H_{n;p,p} \end{bmatrix}. \tag{2.13}$$

More precisely, the relation between $1 \le \beta_1, \beta_2, \cdots, \beta_{q+1} \le p$ and $1 \le i, j \le p^{q+1}$ is as follows. Given $1 \le i, j \le p^{q+1}$, choose $i_l, j_l \in \mathcal{S}_p$, $1 \le l \le q+1$, such that

$$i = \psi(i_1, i_2, \cdots, i_{q+1}) \quad \text{and} \quad j = \psi(j_1, j_2, \cdots, j_{q+1}). \tag{2.14}$$

For $1 \le l \le q+1$, let

$$\beta_l(i, j) = \psi(i_l, j_l). \tag{2.15}$$

Then,

$$H_{n+q;i,j} = H_{n+q;\beta_1(i,j);\beta_2(i,j);\cdots;\beta_{q+1}(i,j)}. \tag{2.16}$$

From (2.11), for $q \ge 1$,

$$H_{n+q;\beta_1;\beta_2;\cdots;\beta_{q+1}} = v_{\beta_1,\beta_2} v_{\beta_2,\beta_3} \cdots v_{\beta_q,\beta_{q+1}} H_{n;\beta_{q+1}} \tag{2.17}$$

can be verified. Hence, let $\mathbb{H}_{q+1} = [h_{q+1;i,j}]_{p^{q+1} \times p^{q+1}}$; by (2.12) and (2.17),

$$H_{n+q;i,j} = h_{q+1;i,j} H_{n;\beta_{q+1}(i,j)} = h_{q+1;i,j} H_{n;i'(i,j),j'(i,j)}, \tag{2.18}$$

where $1 \le i', j' \le p$. Actually, we have

$$i'(i, j) = i'(i) \quad \text{and} \quad j'(i, j) = j'(j). \tag{2.19}$$

Hence, i' (or j') depends only on i (or j).

Before showing the formula for reducing $\mathbb{H}_{n+q}$ to $\mathbb{H}_n$, two products of matrices are introduced as follows. For any two matrices $A = [a_{i,j}]$ and $B = [b_{k,l}]$, the Kronecker product (tensor product) of $A \otimes B$ is defined by

$$A \otimes B = [a_{i,j} B].$$

Next, for any two $m \times m$ matrices $C = [c_{i,j}]$ and $D = [d_{i,j}]$, where $c_{i,j}$ and $d_{i,j}$ are numbers or matrices, the Hadamard product of $C \circ D$ is defined by

$$C \circ D = [c_{i,j} \cdot d_{i,j}],$$

where the product $c_{i,j} \cdot d_{i,j}$ of $c_{i,j}$ and $d_{i,j}$ may be a multiplication between numbers, between numbers and matrices or between matrices whenever it is well-defined.

Now, from (2.18), high-order transition matrices $\mathbb{H}_{n+q}$ can be reduced to lower order transition matrices $\mathbb{H}_n$ as follows.

Proposition 2.1. *For any $n \ge 2$ and $q \ge 1$,*

$$\mathbb{H}_{n+q} = (\mathbb{H}_{q+1})_{p^{q+1} \times p^{q+1}} \circ \left(E_{p^q \times p^q} \otimes [H_{n;i',j'}]_{p \times p} \right), \tag{2.20}$$

where $E_{k \times k}$ is the $k \times k$ full matrix.

Notably, the results for $\mathbb{H}_n$ are also valid for $\mathbb{V}_n$, as easily determined by exchanging the terms $\mathbb{H}_n$ and $\mathbb{V}_n$. Therefore, for simplicity, only the results for $\mathbb{H}_n$ are presented herein.

In the following, $\mathbb{H}_{n+q}^m$ can be expressed in terms of $\mathbb{H}_n^m$ and connecting operator $\mathbb{C}_m$. This result is crucial in establishing finitely checkable conditions for the primitivity of all $\mathbb{H}_k$, $k \geq 2$.

From (2.13), for $m \geq 2$, the elementary pattern of $\mathbb{H}_n^m$ is

$$H_{n;j_1,j_2} H_{n;j_2,j_3} \cdots H_{n;j_m,j_{m+1}},$$

where $1 \leq j_s \leq p$, $1 \leq s \leq m+1$. Let

$$H_{m,n;\alpha}^{(k)} = H_{n;j_1,j_2} H_{n;j_2,j_3} \cdots H_{n;j_m,j_{m+1}}, \tag{2.21}$$

where

$$\alpha = \psi(j_1 - 1, j_{m+1} - 1) \quad \text{and} \quad k = \psi(j_2 - 1, j_3 - 1, \cdots, j_m - 1).$$

From (2.2) and (2.7), the entry $\left(H_{m,n;\alpha}^{(k)}\right)_{i,j}$ is equal to the cardinal number of the set of all $(m+1) \times n$ $\mathcal{B}$-admissible local patterns $U_{(m+1)\times n} = (u_{s,t})_{0 \leq s \leq m, 0 \leq t \leq n-1}$ with

(2.22)
$$\begin{cases} \alpha = \psi(u_{0,0}, u_{m,0}) \\ k = \psi(u_{1,0}, u_{2,0}, \cdots, u_{m-1,0}) \end{cases} \quad \text{and} \quad \begin{cases} i = \psi(u_{0,1}, u_{0,2}, \cdots, u_{0,n-1}) \\ j = \psi(u_{m,1}, u_{m,2}, \cdots, u_{m,n-1}). \end{cases}$$

Clearly, $1 \leq \alpha \leq p^2$ and $1 \leq k \leq p^{m-1}$.

Therefore, for $m \geq 2$,

$$\mathbb{H}_n^m = \begin{bmatrix} H_{m,n;1} & H_{m,n;2} & \cdots & H_{m,n;p} \\ H_{m,n;p+1} & H_{m,n;p+2} & \cdots & H_{m,n;2p} \\ \vdots & \vdots & \ddots & \vdots \\ H_{m,n;p(p-1)+1} & H_{m,n;p(p-1)+2} & \cdots & H_{m,n;p^2} \end{bmatrix}, \tag{2.23}$$

where

$$H_{m,n;\alpha} = \sum_{k=1}^{p^{m-1}} H_{m,n;\alpha}^{(k)}.$$

Furthermore, denote by

$$\widehat{H}_{m,n;\alpha} = \left(H_{m,n;\alpha}^{(k)}\right)_{1 \leq k \leq p^{m-1}}^{t} \tag{2.24}$$

a p^{m-1} column-vector that consists of all $H_{m,n;\alpha}^{(k)}$ in $H_{m,n;\alpha}$, which is very useful in deriving the reduction formula.

Now, the connecting operator $\mathbb{C}_m = [C_{m;i,j}]$ that was introduced in [2] is recalled. First, the connecting ordering matrix $\mathbf{C}_m = [\mathbf{C}_{m;i,j}]$, a different arrangement for $\Sigma_{(m+1)\times 2}(p)$ from $\mathbf{Y}_{m+1}$, is introduced. $\mathbf{C}_m = [\mathbf{C}_{m;i,j}]_{p^2 \times p^2}$, where $\mathbf{C}_{m;i,j}$ is a $p^{m-1} \times p^{m-1}$ matrix of local patterns, is defined as follows.

With fixed $1 \leq i, j \leq p^2$, for $1 \leq s, t \leq p^{m-1}$,

$$(\mathbf{C}_{m;i,j})_{s,t} = \begin{array}{|c|c|c|c|} \hline u_{0,1} & u_{1,1} & \cdots & u_{m,1} \\ \hline u_{0,0} & u_{1,0} & \cdots & u_{m,0} \\ \hline \end{array} \tag{2.25}$$

with $i = \psi(u_{0,0}, u_{0,1})$, $j = \psi(u_{m,0}, u_{m,1})$, $s = \psi(u_{1,0}, u_{2,0}, \cdots, u_{m-1,0})$ and $t = \psi(u_{1,1}, u_{2,1}, \cdots, u_{m-1,1})$.

Now, $\mathbf{C}_{m+1;i,j}$ can be obtained in terms of $\mathbf{C}_{m;k,l}$ as follows.

Proposition 2.2. *Let $\mathbf{X}_2 = [x_{i,j}]_{p^2\times p^2}$. For any $m \geq 2$ and $1 \leq i, j \leq p^2$,*

$$\mathbf{C}_{m+1;i,j} = \begin{bmatrix} x_{i,1}\mathbf{C}_{m;1,j} & x_{i,2}\mathbf{C}_{m;2,j} & \cdots & x_{i,p}\mathbf{C}_{m;p,j} \\ x_{i,p+1}\mathbf{C}_{m;p+1,j} & x_{i,p+2}\mathbf{C}_{m;p+2,j} & \cdots & x_{i,2p}\mathbf{C}_{m;2p,j} \\ \vdots & \vdots & \ddots & \ddots \\ x_{i,p(p-1)+1}\mathbf{C}_{m;p(p-1)+1,j} & x_{i,p(p-1)+2}\mathbf{C}_{m;p(p-1)+2,j} & \cdots & x_{i,p^2}\mathbf{C}_{m;p^2,j} \end{bmatrix}.$$

The matrix multiplication of $\mathbf{C}_{m;i,j}$ and $\mathbf{C}_{m;j,k}$ cannot connect local patterns in the vertical direction. However, $\mathbf{S}_{m;\alpha,\beta}$ does so. By changing the index of $\mathbf{C}_m = [\mathbf{C}_{m;i,j}]_{p^2\times p^2}$, the ordering matrix $\mathbf{S}_m = [\mathbf{S}_{m;\alpha,\beta}]_{p^2\times p^2}$ is defined by

$$\mathbf{S}_{m;\alpha,\beta} = \mathbf{C}_{m;\psi(\alpha_1,\beta_1),\psi(\alpha_2,\beta_2)}, \tag{2.26}$$

where $\alpha_k, \beta_k \in \mathcal{S}_p$, $1 \leq k \leq 2$, satisfying $\alpha = \psi(\alpha_1, \alpha_2)$ and $\beta = \psi(\beta_1, \beta_2)$. Indeed, for $1 \leq s, t \leq p^{m-1}$,

$$(\mathbf{S}_{m;\alpha,\beta})_{s,t} = \begin{array}{|c|c|c|c|} \hline u_{0,1} & u_{1,1} & \cdots & u_{m,1} \\ \hline u_{0,0} & u_{1,0} & \cdots & u_{m,0} \\ \hline \end{array} \tag{2.27}$$

with $\alpha = \psi(u_{0,0}, u_{m,0})$, $\beta = \psi(u_{0,1}, u_{m,1})$, $s = \psi(u_{1,0}, u_{2,0}, \cdots, u_{m-1,0})$ and $t = \psi(u_{1,1}, u_{2,1}, \cdots, u_{m-1,1})$. From (2.27), the matrix multiplication of $\mathbf{S}_{m;\alpha,\beta}$ and $\mathbf{S}_{m;\beta,\gamma}$ represents the vertical connection of the patterns on $\mathbb{Z}_{(m+1)\times 2}$.

Now, given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, for $m \geq 2$, the connecting operator $\mathbb{C}_m = [C_{m;i,j}]_{1\leq i,j\leq p^2}$ of $\mathbf{C}_m = [\mathbf{C}_{m;i,j}]_{1\leq i,j\leq p^2}$ is defined as follows. For $1 \leq s, t \leq p^{m-1}$,

$$\begin{cases} (C_{m;i,j})_{s,t} = 1 & \text{if } (\mathbf{C}_{m;i,j})_{s,t} \text{ is } \mathcal{B}\text{-admissible}, \\ (C_{m;i,j})_{s,t} = 0 & \text{otherwise.} \end{cases}$$

In the following, $\mathbb{C}_2$ can be obtained explicitly. For $\mathbb{H}_2 = [h_{i,j}]_{p^2\times p^2}$, define

$$\widetilde{\mathbb{H}}_2 = \begin{bmatrix} \widetilde{H}_{2;1} & \widetilde{H}_{2;2} & \cdots & \widetilde{H}_{2;p} \\ \widetilde{H}_{2;p+1} & \widetilde{H}_{2;p+2} & \cdots & \widetilde{H}_{2;2p} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{H}_{2;p(p-1)+1} & \widetilde{H}_{2;p(p-1)+2} & \cdots & \widetilde{H}_{2;p^2} \end{bmatrix}, \tag{2.28}$$

where

$$\widetilde{H}_{2;\alpha} = \begin{bmatrix} h_{1,\alpha} & h_{2,\alpha} & \cdots & h_{p,\alpha} \\ h_{p+1,\alpha} & h_{p+2,\alpha} & \cdots & h_{2p,\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ h_{p(p-1)+1,\alpha} & h_{p(p-1)+2,\alpha} & \cdots & h_{p^2,\alpha} \end{bmatrix}$$

for $1 \le \alpha \le p^2$. Then, for $1 \le i, j \le p^2$,

$$C_{2;i,j} = V_{2;i} \circ \widetilde{H}_{2;j} \tag{2.29}$$

is a $p \times p$ zero-one matrix. By Proposition 2.2, the connecting operator $\mathbb{C}_{m+1}$ can also be obtained from $\mathbb{C}_m$. For $m \ge 2$, $\mathbb{C}_{m+1} = [C_{m+1;i,j}]_{1 \le i,j \le p^2}$ satisfies

(2.30)
$$C_{m+1;i,j} = \begin{bmatrix} h_{i,1}C_{m;1,j} & h_{i,2}C_{m;2,j} & \cdots & h_{i,p}C_{m;p,j} \\ h_{i,p+1}C_{m;p+1,j} & h_{i,p+2}C_{m;p+2,j} & \cdots & h_{i,2p}C_{m;2p,j} \\ \vdots & \vdots & \ddots & \vdots \\ h_{i,p(p-1)+1}C_{m;p(p-1)+1,j} & h_{i,p(p-1)+2}C_{m;p(p-1)+2,j} & \cdots & h_{i,p^2}C_{m;p^2,j} \end{bmatrix}.$$

From (2.26), $\mathbb{S}_m = [S_{m;\alpha,\beta}]_{p^2 \times p^2}$ is defined by

$$S_{m;\alpha,\beta} = C_{m;\psi(\alpha_1,\beta_1),\psi(\alpha_2,\beta_2)}, \tag{2.31}$$

where $0 \le \alpha_1, \alpha_2, \beta_1, \beta_2 \le p-1$ such that $\alpha = \psi(\alpha_1, \alpha_2)$ and $\beta = \psi(\beta_1, \beta_2)$.

Now, the relation between $\mathbb{H}^m_{n+1}$ and $\mathbb{H}^m_n$ is elucidated as follows. Since the sizes of $H^{(k)}_{m,n+1;\alpha}$ and $H^{(l)}_{m,n;\beta}$ are different, the elementary pattern $H^{(k)}_{m,n+1;\alpha}$ can be reduced further as follows.

Let

(2.32)
$$H^{(k)}_{m,n+1;\alpha} = \begin{bmatrix} H^{(k)}_{m,n+1;\alpha;1} & H^{(k)}_{m,n+1;\alpha;2} & \cdots & H^{(k)}_{m,n+1;\alpha;p} \\ H^{(k)}_{m,n+1;\alpha;p+1} & H^{(k)}_{m,n+1;\alpha;p+2} & \cdots & H^{(k)}_{m,n+1;\alpha;2p} \\ \vdots & \vdots & \ddots & \vdots \\ H^{(k)}_{m,n+1;\alpha;p(p-1)+1} & H^{(k)}_{m,n+1;\alpha;p(p-1)+2} & \cdots & H^{(k)}_{m,n+1;\alpha;p^2} \end{bmatrix}$$

and

$$\widehat{H}_{m,n+1;\alpha;\beta} = \left(H^{(k)}_{m,n+1;\alpha;\beta}\right)^t_{1 \le k \le p^{m-1}}. \tag{2.33}$$

As (2.22), the entry $\left(H^{(k)}_{m,n+1;\alpha;\beta}\right)_{i,j}$ can be verified to be the number of the set of all $(m+1) \times (n+1)$ $\mathcal{B}$-admissible local patterns $U_{(m+1)\times(n+1)} = (u_{s,t})_{0 \le s \le m, 0 \le t \le n}$ with

(2.34)
$$\begin{cases} \alpha = \psi(u_{0,0}, u_{m,0}) \\ \beta = \psi(u_{0,1}, u_{m,1}) \\ k = \psi(u_{1,0}, u_{2,0}, \cdots, u_{m-1,0}) \end{cases} \quad \text{and} \quad \begin{cases} i = \psi(u_{0,2}, u_{0,2}, \cdots, u_{0,n}) \\ j = \psi(u_{m,2}, u_{m,2}, \cdots, u_{m,n}). \end{cases}$$

In the following proposition, $\widehat{H}_{m,n+1;\alpha;\beta}$ can be obtained as the product of $S_{m;\alpha,\beta}$ and $\widehat{H}_{m,n;\beta}$ [2], i.e., $S_{m;\alpha,\beta}$ reduces $\mathbb{H}^m_{n+1}$ to $\mathbb{H}^m_n$.

Proposition 2.3. *For any $m, n \geq 2$,*

$$\widehat{H}_{m,n+1;\alpha;\beta} = S_{m;\alpha,\beta}\widehat{H}_{m,n;\beta}. \tag{2.35}$$

Furthermore, for $n = 1$, let

$$H^{(k)}_{m,2;\alpha} = \begin{bmatrix} H^{(k)}_{m,2;\alpha;1} & H^{(k)}_{m,2;\alpha;2} & \cdots & H^{(k)}_{m,2;\alpha;p} \\ H^{(k)}_{m,2;\alpha;p+1} & H^{(k)}_{m,2;\alpha;p+2} & \cdots & H^{(k)}_{m,2;\alpha;2p} \\ \vdots & \vdots & \ddots & \vdots \\ H^{(k)}_{m,2;\alpha;p(p-1)+1} & H^{(k)}_{m,2;\alpha;p(p-1)+2} & \cdots & H^{(k)}_{m,2;\alpha;p^2} \end{bmatrix}, \tag{2.36}$$

then

$$H^{(k)}_{m,2;\alpha;\beta} = \sum_{l=1}^{p^{m-1}} \left(S_{m;\alpha,\beta}\right)_{k,l}. \tag{2.37}$$

Furthermore, for $q \geq 2$, q-many $S_{m;\alpha,\beta}$ can reduce $\mathbb{H}^m_{n+q}$ to $\mathbb{H}^m_n$ as follow.

For any positive integer $q \geq 2$, the elementary patterns of $\mathbb{H}^m_{n+q}$ can be decomposed by applying (2.32) q times. Indeed, for $q \geq 2$ and $1 \leq r \leq q-1$, define

$$\begin{aligned} & H^{(k)}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{r+1}} \\ = & \begin{bmatrix} H^{(k)}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};1} & H^{(k)}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};2} & \cdots & H^{(k)}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};p} \\ H^{(k)}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};p+1} & H^{(k)}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};p+2} & \cdots & H^{(k)}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};2p} \\ \vdots & \vdots & \ddots & \vdots \\ H^{(k)}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};p(p-1)+1} & H^{(k)}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};p(p-1)+2} & \cdots & H^{(k)}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{r+1};p^2} \end{bmatrix}. \end{aligned}$$

Therefore, for any $q \geq 1$, $\mathbb{H}^m_{n+q}$ can be represented as a $p^{q+1} \times p^{q+1}$ matrix

$$\mathbb{H}^m_{n+q} \equiv \left[H_{m,n+q;i,j}\right]_{p^{q+1}\times p^{q+1}} = \left[H_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{q+1}}\right] \tag{2.38}$$

where

$$H_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{q+1}} = \sum_{k=1}^{p^{m-1}} H^{(k)}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{q+1}}$$

is a $p^{n-1} \times p^{n-1}$ matrix. Notably, the relation between $1 \leq \beta_1, \beta_2, \cdots, \beta_{q+1} \leq p$ and $1 \leq i, j \leq p^{q+1}$ is the same as (2.14) and (2.15). Define

$$\widehat{H}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{q+1}} = \left(H^{(k)}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{q+1}}\right)^t_{1\leq k\leq p^{m-1}}.$$

As in Proposition 2.3, the elementary patterns of $\mathbb{H}^m_{n+q}$ can be expressed as the product of q-many $S_{m;\alpha,\beta}$ and the elementary patterns of $\mathbb{H}^m_n$.

Proposition 2.4. *For any $m, n \geq 2$ and $q \geq 1$,*

$$\widehat{H}_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{q+1}} = S_{m;\beta_1,\beta_2} S_{m;\beta_2,\beta_3} \cdots S_{m;\beta_q,\beta_{q+1}} \widehat{H}_{m,n;\beta_{q+1}}, \tag{2.39}$$

where $1 \leq \beta_i \leq p^2$, $1 \leq i \leq q+1$. Moreover,

$$H_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{q+1}} = \sum_{k,l=1}^{p^{m-1}} (S_{m;\beta_1,\beta_2} S_{m;\beta_2,\beta_3} \cdots S_{m;\beta_q,\beta_{q+1}})_{k,l} H^{(l)}_{m,n;\beta_{q+1}}. \tag{2.40}$$

Similarly, for $\mathbb{V}_2$, the connecting operators are denoted by $\mathbb{U}_m = [U_{m;i,j}]$ (corresponding to $\mathbb{C}_m = [C_{m;i,j}]$ for $\mathbb{H}_2$) and $\mathbb{W}_m = [W_{m;\alpha,\beta}]$ (corresponding to $\mathbb{S}_m = [S_{m;\alpha,\beta}]$ for $\mathbb{H}_2$). The arguments that hold for $\mathbb{H}_n$ are also valid for $\mathbb{V}_n$.

3. Extendability and topological mixing

This section investigates extendability and mixing of $\Sigma(\mathcal{B})$.

It is clear that primitivity of $\mathbb{H}_n$ and $\mathbb{V}_n$ for all $n \geq 2$ can be interpreted as topological mixing in horizontal and vertical directions respectively. In general, the relative position of lattices R_1 and $R_2 + \mathbf{v}$ are not located horizontally or vertically. Accordingly, mixing in the directions other than horizontal and vertical directions need to be studied. To treat these situations, rectangle-extendability, corner-extendable conditions and crisscross-extendability are introduced.

First, the rectangle-extendability of $\mathcal{B}$ is defined as follows.

Definition 3.1. *For $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, $\mathcal{B}$ is called rectangle-extendable if for every pattern $U_{m\times n} \in \Sigma_{m\times n}(\mathcal{B})$, $m, n \geq 2$, there exists $W \in \Sigma(\mathcal{B})$ such that $W\,|_{\mathbb{Z}_{m\times n}} = U_{m\times n}$.*

Previously, the importance of corners of lattices has been noticed [12, 44]. Indeed, in studying mixing properties, the concept of corner gluing was introduced by Boyle et al. [12]. Similarly, for studying rectangle-extendability and mixing, the corners of the rectangular lattice need to be studied closely. Indeed, let the L-shaped lattices $\mathbb{L}_1 = \mathbb{Z}_{3\times 3} \setminus \{(2,2)\}$, $\mathbb{L}_2 = \mathbb{Z}_{3\times 3} \setminus \{(0,2)\}$, $\mathbb{L}_3 = \mathbb{Z}_{3\times 3} \setminus \{(0,0)\}$ and $\mathbb{L}_4 = \mathbb{Z}_{3\times 3} \setminus \{(2,0)\}$, that is,

$\mathbb{L}_1 =$  , $\mathbb{L}_2 =$  , $\mathbb{L}_3 =$  , $\mathbb{L}_4 =$  .

Figure 3.1.

Definition 3.2. *Let $\mathcal{B} \subset \Sigma_{2\times 2}(p)$. For $1 \leq i \leq 4$, $\mathcal{B}$ satisfies corner-extendable condition $C(i)$ if for any $U \in \Sigma_{\mathbb{L}_i}(\mathcal{B})$, there exists $U' \in \Sigma_{3\times 3}(\mathcal{B})$ such that $U'\,|_{\mathbb{L}_i} = U$.*

Whether or not $\mathbb{H}_2(\mathcal{B})$ or $\mathbb{V}_2(\mathcal{B})$ contains a zero row or a zero column are very much different in studying mixing problem. We begin with the study when there is no zero row and column. We need the following notation.

Definition 3.3. *A matrix $A = [a_{i,j}]_{n\times n}$ is called compressible if contains a zero row or a zero column. A matrix is non-compressible if it is not compressible. An $\mathbb{H}_2$ (or $\mathbb{V}_2$) is degenerated if $H_{2;\alpha}$ (or $V_{2;\alpha}$) is compressible for some $1 \leq \alpha \leq p^2$. An $\mathbb{H}_2$ (or $\mathbb{V}_2$) is non-degenerated if it is not degenerated.*

First, consider the case for $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ when $\mathbb{H}_2(\mathcal{B})$ and $\mathbb{V}_2(\mathcal{B})$ are non-degenerated.

Clearly, if both A and B are non-negative and non-compressible matrices, then AB is non-compressible. From (2.10), (2.11), (2.13) and (2.21), the following result is easily obtained, and the similar result for $\mathbb{V}_2$ also holds.

Proposition 3.4. *If $\mathbb{H}_2$ is non-degenerated, then $H_{n;\alpha}$ are non-compressible for $n \geq 2$ and $1 \leq \alpha \leq p^2$. In particular, $\mathbb{H}_n$ is non-compressible for all $n \geq 2$. Moreover, $H^{(k)}_{m,n;\alpha}$ are also non-compressible for $m, n \geq 2$, $1 \leq \alpha \leq p^2$ and $1 \leq k \leq p^{m-1}$.*

The following lemma is easily proven and the proof is omitted.

Lemma 3.5. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, if $\mathbb{H}_n(\mathcal{B})$ and $\mathbb{V}_n(\mathcal{B})$ are non-compressible for all $n \geq 2$, then $\mathcal{B}$ is rectangle-extendable.*

The non-degeneracy of $\mathbb{H}_2(\mathcal{B})$ and $\mathbb{V}_2(\mathcal{B})$ implies rectangle-extendability and, moreover, three of the corner-extendable conditions, as follows.

Theorem 3.6. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, if $\mathbb{H}_2(\mathcal{B})$ and $\mathbb{V}_2(\mathcal{B})$ are non-degenerated, then*

(i) *$\mathcal{B}$ is rectangle-extendable,*
(ii) *$\mathcal{B}$ satisfies $C(1)$, $C(2)$ and $C(4)$.*

Proof. (i) is obtained directly from Proposition 3.4 and Lemma 3.5.

(ii) Since $\mathbb{H}_2(\mathcal{B})$ is non-degenerated. From (2.2), for any $u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1} \in \mathcal{S}_p$, there exist $a, b \in \mathcal{S}_p$ such that

$$\begin{array}{|c|c|}\hline u_{0,1} & a \\ \hline u_{0,0} & u_{1,0} \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline b & u_{1,1} \\ \hline u_{0,0} & u_{1,0} \\ \hline \end{array}$$

Figure 3.2.

are in $\mathcal{B}$, which implies that conditions $C(1)$ and $C(2)$ are satisfied. Similarly, that $\mathbb{V}_2(\mathcal{B})$ is non-degenerated implies that $\mathcal{B}$ satisfies conditions $C(1)$ and $C(4)$.

The proof is complete. □

Now, $\Sigma(\mathcal{B})$ is mixing follows from the non-degeneracy of $\mathbb{H}_2(\mathcal{B})$ and $\mathbb{V}_2(\mathcal{B})$ and primitivity of $\mathbb{H}_n$ and $\mathbb{V}_n$, $n \geq 2$.

Theorem 3.7. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, if $\mathbb{H}_2(\mathcal{B})$ and $\mathbb{V}_2(\mathcal{B})$ are non-degenerated, then the following statements are equivalent.*

(i) *$\mathbb{H}_n(\mathcal{B})$ and $\mathbb{V}_n(\mathcal{B})$ are primitive for all $n \geq 2$.*
(ii) *$\Sigma(\mathcal{B})$ is mixing.*

Proof. (i)⇒(ii). Let R_1 and R_2 be finite sublattices of $\mathbb{Z}^2$. Then, there exist $N \geq 2$ and (i_1, j_1),$(i_2, j_2) \in \mathbb{Z}^2$ such that $R_l \subset \mathbb{Z}_{N\times N}((i_l, j_l))$, $l = 1, 2$. From (i), there exists $K \geq 1$ such that $\mathbb{H}^K_N(\mathcal{B}) > 0$ and $\mathbb{V}^K_N(\mathcal{B}) > 0$.

Then, take $M = M(R_1, R_2) = \sqrt{2}(2N + K - 2)$. Let $\mathbf{v} = (v_1, v_2) \in \mathbb{Z}^2$ with $d(R_1, R_2 + \mathbf{v}) \geq M$ and any two allowable patterns $U_1 \in \Pi_{R_1}(\Sigma(\mathcal{B}))$ and $U_2 \in \Pi_{R_2+\mathbf{v}}(\Sigma(\mathcal{B}))$. Clearly, U_1 and U_2 can be extended as U'_1 on $\mathbb{Z}_{N\times N}((i_1, j_1))$ and U'_2 on $\mathbb{Z}_{N\times N}((i_2 + v_1, j_2 + v_2))$ by using the local patterns in $\mathcal{B}$, respectively.

It is not difficult to prove that U_1' and U_2' can be connected as the L-shaped pattern U_L by using the local patterns in $\mathcal{B}$, as follows.

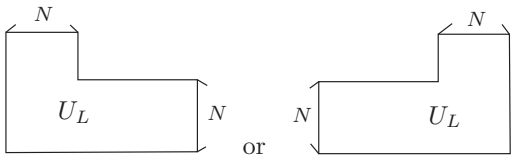


Figure 3.3.

Notably, the L-shaped lattices may degenerate into rectangular lattices.

Since $\mathbb{H}_2(\mathcal{B})$ and $\mathbb{V}_2(\mathcal{B})$ are non-degenerated, by Theorem 3.6, $\mathcal{B}$ satisfies conditions $C(1)$ and $C(2)$. Then, U_L can be extended as U_r on the rectangular lattice by using the local patterns in $\mathcal{B}$, which is obtained by filling the corner of the L-shaped lattices.

From Theorem 3.6, $\mathcal{B}$ is rectangle-extendable. Then, U_r can be extended as $W \in \Sigma(\mathcal{B})$ with $\Pi_{R_1}(W) = U_1$ and $\Pi_{R_2+\mathbf{v}}(W) = U_2$. Therefore, $\Sigma(\mathcal{B})$ is mixing.

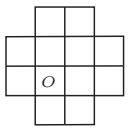
(ii)$\Rightarrow$(i). From Proposition 3.4, $\mathbb{H}_n$ and $\mathbb{V}_n$ are non-compressible for all $n \geq 2$. Then, for $n \geq 2$, any pattern in $\Sigma_{1\times n}(p)$ or $\Sigma_{n\times 1}(p)$ can be extended to $\mathbb{Z}^2$ by using the local patterns in $\mathcal{B}$. It can be easily verified that (ii)$\Rightarrow$(i); the details are omitted. The proof is complete.

□

Next, consider the case for $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ when $\mathbb{H}_2(\mathcal{B})$ or $\mathbb{V}_2(\mathcal{B})$ is degenerated. Theorems 3.6 and 3.7 are to be generalized when $\mathcal{B}$ satisfies corner-extendable conditions and crisscross-extendability, which is introduced as follows. The crisscross lattice $\mathbb{Z}_+$ is defined by

$$\mathbb{Z}_+ = \bigcup_{0\leq |i|+|j|\leq 1} \mathbb{Z}_{2\times 2}((i,j)), \tag{3.1}$$

Indeed,

$\mathbb{Z}_+ =$ 

,

Figure 3.4.

where $O = (0,0)$ is the origin of $\mathbb{Z}^2$. For $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, denote by

$$\Sigma_+(\mathcal{B}) = \Sigma_{\mathbb{Z}_+}(\mathcal{B}).$$

Later, the extendability of $B \in \mathcal{B}$ on $\mathbb{Z}^2$ will be reduced to study the extendability on $\mathbb{Z}_+$. First, the subset $\mathcal{B}_c$ of $\mathcal{B}$, which is the collection of all crisscross-extendable patterns in $\mathcal{B}$, is introduced.

Definition 3.8. *For $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, $\mathcal{B}_c = \mathcal{B}_c(\mathcal{B})$ is the maximal subset of $\mathcal{B}$ such that if $B \in \mathcal{B}_c$, there exists $U_+ \in \Sigma_+(\mathcal{B})$ with $U_+ |_{\mathbb{Z}_{2\times 2}} = B$.*

Clearly, $\mathcal{B}_c$ can be obtained through the following finite processes.

Proposition 3.9. *For $\mathcal{B} \subset \Sigma_{2\times 2}(p)$,*

$$\mathcal{B}_c = \mathcal{B} \setminus \left(\mathcal{N}_h(\mathcal{B}) \bigcup \mathcal{N}_v(\mathcal{B})\right), \tag{3.2}$$

where

$$\mathcal{N}_h(\mathcal{B}) = \left\{ x_{i,j} \in \mathcal{B} \mid \sum_{k=1}^{p^2} h_{k,i} = 0 \ or \sum_{k=1}^{p^2} h_{j,k} = 0 \right\} \tag{3.3}$$

and

$$\mathcal{N}_v(\mathcal{B}) = \left\{ y_{i,j} \in \mathcal{B} \mid \sum_{k=1}^{p^2} v_{k,i} = 0 \ or \sum_{k=1}^{p^2} v_{j,k} = 0 \right\}. \tag{3.4}$$

That the shift spaces $\Sigma(\mathcal{B})$ and $\Sigma(\mathcal{B}_c)$ are equal is proven as follows.

Proposition 3.10. *For $\mathcal{B} \subset \Sigma_{2\times 2}(p)$,*

$$\Sigma(\mathcal{B}_c) = \Sigma(\mathcal{B}). \tag{3.5}$$

In particular, $\Sigma(\mathcal{B}_c)$ is mixing if and only if $\Sigma(\mathcal{B})$ is mixing.

Proof. Clearly, $\Sigma(\mathcal{B}_c) \subseteq \Sigma(\mathcal{B})$.

Suppose that $\Sigma(\mathcal{B}_c) \not\supseteq \Sigma(\mathcal{B})$. Then, there exists $U \in \Sigma(\mathcal{B})$ but $U \notin \Sigma(\mathcal{B}_c)$, that is, $U\mid_{\mathbb{Z}_{2\times 2}((i',j'))} \in (\mathcal{B} \setminus \mathcal{B}_c)$ for some $(i', j') \in \mathbb{Z}^2$. Since $U \in \Sigma(\mathcal{B})$, from the definition of $\mathcal{B}_c$, we have $U\mid_{\mathbb{Z}_{2\times 2}((i',j'))} \in \mathcal{B}_c$. This leads a contradiction. Thus, $\Sigma(\mathcal{B}_c) \supseteq \Sigma(\mathcal{B})$.

The proof is complete. □

The following notation is important in studying rectangle-extendability and mixing for $\Sigma(\mathcal{B})$ when $\mathbb{H}_2(\mathcal{B})$ or $\mathbb{V}_2(\mathcal{B})$ is degenerated.

Definition 3.11. *A basic set $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ is called crisscross-extendable if $\mathcal{B}_c(\mathcal{B}) = \mathcal{B}$, that is, for every $B \in \mathcal{B}$, there exists $U_+ \in \Sigma_+(\mathcal{B})$ with $U_+\mid_{\mathbb{Z}_{2\times 2}} = B$.*

When $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ is not crisscross-extendable, the maximal crisscross-extendable subset $\mathcal{B}^*$ of $\mathcal{B}$ can be obtained as follows. Indeed, let $\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{B}_n = \mathcal{B}_c(\mathcal{B}_{n-1})$ for all $n \geq 1$. Clearly, $\mathcal{B}_n \subseteq \mathcal{B}_{n-1}$ for all $n \geq 1$. We have that the size of $\mathbb{H}_2(\mathcal{B})$ and $\mathbb{V}_2(\mathcal{B})$ is $p^2 \times p^2$. Define

$$\mathcal{B}^* = \mathcal{B}^*(\mathcal{B}) \equiv \bigcap_{n=0}^{\infty} \mathcal{B}_n = \bigcap_{n=0}^{2p^2} \mathcal{B}_n. \tag{3.6}$$

From the construction of $\mathcal{B}^*$, by Proposition 3.10, the following result can be obtained easily and the proof is omitted.

Proposition 3.12. *For $\mathcal{B} \subset \Sigma_{2\times 2}(p)$,*
(i) *$\mathcal{B}^*$ is crisscross-extendable,*
(ii) *$\Sigma(\mathcal{B}^*) = \Sigma(\mathcal{B})$.*

Therefore, if $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ is not crisscross-extendable, then the discussion concerning rectangle-extendability and mixing of $\Sigma(\mathcal{B})$ is the same as that of $\Sigma(\mathcal{B}^*)$. Hereafter, $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ is always assumed to be crisscross-extendable when $\mathbb{H}_2(\mathcal{B})$ or $\mathbb{V}_2(\mathcal{B})$ is degenerated.

Proposition 3.13. *If $\mathcal{B}$ satisfies either $C(1)$ and $C(3)$ or $C(2)$ and $C(4)$, then the following statements are equivalent.*
(i) *$\mathcal{B}$ is rectangle-extendable.*
(ii) *$\mathcal{B}$ is crisscross-extendable.*

Proof. Clearly, (i) implies (ii).

(ii)⇒(i). Assume that $\mathcal{B}$ satisfies $C(1)$ and $C(3)$. The case in which it satisfies $C(2)$ and $C(4)$ is similar. Let $U_{m\times n} \in \Sigma_{m\times n}(\mathcal{B})$, $m, n \geq 2$. Since $\mathcal{B}$ satisfies $C(1)$ and $C(3)$, from (ii), $U_{m\times n}$ can be extended in both positive and and negative vertical directions by using the local patterns in $\mathcal{B}$, as follows.

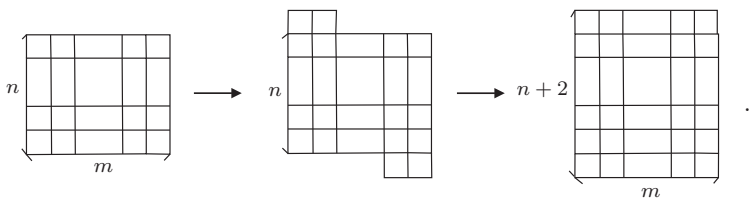


Figure 3.5.

Similarly, the above pattern can be extended in both positive horizontal and negative horizontal directions by using the local patterns in $\mathcal{B}$. Therefore, by the above method, $U_{m\times n}$ can be extended to $\mathbb{Z}^2$ by using the local patterns in $\mathcal{B}$. The proof is complete. □

Theorem 3.7 can now be generalized. Indeed, the following theorem shows that primitivity and mixing are equivalent when $\mathcal{B}$ is crisscross-extendable and satisfies the corner-extendable conditions.

Theorem 3.14. *If*

(i) *$\mathcal{B} \subset \Sigma_{2\times 2}(p)$ is crisscross-extendable,*
(ii) *$\mathcal{B}$ satisfies three of corner-extendable conditions $C(i)$, $1 \leq i \leq 4$,*

then $\mathbb{H}_n(\mathcal{B})$ and $\mathbb{V}_n(\mathcal{B})$ are primitive for all $n \geq 2$ if and only if $\Sigma(\mathcal{B})$ is mixing.

Proof. (⇒). From (ii), without loss of generality, assume that $\mathcal{B}$ satisfies conditions $C(1)$, $C(2)$ and $C(3)$.

Let R_1 and R_2 be finite sublattices of $\mathbb{Z}^2$. Since $\mathcal{B}$ satisfies $C(1)$ and $C(2)$, as in the proof of Theorem 3.7, there exists $M(R_1, R_2) \geq 1$ such that for all $\mathbf{v} = (v_1, v_2) \in \mathbb{Z}^2$ with $d(R_1, R_2 + \mathbf{v}) \geq M$ and any two allowable patterns $U_1 \in \Pi_{R_1}(\Sigma(\mathcal{B}))$ and $U_2 \in \Pi_{R_2+\mathbf{v}}(\Sigma(\mathcal{B}))$, U_1 and U_2 can be extended as U_r on the rectangular lattice by using the local patterns in $\mathcal{B}$.

Since $\mathcal{B}$ is crisscross-extendable and satisfies conditions $C(1)$ and $C(3)$, by Proposition 3.13, $\mathcal{B}$ is rectangle-extendable. Then, U_r can be extended as $W \in \Sigma(\mathcal{B})$ with $\Pi_{R_1}(W) = U_1$ and $\Pi_{R_2+\mathbf{v}}(W) = U_2$. Therefore, $\Sigma(\mathcal{B})$ is mixing.

(⇐). From (i) and (ii), by Proposition 3.13, $\mathcal{B}$ is rectangle-extendable. Then, for $n \geq 2$, any pattern in $\Sigma_{2\times n}(\mathcal{B})$ can be extended to $\mathbb{Z}^2$ by using the local patterns

in $\mathcal{B}$. Therefore, that $\Sigma(\mathcal{B})$ is mixing implies that $\mathbb{H}_n(\mathcal{B})$ is primitive for all $n \geq 2$. Similarly, $\mathbb{V}_n(\mathcal{B})$ is primitive for all $n \geq 2$.

The proof is complete. □

The following example demonstrates that the corner-extendable conditions of Theorem 3.14 are crucial: if corner-extendable conditions fail, then crisscross-extendability (or rectangle-extendability) and primitivity may not imply mixing.

Example 3.15. *Let*

$$\mathcal{B}_{\pi/4} = \left\{ \begin{array}{|c|c|}\hline u_2 & u_4 \\ \hline u_1 & u_3 \\ \hline \end{array} : u_4 \geq u_1 \text{ and } u_1, u_2, u_3, u_4 \in \{0,1\} \right\}.$$

Clearly,

$$\mathbb{H}_2(\mathcal{B}_{\pi/4}) = \mathbb{V}_2(\mathcal{B}_{\pi/4}) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

From (2.10) and (2.11), it can be verified that $\mathbb{H}_n$ is non-compressible for all $n \geq 2$. Since $\mathbb{V}_2 = \mathbb{H}_2$, $\mathbb{V}_n$ is also non-compressible for all $n \geq 2$. By Lemma 3.5, $\mathcal{B}_{\pi/4}$ is rectangle-extendable. In particular, $\mathcal{B}_{\pi/4}$ is crisscross-extendable.

For $n \geq 2$, any two $1 \times n$ local patterns $U_{1\times n} = (u_{0,j})_{0\leq j\leq n-1} = (u_j)_{0\leq j\leq n-1}$ and $U'_{1\times n} = (u'_{0,j})_{0\leq j\leq n-1} = (u'_j)_{0\leq j\leq n-1}$, $u_j, u'_j \in \{0,1\}$, $0 \leq j \leq n-1$, can be connected in the horizontal direction by using the local patterns in $\mathcal{B}_{\pi/4}$, as follows.

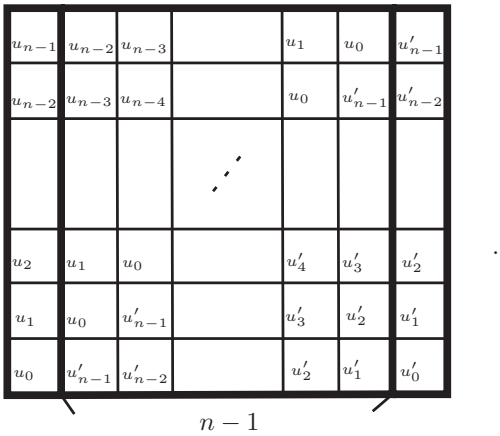


Figure 3.6.

Then, $\mathbb{H}_n^n > 0$ for all $n \geq 2$. Since $\mathbb{V}_2 = \mathbb{H}_2$, $\mathbb{V}_n^n = \mathbb{H}_n^n > 0$ for all $n \geq 2$.

Let $R_1 = R_2 = \mathbb{Z}_{2\times 2}$. Consider $U_0 = \{0\}^{\mathbb{Z}^2}$ and $U_1 = \{1\}^{\mathbb{Z}^2}$. Clearly, $U_0, U_1 \in \Sigma(\mathcal{B}_{\pi/4})$, but $\Pi_{R_1}(U_1)$ cannot connect with $\Pi_{R_2+(i,i)}(U_0) = \Pi_{\mathbb{Z}_{2\times 2}((i,i))}(U_0)$ by using the local patterns in $\mathcal{B}_{\pi/4}$ for all $i \geq 2$. Therefore, $\Sigma(\mathcal{B}_{\pi/4})$ is not mixing. This claim does not contradict Theorem 3.14 since $\mathcal{B}_{\pi/4}$ does not satisfy conditions $C(2)$

and $C(4)$: neither

$$\begin{array}{|c|c|c|}\hline & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \in \Sigma_{\mathbb{L}_2}(\mathcal{B}_{\pi/4}) \qquad \textit{nor} \qquad \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 1 & \\ \hline \end{array} \in \Sigma_{\mathbb{L}_4}(\mathcal{B}_{\pi/4})$$

can be extended to $\mathbb{Z}_{3\times 3}$ by using the local patterns in $\mathcal{B}_{\pi/4}$.

The following corollary follows directly from the proof of Theorem 3.14 and is useful for checking that $\Sigma(\mathcal{B})$ is not mixing.

Corollary 3.16. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, assume that $\mathcal{B}$ is rectangle-extendable. If there exists $N \geq 2$ such that $\mathbb{H}_N(\mathcal{B})$ or $\mathbb{V}_N(\mathcal{B})$ is not primitive, then $\Sigma(\mathcal{B})$ is not mixing.*

At the end of this section, we recall and compare various mixing properties which are described in the introduction.

Definition 3.17. *Suppose Σ is a $\mathbb{Z}^2$ shift.*

(i) *Σ has the uniform filling property (UFP) if a number $M(\Sigma) \geq 1$ exists such that for any two allowable patterns $U_1 \in \Pi_{R_1}(\Sigma)$ and $U_2 \in \Pi_{R_2}(\Sigma)$ with $d(R_1, R_2) \geq M$, where $R_1 = \mathbb{Z}_{m\times n}((i,j))$, $m, n \geq 1$ and $(i,j) \in \mathbb{Z}^2$, and $R_2 \subset \mathbb{Z}^2$, there exists a global pattern $W \in \Sigma$ with $\Pi_{R_1}(W) = U_1$ and $\Pi_{R_2}(W) = U_2$.*

(ii) *Σ is strongly irreducible if a number $M(\Sigma) \geq 1$ exists such that for any two allowable patterns $U_1 \in \Pi_{R_1}(\Sigma)$ and $U_2 \in \Pi_{R_2}(\Sigma)$ with $d(R_1, R_2) \geq M$, where $R_1 \subset \mathbb{Z}^2$ is finite and $R_2 \subset \mathbb{Z}^2$, there exists a global pattern $W \in \Sigma$ with $\Pi_{R_1}(W) = U_1$ and $\Pi_{R_2}(W) = U_2$.*

(iii) *Σ is corner gluing if a number $M(\Sigma) \geq 1$ exists such that for any two allowable patterns $U_1 \in \Pi_{R_1}(\Sigma)$ and $U_2 \in \Pi_{R_2}(\Sigma)$ with $d(R_1, R_2) \geq M$, where $R_1 = \mathbb{Z}_{m\times n}((i,j))$, $m, n \geq 1$ and $(i,j) \in \mathbb{Z}^2$, and $R_2 = \mathbb{Z}_{m_1\times n_1}((i+m-m_1, j+n-n_1)) \setminus \mathbb{Z}_{m_2\times n_2}((i+m-m_2, j+n-n_2))$, $m_1 > m_2 \geq m+M$ and $n_1 > n_2 \geq n+M$, there exists a global pattern $W \in \Sigma$ with $\Pi_{R_1}(W) = U_1$ and $\Pi_{R_2}(W) = U_2$.*

(iv) *Σ is block gluing if a number $M(\Sigma) \geq 1$ exists such that for any two allowable patterns $U_1 \in \Pi_{R_1}(\Sigma)$ and $U_2 \in \Pi_{R_2}(\Sigma)$ with $d(R_1, R_2) \geq M$, where $R_1 = \mathbb{Z}_{m_1\times n_1}((i_1, j_1))$ and $R_2 = \mathbb{Z}_{m_2\times n_2}((i_2, j_2))$, $m_l, n_l \geq 1$ and $(i_l, j_l) \in \mathbb{Z}^2$, $l \in \{1, 2\}$, there exists a global pattern $W \in \Sigma$ with $\Pi_{R_1}(W) = U_1$ and $\Pi_{R_2}(W) = U_2$.*

(v) *Σ has periodic specification if a number $M(\Sigma) \geq 1$ exists such that for any two allowable patterns $U_1 \in \Pi_{R_1}(\Sigma)$ and $U_2 \in \Pi_{R_2}(\Sigma)$ with $d(R_1, R_2) \geq M$, where $R_1, R_2 \subset \mathbb{Z}^2$ are finite, there exists a periodic pattern $W \in \mathcal{P}_{m\times n}(\Sigma)$, $m, n \geq 1$, with $\Pi_{R_1}(W) = U_1$ and $\Pi_{R_2}(W) = U_2$, where $\mathcal{P}_{m\times n}(\Sigma)$ is the set of all periodic patterns in Σ with period m in horizontal direction and period n in vertical direction.*

Notably, (i)$\sim$(iv) are introduced in Boyle et al. [12] and (iv) is newly introduced.

Remark 3.18. *Boyle et al.* [12] *showed that the strong irreducibility for both R_1 and R_2 are finite is equivalent to R_1 is finite and R_2 is arbitrary. Hence, it is clear that periodic specification implies strong irreducibility. By a similar argument as in Ward* [54] *in which $\Sigma(\mathcal{B})$ with strong specification has dense periodic patterns, it can be shown that strong specification for $\Sigma(\mathcal{B})$ implies periodic specification for $\Sigma(\mathcal{B})$. Recently, Ceccherini-Silberstein and Coornaert* [15, 16] *proved that strong specification is equivalent to strong irreducibility. Therefore, strong specification, periodic specification and strong irreducibility for $\Sigma(\mathcal{B})$ are all equivalent.*

As Theorem 3.14, it can be proven that N-primitivity and block gluing are equivalent when $\mathcal{B}$ is rectangle-extendable. For brevity, the proof is omitted

Theorem 3.19. *If $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ is rectangle-extendable, then the following statements are equivalent.*
(i) *there exists $N \geq 1$ such that $\mathbb{H}_n$ and $\mathbb{V}_n$ are N-primitive for all $n \geq 2$*
(ii) *$\Sigma(\mathcal{B})$ is block gluing.*

4. Invariant diagonal cycles and primitive commutative cycles

This section introduces invariant diagonal cycles and primitive commutative cycles to provide finitely checkable conditions for the primitivity of $\mathbb{H}_n$ or $\mathbb{V}_n$ for $n \geq 2$. For brevity, the discussion for $\mathbb{H}_n$ is more addressed. The discussion for $\mathbb{V}_n$ is similar to that for $\mathbb{H}_n$.

4.1. **Invariant diagonal cycles.** This subsection introduces invariant diagonal cycles to provide finitely checkable conditions for the primitivity of $\mathbb{H}_n$ or $\mathbb{V}_n$.

First, the diagonal index set is defined by

$$\mathcal{D}_p = \{1 + j(p+1) | j \in \mathcal{S}_p\}.$$

Clearly, if $\beta_1, \beta_2, \cdots, \beta_{q+1} \in \mathcal{D}_p$, then $H_{m,n+q;\beta_1;\beta_2;\cdots;\beta_{q+1}}$ lies on the diagonal of $\mathbb{H}^m_{n+q}$ in (2.38).

Definition 4.1.
(i) *For $q \geq 1$, a finite sequence $\overline{\beta}_q = \beta_1\beta_2\cdots\beta_q\beta_1$ is called a diagonal cycle with length q if $\beta_j \in \mathcal{D}_p$ for $1 \leq j \leq q$.*
(ii) *A diagonal cycle $\overline{\beta}_q = \beta_1\beta_2\cdots\beta_q\beta_1$ is called an S-invariant diagonal cycle of order (m,q) if there exist $m \geq 2$ and an invariant index set $\mathcal{K} \subseteq \{1, 2, \cdots, p^{m-1}\}$ such that*

$$\sum_{k\in\mathcal{K}} \left(S_{m;\beta_1,\beta_2} S_{m;\beta_2,\beta_3} \cdots S_{m;\beta_q,\beta_1}\right)_{k,l} \geq 1 \tag{4.1}$$

for all $l \in \mathcal{K}$.
(iii) *A diagonal cycle $\overline{\beta}_q = \beta_1\beta_2\cdots\beta_q\beta_1$ is called a W-invariant diagonal cycle of order (m,q) if there exist $m \geq 2$ and an invariant index set $\mathcal{K} \subseteq \{1, 2, \cdots, p^{m-1}\}$ such that*

$$\sum_{k\in\mathcal{K}} \left(W_{m;\beta_1,\beta_2} W_{m;\beta_2,\beta_3} \cdots W_{m;\beta_q,\beta_1}\right)_{k,l} \geq 1 \tag{4.2}$$

for all $l \in \mathcal{K}$.

Notably, it is easy to show that for any $n \geq 1$,

$$\sum_{k\in\mathcal{K}} \left((S_{m;\beta_1,\beta_2} S_{m;\beta_2,\beta_3} \cdots S_{m;\beta_q,\beta_1})^n\right)_{k,l} \geq 1 \tag{4.3}$$

for all $l \in \mathcal{K}$ if (4.1) holds. The case for W-invariant diagonal cycles is similar.

In the case of non-degeneracy, the following simpler sufficient condition for primitivity holds. The proof is postponed until after Theorem 4.12 which treats more general situation.

Theorem 4.2. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, if*

(i) $\mathbb{H}_2(\mathcal{B})$ *is non-degenerated,*

(ii) *there exists an S-invariant diagonal cycle $\overline{\beta}_{\bar{q}} = \bar{\beta}_1\bar{\beta}_2\cdots\bar{\beta}_{\bar{q}}\bar{\beta}_1$ of order $(\bar{m},\bar{q})$ with its invariant index set $\mathcal{K}$,*

(iii) $\sum_{l\in\mathcal{K}} H^{(l)}_{\bar{m},n;\bar{\beta}_1}$ *is primitive for $2 \le n \le \bar{q}+1$,*

then $\mathbb{H}_n$ is primitive for all $n \ge 2$. Similarly, if

(i)' $\mathbb{V}_2(\mathcal{B})$ *is non-degenerated,*

(ii)' *there exists a W-invariant diagonal cycle $\overline{\beta}_{\bar{q}} = \bar{\beta}_1\bar{\beta}_2\cdots\bar{\beta}_{\bar{q}}\bar{\beta}_1$ of order $(\bar{m},\bar{q})$ with its invariant index set $\mathcal{K}$,*

(iii)' $\sum_{l\in\mathcal{K}} V^{(l)}_{\bar{m},n;\bar{\beta}_1}$ *is primitive for $2 \le n \le \bar{q}+1$,*

then $\mathbb{V}_n$ is primitive for all $n \ge 2$. Furthermore, if (i)$\sim$(iii) and (i)'$\sim$(iii)' hold, then $\Sigma(\mathcal{B})$ is mixing.

The following example illustrates the application of Theorem 4.2.

Example 4.3. *Consider*

$$\mathbb{H}_2(\mathcal{B}) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Clearly, $\mathbb{H}_2$ is non-degenerated. From (2.30),

$$S_{3;1,1} = C_{3;1,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let $\overline{\beta}_1 = 11$ and $\mathcal{K} = \{3,4\}$. Since

$$\sum_{k\in\mathcal{K}} (S_{3;1,1})_{k,l} \ge 1$$

for $l \in \mathcal{K}$, $\overline{\beta}_1$ is an S-invariant diagonal cycle of order $(3,1)$ with index set $\mathcal{K}$. Clearly,

$$\sum_{l\in\mathcal{K}} H^{(l)}_{3,2;1} = H_{2;1,2}H_{2;2,1}H_{2;1,1} + H_{2;1,2}H_{2;22}H_{2;21} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

is primitive. By using Theorem 4.2, $\mathbb{H}_n$ is primitive for all $n \ge 2$. Mixing of $\Sigma(\mathcal{B})$ will be proven in Example 4.29.

To study the general cases, which includes degenerated case, more notation and lemmas are required. First, the crisscross lattice $\mathbb{Z}_+$ is decomposed into four rectangular lattices:

$$\mathbb{Z}_+ = \bigcup_{i=1}^{4} \mathbb{R}(i),$$

where

$$\begin{cases} \mathbb{R}(1) = \mathbb{Z}_{3\times 2}((0,0)), \\ \mathbb{R}(2) = \mathbb{Z}_{2\times 3}((0,0)), \end{cases} \quad \text{and} \quad \begin{cases} \mathbb{R}(3) = \mathbb{Z}_{3\times 2}((-1,0)), \\ \mathbb{R}(4) = \mathbb{Z}_{2\times 3}((0,-1)). \end{cases}$$

Definition 4.4. *For $1 \le i \le 4$, a basic set $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ is called $R(i)$-extendable if for any $B \in \mathcal{B}$, there exists $U \in \Sigma_{\mathbb{R}(i)}(\mathcal{B})$ such that $U|_{\mathbb{Z}_{2\times 2}} = B$.*

Clearly, a basic set $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ is crisscross-extendable if and only if $\mathcal{B}$ is $R(i)$-extendable for all $1 \le i \le 4$.

Let $A = [a_{i,j}]_{n\times n}$ be a non-negative matrix; the index set of non-zero rows of A and the index set of non-zero columns of A are denoted by

$$r(A) = \left\{ i \mid \sum_{j=1}^{n} a_{i,j} > 0 \right\} \quad \text{and} \quad c(A) = \left\{ j \mid \sum_{i=1}^{n} a_{i,j} > 0 \right\}, \tag{4.4}$$

respectively. From (2.9), it is easy to show that

$$\begin{cases} \mathcal{B} \text{ is } R(1)\text{-extendable if and only if } r(\mathbb{H}_2(\mathcal{B})) \supseteq c(\mathbb{H}_2(\mathcal{B})); \\ \mathcal{B} \text{ is } R(2)\text{-extendable if and only if } r(\mathbb{V}_2(\mathcal{B})) \supseteq c(\mathbb{V}_2(\mathcal{B})); \\ \mathcal{B} \text{ is } R(3)\text{-extendable if and only if } c(\mathbb{H}_2(\mathcal{B})) \supseteq r(\mathbb{H}_2(\mathcal{B})); \\ \mathcal{B} \text{ is } R(4)\text{-extendable if and only if } c(\mathbb{H}_2(\mathcal{B})) \supseteq r(\mathbb{H}_2(\mathcal{B})). \end{cases} \tag{4.5}$$

Now, the following lemma is obtained.

Lemma 4.5. *Assume that $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ is $R(2)$-extendable. For $n \ge 2$ and $q \ge 1$, denote $\mathbb{H}_{n+q} = [H_{n+q;i,j}]_{p^{q+1}\times p^{q+1}}$ and $\mathbb{H}_{q+1} = [h_{q+1;i,j}]_{p^{q+1}\times p^{q+1}}$. If $h_{q+1;i,j} = 1$, then $H_{n+q;i,j}$ is not a zero matrix.*

Proof. Since $\mathcal{B}$ is $R(2)$-extendable, any pattern in $\Sigma_{2\times(q+1)}(\mathcal{B})$ can be extended to $\mathbb{Z}_{2\times(n+q)}$ by using the local patterns in $\mathcal{B}$. From (2.17), the result follows. □

In degenerated case, the weak non-degeneracy of $\mathbb{H}_2(\mathcal{B})$ (or $\mathbb{V}_2(\mathcal{B})$) is introduced and is useful in establishing the primitivity of $\mathbb{H}_n(\mathcal{B})$ (or $\mathbb{V}_n(\mathcal{B})$).

Definition 4.6. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, $\mathbb{H}_2(\mathcal{B}) = [H_{2;i,j}]_{p\times p}$ is weakly non-degenerated if*

(i) *when both $H_{2;i,j_1}$ and $H_{2;i,j_2}$ are not zero matrices, $1 \le i, j_1, j_2 \le p$,*
$$r(H_{2;i,j_1}) = r(H_{2;i,j_2});$$
(ii) *when both $H_{2;i_1,j}$ and $H_{2;i_2,j}$ are not zero matrices, $1 \le i_1, i_2, j \le p$,*
$$c(H_{2;i_1,j}) = c(H_{2;i_2,j}).$$

Weak non-degeneracy of $\mathbb{V}_2(\mathcal{B})$ is defined analogously.

Clearly, if $\mathbb{H}_2(\mathcal{B})$ (or $\mathbb{V}_2(\mathcal{B})$) is non-degenerated, then $\mathbb{H}_2(\mathcal{B})$ (or $\mathbb{V}_2(\mathcal{B})$) is weakly non-degenerated.

The following lemma can be proven straightforwardly from Lemma 4.5 and the proof is omitted.

Lemma 4.7. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, if $\mathcal{B}$ is $R(2)$-extendable and $\mathbb{H}_2(\mathcal{B})$ is weakly non-degenerated, then*

(i) *when both $H_{2;i,j_1}$ and $H_{2;i,j_2}$ are not zero matrices, $1 \le i, j_1, j_2 \le p$,*
$$r(H_{n;i,j_1}) = r(H_{n;i,j_2}) \text{ for all } n \ge 2;$$
(ii) *when both $H_{2;i_1,j}$ and $H_{2;i_2,j}$ are not zero matrices, $1 \le i_1, i_2, j \le p$,*
$$c(H_{n;i_1,j}) = c(H_{n;i_2,j}) \text{ for all } n \ge 2.$$

In the following lemma, the corner-extendable conditions $C(i)$ can be obtained from the $R(i)$-extendability and weak non-degeneracy.

Lemma 4.8. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, assume $\mathbb{H}_2(\mathcal{B})$ is weakly non-degenerated, then*

(i) *if $\mathcal{B}$ is $R(1)$- and $R(2)$-extendable, then $\mathcal{B}$ satisfies $C(1)$;*
(ii) *if $\mathcal{B}$ is $R(2)$- and $R(3)$-extendable, then $\mathcal{B}$ satisfies $C(2)$.*

Similarly, assume $\mathbb{V}_2(\mathcal{B})$ is weakly non-degenerated, then

(iii) *if $\mathcal{B}$ is $R(1)$- and $R(2)$-extendable, then $\mathcal{B}$ satisfies $C(1)$;*
(iv) *if $\mathcal{B}$ is $R(1)$- and $R(4)$-extendable, then $\mathcal{B}$ satisfies $C(4)$.*

Proof. For simplicity, only (i) is proven. The other are proven similarly.

For any $U_{\mathbb{L}_1} = (u_{i,j})_{(i,j)\in\mathbb{L}_1} \in \Sigma_{\mathbb{L}_1}(\mathcal{B})$, let $i_1 = u_{1,1}+1$, $i_2 = u_{2,1}+1$ and $i_3 = u_{1,2}+1$. The $R(2)$-extendability of $\mathcal{B}$ implies that $H_{2;i_1,i_2}$ is not a zero matrix. From the $R(1)$-extendability of $\mathcal{B}$, there exist $1 \le i_4, i_5 \le p$ such that $(H_{2;i_1,i_4})_{i_3,i_5} = 1$. Since $\mathbb{H}_2(\mathcal{B})$ is weakly non-degenerated, there exists $1 \le i_6 \le p$ such that $(H_{2;i_1,i_2})_{i_3,i_6} = 1$, that is,

$$\begin{array}{|c|c|}\hline u_{1,2} & a \\ \hline u_{1,1} & u_{2,1} \\ \hline \end{array} \in \mathcal{B},$$

where $a = i_6 - 1$. Therefore, $U_{\mathbb{L}_1}$ can be extended to $\mathbb{Z}_{3\times 3}$ by using the local patterns in $\mathcal{B}$, which implies $\mathcal{B}$ satisfies $C(1)$. The proof is complete. □

For easily expressing the primitivity of compressible matrices, the following definition is introduced.

Definition 4.9. *If $A = [a_{i,j}]_{n\times n}$ is a matrix with $a_{i,j} \in \{0,1\}$, the associated saturated matrix $\mathbb{E}(A) = [e_{i,j}]_{n\times n}$ of A is defined by*

$$\begin{cases} e_{i,j} = 0 & \text{if } \sum_{k=1}^{n} a_{i,k} = 0 \text{ or } \sum_{k=1}^{n} a_{k,j} = 0, \\ e_{i,j} = 1 & \text{otherwise.} \end{cases} \tag{4.6}$$

Clearly, given $A = [a_{i,j}]_{n\times n}$ with $a_{i,j} \in \{0,1\}$, if there exists $n_0 \ge 1$ such that $A^{n_0} \ge \mathbb{E}(A)$, then A is primitive (n_0-primitive); here, if $B = [b_{i,j}]_{n\times n}$ and $C = [c_{i,j}]_{n\times n}$ are two matrices, $B \ge C$ means $b_{i,j} \ge c_{i,j}$ for all $1 \le i,j \le n$.

For $n \ge 2$ and $q \ge 0$, as (2.12), let

$$\mathbb{H}_{n+q} = [H_{n+q;i,j}]_{p^{q+1}\times p^{q+1}} = \left[H_{n+q;\beta_1;\beta_2;\cdots;\beta_{q+1}}\right].$$

For convenience, $\mathbb{E}(\mathbb{H}_{n+q})$ is denoted by

$$\mathbb{E}(\mathbb{H}_{n+q}) = [E_{n+q;i,j}]_{p^{q+1}\times p^{q+1}} = \left[E_{n+q;\beta_1;\beta_2;\cdots;\beta_{q+1}}\right]. \tag{4.7}$$

From (2.20), it is easy to verify that

$$\mathbb{E}(\mathbb{H}_{n+q}) \le (\mathbb{E}(\mathbb{H}_{q+1}))_{p^{q+1}\times p^{q+1}} \circ \left[E_{p^q\times p^q} \otimes [E_{n;i',j'}]_{p\times p}\right]. \tag{4.8}$$

Lemmas 4.7 and 4.8 yield the following lemma.

Lemma 4.10. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, for $n \geq 2$, let $\mathbb{H}_n(\mathcal{B}) = [H_{n;i,j}]_{p\times p}$ and $\mathbb{E}(\mathbb{H}_n) = [E_{n;i,j}]_{p\times p}$. Assume that $\mathcal{B}$ is $R(q)$-extendable, $q \in \{1,2,3\}$, and $\mathbb{H}_2(\mathcal{B})$ is weakly non-degenerated. Then, for $n \geq 2$ and $1 \leq i_1, i_2, i_3 \leq p$,*

$$H_{n;i_1,i_2} E_{n;i_2,i_3} \geq E_{n;i_1,i_3} \tag{4.9}$$

when $H_{2;i_1,i_2}$ is not a zero matrix;

$$E_{n;i_1,i_2} H_{n;i_2,i_3} \geq E_{n;i_1,i_3} \tag{4.10}$$

when $H_{2;i_2,i_3}$ is not a zero matrix.

Proof. By Lemma 4.8, we have that $\mathcal{B}$ satisfies $C(1)$ and $C(2)$.

From the $R(1)$-extendability of $\mathcal{B}$ and $C(1)$, for $n \geq 2$, every pattern $U_{2\times n} \in \Sigma_{2\times n}(\mathcal{B})$ can be extended to $\mathbb{Z}_{3\times n}$ by using the local patterns in $\mathcal{B}$. Therefore, by Lemma 4.7, it can be verified that (4.9) holds; the details of the proof are omitted. Similarly, (4.10) can be shown to hold.

□

We also need following notation in proving the theorem for the primitivity of $\mathbb{H}_n$.

Definition 4.11. *Let $\mathbb{M} = [M_{i,j}]_{N\times N}$, where $M_{i,j}$ is an $M \times M$ non-negative matrix for $1 \leq i, j \leq N$. The indicator matrix $\Lambda(\mathbb{M}) = [m_{i,j}]_{N\times N}$ of $\mathbb{M}$ is defined by*

$$\begin{cases} m_{i,j} = 1 & \text{if } |M_{i,j}| > 0, \\ m_{i,j} = 0 & \text{otherwise,} \end{cases}$$

where $|M_{i,j}|$ is the sum of all entries in $M_{i,j}$.

The following theorem provides a sufficient condition for the primitivity of $\mathbb{H}_n$ when $\mathbb{H}_2$ is weakly non-degenerated with some $R(i)$-extendability.

Theorem 4.12. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, if*

(i) *$\mathbb{H}_2(\mathcal{B})$ is weakly non-degenerated,*
(ii) *$\mathcal{B}$ is $R(i)$-extendable, $i \in \{1,2,3\}$,*
(iii) *there exists an S-invariant diagonal cycle $\overline{\beta}_{\bar{q}} = \bar{\beta}_1\bar{\beta}_2\cdots\bar{\beta}_{\bar{q}}\bar{\beta}_1$ of order $(\bar{m}, \bar{q})$ with its invariant index set $\mathcal{K}$,*
(iv) *for $2 \leq n \leq \bar{q}+1$, there exists $\bar{a} = \bar{a}(n) \geq 1$ such that*

$$\left(\sum_{l\in\mathcal{K}} H^{(l)}_{\bar{m},n;\bar{\beta}_1}\right)^{\bar{a}} \geq E_{n;\bar{\beta}_1},$$

(v) *$\mathbb{H}_n$ is primitive for $2 \leq n \leq \bar{q}+1$,*

then $\mathbb{H}_n$ is primitive for all $n \geq 2$.

Proof. The result that $\mathbb{H}_n$ is primitive for $n \geq 2$ is proven by induction, as follows. For $s \geq 0$, the statement $P(s)$ means that $\mathbb{H}_n$ is primitive for $s\bar{q}+2 \leq n \leq (s+1)\bar{q}+1$.

From (v), $P(0)$ is true. Assume that $P(t)$ follows for some $t \geq 0$, that is, $\mathbb{H}_n$ is primitive for $t\bar{q}+2 \leq n \leq (t+1)\bar{q}+1$.

For $(t+1)\bar{q}+2 \leq n \leq (t+2)\bar{q}+1$, let $n = (t+1)\bar{q}+r$, where $2 \leq r \leq \bar{q}+1$. Let $N = (t+1)\bar{q}+1$, define $\overline{\beta}_{N-1} = \overline{\beta}_{(t+1)\bar{q}} = \left(\bar{\beta}_1\bar{\beta}_2\cdots\bar{\beta}_{\bar{q}}\right)^{t+1}\bar{\beta}_1$. From (4.3), $\overline{\beta}_{N-1}$ is an S-invariant diagonal cycle of order $(\bar{m}, N-1)$ with invariant index set $\mathcal{K}$.

From (2.38), let $\mathbb{H}_n = [H_{n;i_N,j_N}]_{p^N\times p^N}$ and $\mathbb{H}_n^m = [H_{m,n;i_N,j_N}]_{p^N\times p^N}$ for $m \geq 1$. Then, from (2.20),

$$\mathbb{H}_n = [H_{n;i_N,j_N}] = (\mathbb{H}_N)_{p^N\times p^N} \circ \left[E_{p^{N-1}\times p^{N-1}} \otimes [H_{r;i,j}]_{p\times p}\right].$$

By Lemma 4.5, $\mathbb{H}_N$ is the indicative matrix of $\mathbb{H}_n = [H_{n;i_N,j_N}]$. By the assumption for $P(t)$, $\mathbb{H}_N$ is primitive; then suppose $\mathbb{H}_N^{m'} \geq \mathbb{E}(\mathbb{H}_N)$ for some $m' \geq 1$. Therefore, for any $1 \leq i_N, j_N \leq p^N$ with $(\mathbb{E}(\mathbb{H}_N))_{i_N,j_N} = 1$, there exist $1 \leq l_0(i_N,j_N), l_1(i_N,j_N), \cdots, l_{m'}(i_N,j_N) \leq p$ such that

$$H_{m',n;i_N,j_N} \geq H_{r;l_0,l_1}H_{r;l_1,l_2}\cdots H_{r;l_{m'-1},l_{m'}},$$

where $H_{r;l_q,l_{q+1}}$ is not a zero matrix for all $0 \leq q \leq m'-1$.

From (2.40),

$$\begin{aligned} H_{\bar{m},n;\overline{\beta}_{N-1}} &\equiv \underbrace{H_{\bar{m},n;\bar{\beta}_1;\bar{\beta}_2;\cdots;\bar{\beta}_{\bar{q}};\cdots;\bar{\beta}_1;\bar{\beta}_2;\cdots;\bar{\beta}_{\bar{q}};\bar{\beta}_1}}_{(t+1)\text{ times}} \\ &= \sum_{k,l=1}^{p^{\bar{m}-1}} ((S_{\bar{m};\bar{\beta}_1,\bar{\beta}_2}S_{\bar{m};\bar{\beta}_2,\bar{\beta}_3}\cdots S_{\bar{m};\bar{\beta}_q,\bar{\beta}_1})^{t+1})_{k,l}H^{(l)}_{\bar{m},r;\bar{\beta}_1} \\ &\geq \sum_{l\in\mathcal{K}} H^{(l)}_{\bar{m},r;\bar{\beta}_1}. \end{aligned}$$

Notably, $H_{\bar{m},n;\overline{\beta}_{N-1}}$ is on the diagonal of $\mathbb{H}_n^{\bar{m}}$; then let $H_{\bar{m},n;\bar{k},\bar{k}} = H_{\bar{m},n;\overline{\beta}_{N-1}}$ for some $1 \leq \bar{k} \leq p^N$. Hence, form (iv), $H_{\bar{a}\bar{m},n;\overline{\beta}_{N-1}} \geq E_{r;\bar{\beta}_1}$. Since $\bar{\beta}_1 \in \mathcal{D}_p$, $E_{r;\bar{\beta}_1} = E_{r;k',k'}$ for some $1 \leq k' \leq p$.

Let $\bar{N} = \bar{a}\bar{m} + 2m'$ and $\mathbb{H}_n^{\bar{N}} = \left[H_{\bar{N},n;i_N,j_N}\right]_{p^N\times p^N}$. For $1 \leq i_N, j_N \leq p^N$ with $(\mathbb{E}(\mathbb{H}_N))_{i_N,j_N} = 1$, from (4.9) and (4.10),

$$\begin{aligned} H_{\bar{N},n;i_N,j_N} &\geq H_{m',n;i_N,\bar{k}}H_{\bar{a}\bar{m},n;\bar{k},\bar{k}}H_{m',n;\bar{k},j_N} \\ &\geq H_{r;l_0(i_N,\bar{k}),l_1(i_N,\bar{k})}\cdots H_{r;l_{m'-1}(i_N,\bar{k}),k'}E_{r;k',k'}H_{r;k',l_1(\bar{k},j_N)}\cdots H_{r;l_{m'-1}(\bar{k},j_N),l_{m'}(\bar{k},j_N)} \\ &\geq E_{r;l_0(i_N,\bar{k}),l_{m'}(\bar{k},j_N)}. \end{aligned}$$

Hence, from (4.8),

$$\mathbb{H}_n^{\bar{N}} \geq (\mathbb{E}(\mathbb{H}_N))_{p^N\times p^N} \circ \left[E_{p^{N-1}\times p^{N-1}} \otimes [E_{r;i,j}]_{p\times p}\right] \geq \mathbb{E}(\mathbb{H}_n).$$

Then, $\mathbb{H}_n$ is primitive for $(t+1)\bar{q}+2 \leq n \leq (t+2)\bar{q}+1$, that is, $P(t+1)$ holds.

Therefore, $P(s)$ is true for all $s \geq 0$, implying that $\mathbb{H}_n$ is primitive for all $n \geq 2$. The proof is complete.

□

Now, the proof of Theorem 4.2 is given as follows.

Proof of Theorem 4.2. From (2.4), (2.5) and (4.5), clearly, $\mathcal{B}$ is $R(i)$-extendable for $i \in \{1,2,3\}$. From (iii), by Proposition 3.4, it can be verified that $\mathbb{H}_n$ is primitive for $2 \leq n \leq \bar{q}+1$ and the detail is omitted for brevity. Therefore, by Theorem 4.12, $\mathbb{H}_n$ is primitive for all $n \geq 2$. □

For $n \geq 2$, let $\mathbb{V}_n(\mathcal{B}) = [V_{n;\alpha}]_{1\leq\alpha\leq p^2}$ and $\mathbb{E}(\mathbb{V}_n) = [E'_{n;\alpha}]_{1\leq\alpha\leq p^2}$. Like Theorem 4.12, the following theorem provides a sufficient condition for the primitivity of $\mathbb{V}_n$.

Theorem 4.13. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, if*

(i) *$\mathbb{V}_2(\mathcal{B})$ is weakly non-degenerated,*
(ii) *$\mathcal{B}$ is $R(i)$-extendable, $i \in \{1, 2, 4\}$,*
(iii) *there exists a W-invariant diagonal cycle $\overline{\beta}_{\bar{q}} = \bar{\beta}_1\bar{\beta}_2\cdots\bar{\beta}_{\bar{q}}\bar{\beta}_1$ of order $(\bar{m}, \bar{q})$ with its invariant index set $\mathcal{K}$,*
(iv) *for $2 \leq n \leq \bar{q}+1$, there exists $\bar{b} = \bar{b}(n) \geq 1$ such that*

$$\left(\sum_{l\in\mathcal{K}} V^{(l)}_{\bar{m},n;\bar{\beta}_1}\right)^{\bar{b}} \geq E'_{n;\bar{\beta}_1},$$

(v) *$\mathbb{V}_n$ is primitive for $2 \leq n \leq \bar{q}+1$,*

then $\mathbb{V}_n$ is primitive for all $n \geq 2$.

In viewing the primitive properties in Theorem 4.12 and Theorem 4.13, we introduce the $H(1)$- and $V(1)$-primitive conditions for $\mathcal{B}$ has invariant diagonal cycles as follows. Later, $H(2)$- and $V(2)$-primitive conditions are introduced for $\mathcal{B}$ has primitive commutative cycles; see Definition 4.26.

Definition 4.14. *Let $\mathcal{B} \subset \Sigma_{2\times 2}(p)$.*
(i) *$\mathcal{B}$ satisfies $H(1)$-primitive condition if the conditions (iii)∼(v) of Theorem 4.12 are satisfied.*
(ii) *$\mathcal{B}$ satisfies $V(1)$-primitive condition if the condition (iii)∼(v) of Theorem 4.13 are satisfied.*

Remark 4.15. *From the proof of Theorem 4.2, if $\mathbb{H}_2(\mathcal{B})$ is non-degenerated and the conditions (ii)∼(iii) of Theorem 4.2 are satisfied, then $\mathcal{B}$ satisfies the $H(1)$-primitive condition. A similar result holds when $\mathbb{V}_2(\mathcal{B})$ is non-degenerated.*

From Lemma 4.8 and Theorems 3.14, 4.12 and 4.13, mixing of $\Sigma(\mathcal{B})$ follows.

Theorem 4.16. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, if*

(i) *$\mathbb{H}_2(\mathcal{B})$ and $\mathbb{V}_2(\mathcal{B})$ are weakly non-degenerated,*
(ii) *$\mathcal{B}$ is crisscross-extendable,*
(iii) *$\mathcal{B}$ satisfies $H(1)$- and $V(1)$-primitive conditions,*

then $\Sigma(\mathcal{B})$ is mixing.

The following well-known examples illustrate Theorem 4.16. The first example is the Golden-Mean shift (or the hard square model).

Example 4.17. *The rule of the Golden-Mean shift $\Sigma(\mathcal{B}_G)$, $\mathcal{B}_G \subset \Sigma_{2\times 2}(2)$, is that there is no two 1's next to each other in horizontal or vertical direction. From (2.3),*

$$\mathbb{H}_2(\mathcal{B}_G) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} H_{2;1,1} & H_{2;1,2} \\ H_{2;2,1} & H_{2;2,2} \end{bmatrix}$$

and

$$\mathbb{E}_2(\mathbb{H}_2) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} E_{2;1,1} & E_{2;1,2} \\ E_{2;2,1} & E_{2;2,2} \end{bmatrix}.$$

Clearly, $\mathbb{V}_2(\mathcal{B}_G) = \mathbb{H}_2(\mathcal{B}_G)$*. From (4.5),* $\mathcal{B}_G$ *is crisscross-extendable. That* $\mathbb{H}_2(\mathcal{B}_G)$ *and* $\mathbb{V}_2(\mathcal{B}_G)$ *are weakly non-degenerated are easily seen.*

Now,

$$S_{2;1,1} = C_{2;1,1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, $\overline{\beta}_1 = 11$ *is an S-invariant diagonal cycle of order* $(2,1)$ *with index set* $\mathcal{K} = \{1,2\}$*. Clearly,*

$$\sum_{l\in\mathcal{K}} H^{(l)}_{2,2;1} = H_{2;1,1}H_{2;1,1} + H_{2;1,2}H_{2;2,1} = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \geq E_{2;1,1} = E_{2;1}.$$

Since $\mathbb{H}_2^2 \geq \mathbb{E}(\mathbb{H}_2)$*,* $\mathbb{H}_2$ *is primitive. Therefore,* $\mathcal{B}_G$ *satisfies the* $H(1)$*-primitive condition.*

Since $\mathbb{V}_2 = \mathbb{H}_2$*,* $\mathcal{B}_G$ *also satisfies* $V(1)$*-primitive condition. Hence, by Theorem 4.16, the Golden-Mean shift* $\Sigma(\mathcal{B}_G)$ *is mixing. In fact, that the Golden-Mean shift has strong specification will be shown in Example 5.6.*

The next example concerns the three-coloring of the square lattice, which is closely related to the six-vertex ice model in statistical physics [7], see also Example 6.7 of this paper.

Example 4.18. *The three-coloring of the square lattice is the coloring of the square lattice* $\mathbb{Z}^2$ *with three colors such that no two adjacent vertices have the same color.*

Let $\mathcal{B}_T \subset \Sigma_{2\times 2}(3)$ *be the basic set of the three-colouring of the square lattice. We have that*

$$\mathbb{H}_2(\mathcal{B}_T) = \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} H_{2;1,1} & H_{2;1,2} & H_{2;1,3} \\ H_{2;2,1} & H_{2;2,2} & H_{2;2,3} \\ H_{2;3,1} & H_{2;3,2} & H_{2;3,3} \end{bmatrix}.$$

Clearly, $\mathbb{V}_2 = \mathbb{H}_2$*. It is easy to verify that* $\mathcal{B}_T$ *is crisscross-extendable and* $\mathbb{H}_2 = \mathbb{V}_2$ *is weakly non-degenerated. We have that*

$$S_{2;1,5} = C_{2;2,2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad S_{2;5,1} = C_{2;4,4} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since $S_{2;1,5}S_{2;5,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $\overline{\beta}_2 \equiv 151$ *is an* S*-invariant diagonal cycle of order* $(2,2)$ *with its invariant index set* $\mathcal{K} = \{2,3\}$.

That the conditions (iv) and (v) of Theorem 4.12 are satisfied can be checked straightforwardly; the details are omitted for brevity. Hence, $\mathcal{B}_T$ *satisfies the* $H(1)$*-primitive condition. Since* $\mathbb{V}_2 = \mathbb{H}_2$, $\mathcal{B}_T$ *satisfies the* $V(1)$*-primitive condition. Therefore, by Theorem 4.16,* $\Sigma(\mathcal{B}_T)$ *is mixing. It can be proved that* $\Sigma(\mathcal{B}_T)$ *does not have strong specification; the detail is omitted.*

4.2. **Primitive commutative cycles.** This subsection introduces primitive commutative cycles in order to obtain another finitely checkable sufficient condition for the primitivity of $\mathbb{H}_n$ or $\mathbb{V}_n$ when invariant diagonal cycles are unavailable.

For $q, q' \geq 1$, let $I_q = i_1 i_2 \cdots i_q i_1$ and $J_{q'} = j_1 j_2 \cdots j_{q'} j_1$ are two cycles, where $i_k, j_l \in \{1, 2, \cdots, p\}$ for $1 \leq k \leq q$ and $1 \leq l \leq q'$.

Definition 4.19. *If* $j_1 = i_1$, *let* $(I_q J_{q'}) = i_1 i_2 \cdots i_q i_1 j_2 \cdots j_{q'} i_1$ *and* $(J_{q'} I_q) = i_1 j_2 \cdots j_{q'} i_1 i_2 \cdots i_q i_1$. *The pair* $(I_q J_{q'})$ *and* $(J_{q'} I_q)$ *is called a commutative cycle pair.*

Given a commutative cycle pair $(I_q J_{q'})$ and $(J_{q'} I_q)$, denote the index of $(I_q J_{q'})$ and $(J_{q'} I_q)$ by $\langle m, \bar{\alpha}; K, L\rangle$, where

$$\begin{cases} m = q + q' \\ \bar{\alpha} = \psi(i_1 - 1, i_1 - 1) \\ K = \psi(i_2 - 1, \cdots, i_q - 1, i_1 - 1, j_2 - 1, \cdots, j_{q'} - 1) \\ L = \psi(j_2 - 1, \cdots, j_{q'} - 1, i_1 - 1, i_2 - 1, \cdots, i_q - 1). \end{cases} \tag{4.11}$$

From (2.21), it is easy to check that

$$\begin{cases} H_{n;i_1,i_2} H_{n;i_2,i_3} \cdots H_{n;i_q,i_1} H_{n;i_1,j_2} H_{n;j_2,j_3} \cdots H_{n;j_{q'},i_1} = H^{(K)}_{m,n;\bar{\alpha}} \\ \\ H_{n;i_1,j_2} H_{n;j_2,j_3} \cdots H_{n;j_{q'},i_1} H_{n;i_1,i_2} H_{n;i_2,i_3} \cdots H_{n;i_q,i_1} = H^{(L)}_{m,n;\bar{\alpha}}. \end{cases} \tag{4.12}$$

Moreover, the number $\bar{\alpha}$ is a member of the diagonal index set $\mathcal{D}_p$, and then $H_{m,n;\bar{\alpha}}$ lies on the diagonal of $\mathbb{H}^m_n$.

Definition 4.20. *A commutative cycle pair* $(I_q J_{q'})$ *and* $(J_{q'} I_q)$ *with index* $\langle m, \bar{\alpha}; K, L\rangle$ *is called an* H*-primitive commutative cycle pair if there exists* $N \geq 1$ *such that*

$$\text{either} \quad \left(H^{(K)}_{m,2;\bar{\alpha}}\right)^N \geq E_{2;\bar{\alpha}} \quad \text{or} \quad \left(H^{(L)}_{m,2;\bar{\alpha}}\right)^N \geq E_{2;\bar{\alpha}}. \tag{4.13}$$

A V-primitive commutative cycle pair is similarly specified, and the details are omitted.

The following theorem provides a sufficient condition for the primitivity of $\mathbb{H}_n$ when $\mathbb{H}_2$ is weakly non-degenerated.

Theorem 4.21. *Given* $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, *if*

(i) $\mathbb{H}_2$ *is weakly non-degenerated,*
(ii) $\mathcal{B}$ *is* $R(i)$*-extendable,* $i \in \{1, 2, 3\}$,
(iii) *there exists an* H*-primitive commutative cycle pair* $(I_q J_{q'})$ *and* $(J_{q'} I_q)$ *with index* $\langle m, \bar{\alpha}; K, L\rangle$ *such that* $(S_{m;\bar{\alpha},\bar{\alpha}})_{K,L} = 1$ *or* $(S_{m;\bar{\alpha},\bar{\alpha}})_{L,K} = 1$,

(iv) $\mathbb{H}_2$ *is primitive,*

then $\mathbb{H}_n$ *is primitive for all* $n \geq 2$.

Proof. By Lemma 4.8, $\mathcal{B}$ satisfies $C(1)$ and $C(2)$. Suppose $(S_{m;\bar{\alpha},\bar{\alpha}})_{K,L} = 1$. The case for $(S_{m;\bar{\alpha},\bar{\alpha}})_{L,K} = 1$ is similar. From (i) and (ii), by Lemma 4.10, (4.9) and (4.10) follow.

First, we show that for $n \geq 2$, there exists $N(n) \geq 1$ such that

$$\left(H^{(K)}_{m,n;\bar{\alpha}}\right)^{N(n)} \geq E_{n;\bar{\alpha}} \quad \text{and} \quad \left(H^{(L)}_{m,n;\bar{\alpha}}\right)^{N(n)} \geq E_{n;\bar{\alpha}} \tag{4.14}$$

by induction. From (4.9), (4.10) and (4.13), it is clear that (4.14) holds for $n = 2$.

Assume that the case for $n = t$ is true, $t \geq 2$. Let $H^{(K)}_{m,2;\bar{\alpha}} = \left[h^*_{i,j}\right]_{p\times p} = \left[H^{(K)}_{m,2;\bar{\alpha};\alpha}\right]_{1\leq\alpha\leq p^2}$, from (2.37), it is clear that

$$h^*_{i,j} = h^*_{i(\alpha),j(\alpha)} = H^{(K)}_{m,2;\bar{\alpha};\alpha} = \sum_{l=1}^{p^{m-1}} (S_{m;\bar{\alpha},\alpha})_{K,l}$$

for all $1 \leq i, j \leq p$. Let $\Lambda\left(H^{(K)}_{m,2;\bar{\alpha}}\right)$ be the indicator matrix of $H^{(K)}_{m,2;\bar{\alpha}} = \left[h^*_{i,j}\right]_{p\times p}$. Since $\left(H^{(K)}_{m,2;\bar{\alpha}}\right)^{N(2)} \geq E_{2;\bar{\alpha}}$, we have $\left(\Lambda\left(H^{(K)}_{m,2;\bar{\alpha}}\right)\right)^{N(2)} \geq E_{2;\bar{\alpha}}$.

Let $H^{(K)}_{m,t+1;\bar{\alpha}} = \left[H^{(K)}_{m,t+1;\bar{\alpha};i,j}\right]_{p\times p} = \left[H^{(K)}_{m,t+1;\bar{\alpha};\alpha}\right]_{1\leq\alpha\leq p^2}$, by Proposition 2.3,

$$H^{(K)}_{m,t+1;\bar{\alpha};i,j} = H^{(K)}_{m,t+1;\bar{\alpha};\alpha} = \sum_{l=1}^{p^{m-1}} (S_{m;\bar{\alpha},\alpha})_{K,l} H^{(l)}_{m,t;\alpha}$$

for all $1 \leq i, j \leq p$. Since $\mathcal{B}$ is $R(2)$-extendable and satisfies condition $C(1)$, for $m \geq 2$, every pattern $U_{m\times 2} \in \Sigma_{m\times 2}(\mathcal{B})$ can be extended to $\mathbb{Z}_{m\times 3}$ by using the local patterns in $\mathcal{B}$. Thus, if $(S_{m;\bar{\alpha},\alpha})_{K,l} = 1$, then $H^{(l)}_{m,t;\alpha}$ is not a zero matrix for $1 \leq \alpha \leq p^2$ and $1 \leq l \leq p^{m-1}$. Hence, $\Lambda\left(H^{(K)}_{m,2;\bar{\alpha}}\right)$ is also the indicator matrix of $H^{(K)}_{m,t+1;\bar{\alpha}} = \left[H^{(K)}_{m,t+1;\bar{\alpha};i,j}\right]_{p\times p}$.

Since $(S_{m;\bar{\alpha},\bar{\alpha}})_{K,L} = 1$ and $\left(H^{(L)}_{m,t;\bar{\alpha}}\right)^{N(t)} \geq E_{t;\bar{\alpha}}$, $\left(H^{(K)}_{m,t+1;\bar{\alpha};\bar{\alpha}}\right)^{N(t)} \geq E_{t;\bar{\alpha}}$. Notably, $H^{(K)}_{m,t+1;\bar{\alpha};\bar{\alpha}}$ lies on the diagonal of $H^{(K)}_{m,t+1;\bar{\alpha}}$. Let $N' = N(t) + 2N(2)$. Since $\left(\Lambda\left(H^{(K)}_{m,2;\bar{\alpha}}\right)\right)^{N(2)} \geq E_{2;\bar{\alpha}}$ and $\left(H^{(K)}_{m,t+1;\bar{\alpha};\bar{\alpha}}\right)^{N(t)} \geq E_{t;\bar{\alpha}}$, from (4.8)$\sim$(4.10),

$$\left(H^{(K)}_{m,t+1;\bar{\alpha}}\right)^{N'} \geq (E_{2;\bar{\alpha}})_{p\times p} \circ \left[[E_{t;\alpha}]_{1\leq\alpha\leq p^2}\right] \geq E_{t+1;\bar{\alpha}}.$$

From (4.9) and (4.10), $\left(H^{(L)}_{m,t+1;\bar{\alpha}}\right)^{N'+1} \geq E_{t+1;\bar{\alpha}}$. Hence, the case for $n = t + 1$ holds. Indeed, $N(t + 1) = N' + 1$. Therefore, (4.14) follows.

Now, we want to show that $\mathbb{H}_n$ is primitive for all $n \geq 2$. First, the case for $n = 2$ directly follows from (iv).

For $n \geq 3$, from (2.20),

$$\mathbb{H}_n = [H_{n;i,j}]_{p^2\times p^2} = (\mathbb{H}_2)_{p^2\times p^2} \circ \left[E_{p\times p} \otimes [H_{n-1;\alpha}]_{1\leq\alpha\leq p^2}\right].$$

From (ii), by Lemma 4.5, if $(\mathbb{H}_2)_{i,j} = 1$, $H_{n;i,j}$ is not a zero matrix. Hence, $\mathbb{H}_2$ is the indicator matrix of $\mathbb{H}_n = [H_{n;i,j}]_{p^2\times p^2}$.

Let $\mathbb{H}_n^m = [H_{m,n;i,j}]_{p^2\times p^2} = [H_{m,n;\beta_1;\beta_2}]$. Since $(S_{m;\bar{\alpha},\bar{\alpha}})_{K,L} = 1$,

$$H_{m,n;\bar{\alpha};\bar{\alpha}} = \sum_{k,l=1}^{p^{m-1}} (S_{m;\bar{\alpha};\bar{\alpha}})_{k,l} H^{(l)}_{m,n-1;\bar{\alpha}} \geq H^{(L)}_{m,n-1;\bar{\alpha}}.$$

From (4.14), clearly, $(H_{m,n;\bar{\alpha};\bar{\alpha}})^{N(n-1)} \geq E_{n-1;\bar{\alpha}}$. Notably, $H_{m,n;\bar{\alpha};\bar{\alpha}}$ is on the diagonal of $\mathbb{H}_n^m$. Therefore, from (iv) and (4.8)$\sim$(4.10), it can be verified that there exists $\bar{N} \geq 1$ such that

$$\mathbb{H}_n^{\bar{N}} \geq (\mathbb{E}(\mathbb{H}_2))_{p^2\times p^2} \circ \left[E_{p\times p} \otimes [E_{n-1;i,j}]_{p\times p}\right] \geq \mathbb{E}(\mathbb{H}_n).$$

Therefore, $\mathbb{H}_n$ is primitive for all $n \geq 2$. The proof is complete. □

Similarly, the following theorem provides a sufficient condition for the primitivity of $\mathbb{V}_n$, and the proof is omitted.

Theorem 4.22. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, if*

(i) *$\mathbb{V}_2$ is weakly non-degenerated,*
(ii) *$\mathcal{B}$ is $R(i)$-extendable, $i \in \{1, 2, 4\}$,*
(iii) *there exists a V-primitive commutative cycle pair $(I_q J_{q'})$ and $(J_{q'} I_q)$ with index $\langle m, \bar{\alpha}; K, L\rangle$ such that $(W_{m;\bar{\alpha},\bar{\alpha}})_{K,L} = 1$ or $(W_{m;\bar{\alpha},\bar{\alpha}})_{L,K} = 1$,*
(iv) *$\mathbb{V}_2$ is primitive,*

then $\mathbb{V}_n$ is primitive for all $n \geq 2$.

Remark 4.23. *In practice, to find an invariant diagonal cycle is easier than to find a primitive commutative cycle pair. However, it may be more convenient to verify primitivity of $\mathbb{H}_n$ or $\mathbb{V}_n$ by using the method of primitive commutative cycle pair; see Example 4.24 and 4.29.*

The following example illustrates Theorem 4.21.

Example 4.24. *Consider*

$$\mathbb{H}_2(\mathcal{B}) = \left[\begin{array}{ccc|ccc|ccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}\right].$$

Clearly, $\mathbb{H}_2$ is weakly non-degenerated. It can be verified that $\mathcal{B}$ is $R(i)$-extendable, $i \in \{1, 2, 3\}$. We also have that $\mathbb{H}_2$ is primitive.

Let $I_3 = 1311$ and $J_4 = 13331$; the following is easily computed.

$$H^{(513)}_{7,2;1} = H_{2;1,3}H_{2;3,1}H_{2;1,1}H_{2;1,3}H_{2;3,3}H_{2;3,3}H_{2;3,1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$H_{7,2;1}^{(709)} = H_{2;1,3}H_{2;3,3}H_{2;3,3}H_{2;3,1}H_{2;1,3}H_{2;3,1}H_{2;1,1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

We have that

$$E_{2;1} = E_{2;1,1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then,

$$\left(H_{7,2;1}^{(513)}\right)^2 \geq E_{2;1} \quad \text{and} \quad \left(H_{7,2;1}^{(709)}\right)^2 \geq E_{2;1}.$$

Hence, (I_3J_4) *and* (J_4I_3) *is an* H*-primitive commutative cycle pair with index* $\langle 7, 1; 513, 709\rangle$*. Moreover,*

$$(S_{7;1,1})_{513,709} = 1.$$

Therefore, by Theorem 4.21, $\mathbb{H}_n$ *is primitive for all* $n \geq 2$*.*

As Theorem 4.2, Theorem 4.21 can be made simpler when $\mathbb{H}_2(\mathcal{B})$ is non-degenerated. The proof is similar to that of Theorem 4.2 and is omitted here. The result for $\mathbb{V}_2$ is also valid.

Theorem 4.25. *Given* $\mathcal{B} \subset \Sigma_{2\times 2}(p)$*, if*

(i) $\mathbb{H}_2$ *is non-degenerated,*
(ii) *there exists an* H*-primitive commutative cycle pair* $(I_qJ_{q'})$ *and* $(J_{q'}I_q)$ *with index* $\langle m, \bar{\alpha}; K, L\rangle$ *such that* $(S_{m;\bar{\alpha},\bar{\alpha}})_{K,L} = 1$ *or* $(S_{m;\bar{\alpha},\bar{\alpha}})_{L,K} = 1$*,*

then $\mathbb{H}_n$ *is primitive for all* $n \geq 2$*.*

As the primitive conditions in Definition 4.14 for invariant diagonal cycles, the primitive conditions for primitive commutative cycles in Theorem 4.21 and 4.22 are introduced as follows.

Definition 4.26. *Let* $\mathcal{B} \subset \Sigma_{2\times 2}(p)$*.*

(i) $\mathcal{B}$ *satisfies* $H(2)$*-primitive condition if the conditions (iii) and (iv) of Theorem 4.21 are satisfied.*
(ii) $\mathcal{B}$ *satisfies* $V(2)$*-primitive condition if the conditions (iii) and (iv) of Theorem 4.22 are satisfied.*

Remark 4.27. *If* $\mathbb{H}_2(\mathcal{B})$ *is non-degenerated and condition (ii) of Theorem 4.25 is satisfied, then* $\mathcal{B}$ *satisfies the* $H(2)$*-primitive condition. A similar result holds when* $\mathbb{V}_2(\mathcal{B})$ *is non-degenerated.*

Since it may happen that $\mathbb{H}$ (or $\mathbb{V}$) has invariant diagonal cycle and $\mathbb{V}$ (or $\mathbb{H}$) has primitive commutative cycles. Therefore, combining the primitive conditions for invariant diagonal cycles and primitive commutative cycles, Theorem 4.16 is generalized, yielding a finitely checkable condition for mixing of $\Sigma(\mathcal{B})$.

Theorem 4.28. *Given* $\mathcal{B} \subset \Sigma_{2\times 2}(p)$*, if*

(i) $\mathbb{H}_2(\mathcal{B})$ *and* $\mathbb{V}_2(\mathcal{B})$ *are weakly non-degenerated,*
(ii) $\mathcal{B}$ *is crisscross-extendable,*
(iii) $\mathcal{B}$ *satisfies* $H(i)$*- and* $V(j)$*-primitive conditions for some* $i, j \in \{1, 2\}$*,*

then $\Sigma(\mathcal{B})$ is mixing.

Notably, when both $\mathbb{H}_2(\mathcal{B})$ and $\mathbb{V}_2(\mathcal{B})$ are non-degenerated, the condition (ii) of Theorem 4.28 automatically holds.

The following example illustrates the application of Theorem 4.28.

Example 4.29. *(continued)*

In Example 4.3,

$$\mathbb{V}_2(\mathcal{B}) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Clearly, $\mathbb{V}_2(\mathcal{B})$ is non-degenerated.

Let $I_5 = 211212$ and $J_2 = 222$. That

$$V^{(12)}_{7,2;4} = V_{2;2,1}V_{2;1,1}V_{2;1,2}V_{2;2,1}V_{2;1,2}V_{2;2,2}V_{2;2,2} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$V^{(51)}_{7,2;4} = V_{2;2,2}V_{2;2,2}V_{2;2,1}V_{2;1,1}V_{2;1,2}V_{2;2,1}V_{2;1,2} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

are primitive can be easily verified. Hence, (I_5J_2) and (J_2I_5) form a V-primitive commutative cycle pair with index $\langle 7, 4; 12, 51\rangle$. Moreover,

$$(W_{7;4,4})_{12,51} = 1.$$

Therefore, $\mathcal{B}$ satisfies the $V(2)$-primitive condition.

From the result in Example 4.3, $\mathcal{B}$ satisfies the $H(1)$-primitive condition. Therefore, by Theorem 4.28, $\Sigma(\mathcal{B})$ is mixing.

The following example demonstrates that the weakly non-degenerated condition is crucial in order to have topological mixing.

Example 4.30. *(continued) In Example 3.15, clearly, $\mathbb{H}_2(\mathcal{B}_{\pi/4}) = \mathbb{V}_2(\mathcal{B}_{\pi/4})$ is not weakly non-degenerated. From (2.29),*

$$S_{2;1,1} = C_{2;1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $\overline{\beta}_1 = 11$ is an S-invariant diagonal cycle of order $(2, 1)$ with index set $\mathcal{K} = \{1\}$, and

$$H^{(1)}_{2,2;1} = H_{2;1,1}H_{2;1,1} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

is primitive. Therefore, $\mathcal{B}_{\pi/4}$ satisfies the $H(1)$-primitive condition. Since $\mathbb{H}_2(\mathcal{B}_{\pi/4}) = \mathbb{V}_2(\mathcal{B}_{\pi/4})$, $\mathcal{B}_{\pi/4}$ also satisfies the $V(1)$-primitive condition.

From Example 3.15, $\mathcal{B}_{\pi/4}$ is crisscross-extendable. Thus, $\mathcal{B}_{\pi/4}$ satisfies conditions (ii) and (iii) of Theorem 4.28. However, $\Sigma(\mathcal{B}_{\pi/4})$ is not mixing. Therefore, the importance of the weakly non-degenerated condition is established.

Remark 4.31. *When invariant diagonal cycles and primitive commutative cycles are unavailable, some examples can also be shown to be mixing.*

For example, consider

$$\mathcal{B}'_L = \left\{ \begin{array}{|c|c|}\hline u_2 & u_4 \\ \hline u_1 & u_3 \\ \hline \end{array} : u_2 + u_3 + u_4 \equiv 0 \ (mod\ 2) \ and\ u_1, u_2, u_3, u_4 \in \{0,1\} \right\}.$$

This shift space $\Sigma(\mathcal{B}'_L)$ is related to the shift space $\Sigma(\mathcal{B}_L)$ that is given by

$$\mathcal{B}_L = \left\{ \begin{array}{|c|c|}\hline u_2 & u_4 \\ \hline u_1 & u_3 \\ \hline \end{array} : u_1 + u_2 + u_3 \equiv 0 \ (mod\ 2) \ and\ u_1, u_2, u_3, u_4 \in \{0,1\} \right\},$$

which was first investigated by Ledrappier [35] *who showed that $\Sigma(\mathcal{B}_L)$ was mixing with zero entropy. Clearly,*

$$\mathbb{H}_2(\mathcal{B}'_L) = \mathbb{V}_2(\mathcal{B}'_L) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

is non-degenerated. $\mathbb{H}_n = \mathbb{V}_n$ can be shown to be primitive for all $n \geq 2$. Then, $\Sigma(\mathcal{B}'_L)$ is mixing.

A systematic method for solving this type of problem is being developed.

Remark 4.32. *For studying mixing problem, the ideas in this work can be applied to higher-dimensional shifts of finite type; see* [5, 31].

5. Strong specification

This section introduces the k hole-filling condition as in Step (5) in the introduction, and provides finitely checkable conditions for the strong specification of $\Sigma(\mathcal{B})$.

First, for $M, N \geq 1$ and $i, j \in \mathbb{Z}$, the rectangularly annular lattice $\mathcal{A}_{M\times N;d}((i,j))$ with hole $\mathbb{Z}_{M\times N}((i,j))$ and width d (called the annular lattice for short) is defined by

$$\mathcal{A}_{M\times N;d}((i,j)) = \mathbb{Z}_{(M+2d)\times(N+2d)}((i-d, j-d)) \setminus \mathbb{Z}_{M\times N}((i,j)). \tag{5.1}$$

In particular,

$$\mathcal{A}_{M\times N;2}((i,j)) = \mathcal{A}_{M\times N}((i,j)) \quad \text{and} \quad \mathcal{A}_{M\times N} = \mathcal{A}_{M\times N;2}((0,0)). \tag{5.2}$$

Hole-filling condition is defined as follows.

Definition 5.1. *For $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ and $k \geq 2$, $\mathcal{B}$ satisfies k hole-filling ($(HFC)_k$) with size (M,N), $M, N \geq 2k-3$, if for any $U \in \Sigma_{\mathcal{A}_{(M+4-2k)\times(N+4-2k);k}((k-2,k-2))}(\mathcal{B})$, there exists $U' \in \Sigma_{\mathbb{Z}_{(M+4)\times(N+4)}((-2,-2))}(\mathcal{B})$ such that $U' \mid_{\mathcal{A}_{M\times N}} = U \mid_{\mathcal{A}_{M\times N}}$. In particular, $(HFC)_2$ is also called hole-filling condition (HFC).*

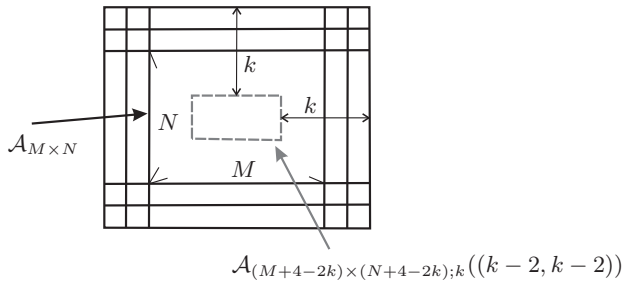


Figure 5.1.

$(\mathrm{HFC})_k$ can be expressed in terms of the horizontal transition matrices $\mathbb{H}_n$ and the connecting operators $S_{m;\alpha,\beta}$ and $W_{m;\alpha,\beta}$. Therefore, the condition can be easily checked, especially using computer programs. The following theorem presents only the case in which $\mathcal{B}$ satisfies hole-filling; for brevity, the general case in which $\mathcal{B}$ satisfies $(\mathrm{HFC})_k$, $k \geq 2$, is presented in Theorem A.1

Theorem 5.2. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, for $M, N \geq 1$, $\mathcal{B}$ satisfies HFC with size (M,N) if and only if for $1 \leq i,j \leq p^M$ and $1 \leq \alpha_l \leq p^2$, $1 \leq l \leq N+4$, if*

$$\text{(i)} \sum_{k=1}^{p^M} \left(S_{M+1;\alpha_1,\alpha_2}\right)_{k,i} \geq 1, \qquad \text{(ii)} \sum_{k=1}^{p^M} \left(S_{M+1;\alpha_{N+3},\alpha_{N+4}}\right)_{j,k} \geq 1,$$

$$\text{(iii)} \sum_{k=1}^{p^{N+4}} \left(\mathbb{H}_{N+4}\right)_{k,s} \geq 1, \qquad \text{(iv)} \sum_{k=1}^{p^{N+4}} \left(\mathbb{H}_{N+4}\right)_{t,k} \geq 1,$$

where

$$\left\{ \begin{array}{l} s = \psi\left(\alpha_{1,1}, \alpha_{2,1}, \cdots, \alpha_{N+4,1}\right) \text{ and } t = \psi\left(\alpha_{1,2}, \alpha_{2,2}, \cdots, \alpha_{N+4,2}\right), \\ \alpha_{l,1}, \alpha_{l,2} \in \mathcal{S}_p \text{ such that } \psi\left(\alpha_{l,1}, \alpha_{l,2}\right) = \alpha_l, 1 \leq l \leq N+4, \end{array} \right. \tag{5.3}$$

then

$$\left(S_{M+1;\alpha_2,\alpha_3} S_{M+1;\alpha_3,\alpha_4}, \cdots, S_{M+1;\alpha_{N+2},\alpha_{N+3}}\right)_{i,j} \geq 1. \tag{5.4}$$

Proof. For $1 \leq i,j \leq p^M$, choose $i_l, j_l \in \mathcal{S}_p$, $1 \leq l \leq M$, such that $i = \psi\left(i_1, i_2, \cdots, i_M\right)$ and $j = \psi\left(j_1, j_2, \cdots, j_M\right)$. Let $N_1 = N+3$ and $N_2 = N+4$. Clearly, conditions (i), (ii), (iii) and (iv) imply that the empty places in the patterns U_b, U_t, U_l and U_r can can be filled with some colors in $\mathcal{S}_p$, such that the patterns U_b, U_t, U_l and U_r are $\mathcal{B}$-admissible, respectively. Furthermore, the following annular pattern is $\mathcal{B}$-admissible.

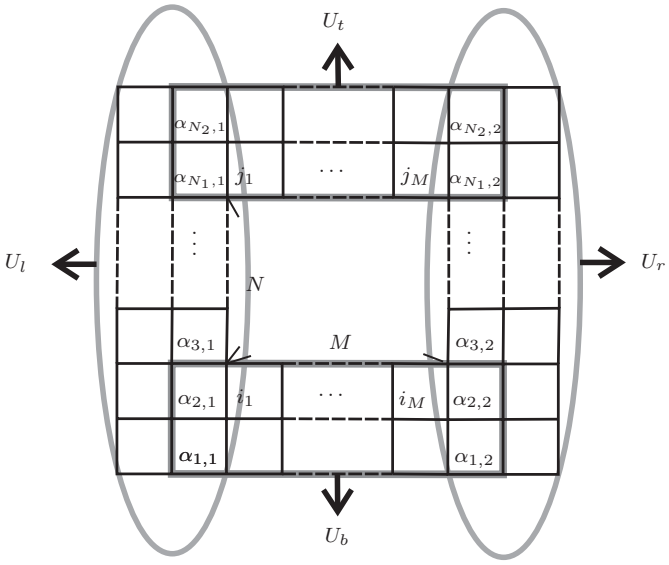


Figure 5.2.

Therefore, by the construction of connecting operators, (5.4) is equivalent to the hole-filling condition with size (M, N). The proof is complete. □

Before showing the main theorem, the following notation is needed..

Definition 5.3. *For $k \geq 2$, $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ is called k crisscross-extendable if $r(\mathbb{H}_k(\mathcal{B})) = c(\mathbb{H}_k(\mathcal{B}))$ and $r(\mathbb{V}_k(\mathcal{B})) = c(\mathbb{V}_k(\mathcal{B}))$. In particular, 2 crisscross-extendability is the crisscross-extendability.*

In the following, the k crisscross-extendability, $(\mathrm{HFC})_k$ and primitivity of $\mathbb{H}_k$ and $\mathbb{V}_k$ are shown to provide sufficient conditions for strong specification of $\Sigma(\mathcal{B})$. Since all conditions are finitely checkable, the theorem provides a finitely checkable sufficient condition for the strong specification of $\Sigma(\mathcal{B})$.

Theorem 5.4. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, if there exists $k \geq 2$ such that*

(i) *$\mathcal{B}$ is k crisscross-extendable,*
(ii) *$\mathcal{B}$ satisfies $(HFC)_k$ with size (M, N) for some $M, N \geq 2k-3$,*
(iii) *$\mathbb{H}_k$ is $(M-2k+5)$-primitive and $\mathbb{V}_k$ is $(N-2k+5)$-primitive,*

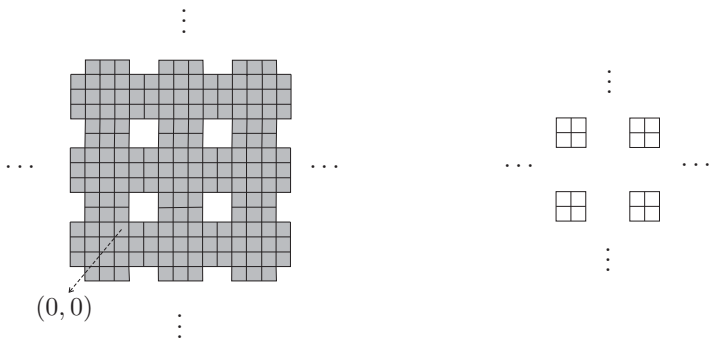
then $\Sigma(\mathcal{B})$ has strong specification.

Proof. Let $M' = M-k+4$ and $N' = N-k+4$. First, define the lattice $\mathbb{L}_{g;k} = \mathbb{L}_{g;k}(M, N)$, which is like the grid on a checkerboard with line width k and $(M+4-2k)\times(N+4-2k)$ blank spaces, as

$$\mathbb{L}_{g;k} = \bigcup_{i,j\in\mathbb{Z}} \mathcal{A}_{(M+4-2k)\times(N+4-2k);k}((iM'+k-2, jN'+k-2)).$$

Denote the lattice of blank spaces on the checkerboard by

$$\mathbb{L}_{b;k} = \mathbb{Z}^2 \setminus \mathbb{L}_{g;k}.$$



(a) The shadowed lattice is $\mathbb{L}_{g;3}$. for $M = N = 4$.

(b) The white lattice is $\mathbb{L}_{b;3}$. for $M = N = 4$.

Figure 5.3.

For $i, j \in \mathbb{Z}$, define

$$\begin{cases} \mathbb{L}(i,j) = \mathbb{L}_{k;M,N}(i,j) = \mathbb{Z}_{M' \times N'}\left((iM'-2, jN'-2)\right) \\ \widehat{\mathbb{L}}(i,j) = \widehat{\mathbb{L}}_{k;M,N}(i,j) = \mathbb{Z}_{(M+4)\times(N+4)}\left((iM'-2, jN'-2)\right). \end{cases}$$

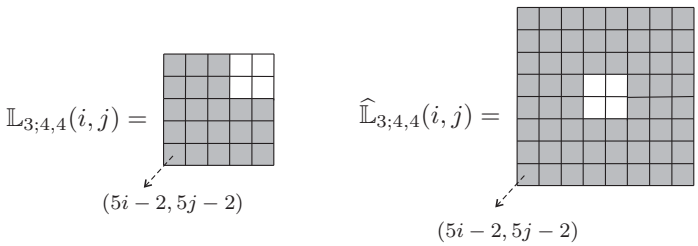


Figure 5.4. The lattices $\mathbb{L}_{3;4,4}(i,j)$ and $\widehat{\mathbb{L}}_{3;4,4}(i,j)$.

Clearly, $\widehat{\mathbb{L}}(i,j) \supset \mathbb{L}(i,j)$ and $\mathbb{L}(i_1,j_1) \bigcap \mathbb{L}(i_2,j_2) = \emptyset$ if $(i_1,j_1) \neq (i_2,j_2)$. Then, $\mathbb{Z}^2$ lattice can be decomposed into disjoint sublattices:

$$\mathbb{Z}^2 = \bigcup_{i,j \in \mathbb{Z}} \mathbb{L}(i,j).$$

Take

$$\bar{d} = 3\sqrt{(M')^2 + (N')^2}. \tag{5.5}$$

Let $R_1, R_2 \subset \mathbb{Z}^2$ with $d(R_1, R_2) \geq \bar{d}$. For any $U_l = \Pi_{R_l}(W_l)$ with $W_l \in \Sigma(\mathcal{B})$, $l = 1, 2$, let

$$R_l' = \bigcup_{\substack{R_l \bigcap \mathbb{L}(i,j) \neq \emptyset \\ (i',j') \in \mathbb{Z}_{2\times 2}((i-1,j-1))}} \widehat{\mathbb{L}}(i',j')$$

for $l = 1, 2$. Hence, U_l can be extended as $U_l' = \Pi_{R_l'}(W_l)$, $l = 1, 2$. Clearly, for $l = 1, 2$,

$$\text{if } (i,j) \in R_l, \text{ then } \mathbb{Z}_{(2k+1)\times(2k+1)}((i-k, j-k)) \subseteq R'_l. \tag{5.6}$$

From (5.5), it can be verified that it never occurs both $\widehat{\mathbb{L}}(i,j) \bigcap R'_1 \neq \emptyset$ and $\widehat{\mathbb{L}}(i,j) \bigcap R'_2 \neq \emptyset$ for all $(i,j) \in \mathbb{Z}^2$.

Now, from conditions (i) and (iii), there exists a $\mathcal{B}$-admissible pattern U'' on $R'_1 \bigcup R'_2 \bigcup \mathbb{L}_{g;k}$ such that $U'' \mid_{R'_l} = U'_l$, $i = 1, 2$. Clearly, $\mathbb{Z}^2 \setminus (R'_1 \bigcup R'_2 \bigcup \mathbb{L}_c)$ is the union of the discrete $(M + 4 - 2k) \times (N + 4 - 2k)$ rectangular lattices.

Hence, from (5.6) and condition (ii), there exists $W \in \Sigma(\mathcal{B})$ such that $W \mid_{R_i} = U_i$ for $i = 1, 2$. Notably, in general, $W \mid_{R'_1 \bigcup R'_2 \bigcup \mathbb{L}_c}$ is not equal to U'' since condition (ii) may change the colors on the boundary of $R'_1 \bigcup R'_2 \bigcup \mathbb{L}_c$ with width $k - 2$. Therefore, $\Sigma(\mathcal{B})$ has strong specification. The proof is complete.

□

Remark 5.5.

(i) *In the proof of Theorem 5.4, the constant $M(\Sigma(\mathcal{B}))$ of strong specification is less than or equal to $3\sqrt{(M')^2 + (N')^2}$, which is given by (5.5). It is of interest to know the optimal (least) value of $M(\Sigma(\mathcal{B}))$.*

(ii) *Lightwood* [42] *showed that if a $\mathbb{Z}^2$ shift of finite type Σ is square filling and topologically mixing, then Σ has the UFP.*

The following two well-known examples illustrate Theorem 5.4. The first is the Golden-Mean shift, which is considered in Example 4.17.

Example 5.6. *From Example 4.17,*

$$\mathbb{H}_2(\mathcal{B}_G) = \mathbb{V}_2(\mathcal{B}_G) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, $\mathcal{B}_G$ is crisscross-extendable and $\mathbb{H}_2 = \mathbb{V}_2$ is 2-primitive.

It can be computed that

$$S_{2;1,1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad S_{2;1,2} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad S_{2;1,3} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

$$S_{2;1,4} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad S_{2;2,1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad S_{2;2,3} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$S_{2;3,1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad S_{2;3,2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S_{2;4,1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

The other $S_{2;\alpha,\beta}$ are zero matrices.

Then, by Theorem 5.2, it is easy to verify that $\mathcal{B}_G$ satisfies HFC with size $(1, 1)$. Therefore, by Theorem 5.4, $\Sigma(\mathcal{B}_G)$ has strong specification.

Burton and Steif [13, 14] introduced the following example, which is closely related to the ferromagnetic Ising model in statistical physics.

Example 5.7. *Consider the color set $\mathcal{S}'_4 = \{-2, -1, 1, 2\}$. The rule of $\mathbf{X}_{BS} \subseteq \mathcal{S}'^{\mathbb{Z}^2}_4$ is that a negative is disallowed to sit to a positive unless they are both ± 1. To fit $\mathcal{S}'_4$ to the color set $\mathcal{S}_4 = \{0, 1, 2, 3\}$ used in this work, -2, -1, 1 and 2 are replaced with 0, 1, 2 and 3, respectively. Then, the following can be shown;*

$$\mathbb{H}_2(\mathcal{B}_{BS}) = \left[\begin{array}{cccc|cccc|cccc|cccc}
1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
1&1&0&0&1&1&0&0&0&1&0&0&0&0&0&0\\
1&1&0&0&1&1&1&0&0&1&1&0&0&0&0&0\\
0&1&0&0&0&1&1&0&0&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&0&0&1&1&0&0&0&1&0\\
0&0&0&0&0&1&1&0&0&1&1&1&0&0&1&1\\
0&0&0&0&0&0&1&0&0&0&1&1&0&0&1&1\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1
\end{array}\right].$$

Clearly, $\mathbb{H}_2 = \mathbb{V}_2$ *is* 3*-primitive and* $\mathcal{B}_{BS}$ *is crisscross-extendable.*

That $\mathcal{B}_{BS}$ *satisfies HFC with size* $(2,2)$ *can be proven and the details are omitted. Therefore, by Theorem 5.4,* $\Sigma(\mathcal{B}_{BS})$ *has strong specification.*

The size (M,N) of hole-filling condition may be even larger than $(2,2)$, as in the following example.

Example 5.8. *It can be verified that*

$$\mathbb{H}_2(\mathcal{B}_1) = \begin{bmatrix} 1&1&1&1\\ 1&1&0&1\\ 1&0&1&1\\ 1&1&1&0 \end{bmatrix}$$

satisfies HFC with size $(3,3)$ *and*

$$\mathbb{H}_2(\mathcal{B}_2) = \begin{bmatrix} 1&1&1&1\\ 1&0&1&1\\ 1&1&1&1\\ 0&1&1&1 \end{bmatrix}$$

satisfies HFC with size $(4,4)$*. Both have strong specification. The details are omitted.*

The following example concerns the Simplified Golden Mean (SGM), which does not satisfy HFC but satisfies $(\mathrm{HFC})_3$.

Example 5.9. *(Simplified Golden-Mean) Consider* $\mathcal{S}_2 = \{0,1\}$ *and*

$$\mathbb{H}_2(\mathcal{B}_s) = \mathbb{V}_2(\mathcal{B}_s) = \begin{bmatrix} 1&1&1&0\\ 1&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{bmatrix}.$$

That $\Sigma(\mathcal{B}_s)$ *has strong specification can be easily shown. Indeed, for any two patterns* $U_1 \in \Pi_{R_1}(\Sigma(\mathcal{B}_s))$ *and* $U_2 \in \Pi_{R_2}(\Sigma(\mathcal{B}_s))$*,* $R_1, R_2 \subseteq \mathbb{Z}^2$*, with* $d(R_1,R_2) \geq 2$*,*

coloring the vertices in $\mathbb{Z}^2 \setminus (R_1 \bigcup R_2)$ *by* 0 *yields a* $\mathcal{B}_s$*-admissible global pattern. Therefore, SGM has strong specification with size* $M = 2$.

However, consider the $\mathcal{B}_s$*-admissible pattern* U *in Fig. 5.5.*

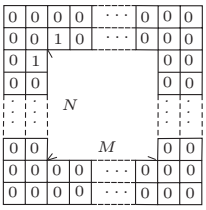


Figure 5.5.

Clearly, U *cannot be extended any further on the corner* $\begin{smallmatrix} 0 & 1 \\ 1 & \end{smallmatrix}$ *since the local patterns* $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ *and* $\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}$ *are forbidden. Therefore, SGM does not satisfy HFC.*

Now, by Theorem A.1, it can be verified that $\mathcal{B}_s$ *satisfies* $(HFC)_3$ *with size* $(3, 3)$. *Clearly,* $\mathcal{B}_s$ *is* 3 *crisscross-extendable, and* $\mathbb{H}_3 = \mathbb{V}_3$ *is* 2*-primitive. Therefore, by Theorem 5.4, SGM has strong specification.*

The following example demonstrates a genuine failure of the hole-filling condition, which causes the failure of strong specification. Indeed, in [12], Boyle at el. consider this example to prove that block gluing is strictly weaker than corner gluing. Hence, this example does not have strong specification.

Example 5.10. *Consider* $\mathcal{B}_B = \mathcal{S}_2^{\mathbb{Z}_{2\times 2}} \setminus \left\{ \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right\}$, *meaning that*

$$\mathbb{H}_2 = \mathbb{V}_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Now, $\Sigma(\mathcal{B}_B)$ *lacks the corner gluing property, as follows. Consider* $W_1 = \{0\}^{\mathbb{Z}^2}$ *and* $W_2 = \{1\}^{\mathbb{Z}^2}$. *Let* $U_i = W_i \mid_{R_i}$, $i = 1, 2$, *where* R_1 *is the L-shaped lattice and* R_2 *is the rectangular lattice; see Fig. 5.6.*

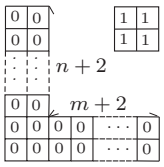


Figure 5.6.

Therefore, the following pattern on $\mathcal{A}_{m\times n}$, *in Fig. 5.7 for all* $m, n \geq 1$ *where* $\bullet \in \{0, 1\}$, *cannot be filled without* $\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}$, *which is forbidden.*

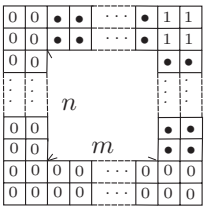


Figure 5.7.

Similarly, the corner gluing property fails in one of the four corners when $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}, \begin{smallmatrix}1&1\\1&1\end{smallmatrix} \in \mathcal{B}$ *and* $B \notin \mathcal{B}$ *for some*

$$B \in \left\{ \begin{smallmatrix}1&0\\0&0\end{smallmatrix}, \begin{smallmatrix}0&0\\1&0\end{smallmatrix}, \begin{smallmatrix}0&0\\0&1\end{smallmatrix}, \begin{smallmatrix}1&0\\1&1\end{smallmatrix}, \begin{smallmatrix}0&1\\1&1\end{smallmatrix}, \begin{smallmatrix}1&1\\0&1\end{smallmatrix}, \begin{smallmatrix}1&1\\1&0\end{smallmatrix} \right\},$$

and then $\Sigma(\mathcal{B})$ *does not have strong specification.*

Remark 5.11. *The ideas for strong specification in this section can be applied to higher-dimensional shifts of finite type; see* [5, 31].

6. Edge coloring

Edge coloring models are very common in statistical physics and other fields [6, 7, 36, 37, 38, 39, 40]. For completeness, this section briefly discusses edge coloring. The ideas of corner coloring in the previous sections apply to edge coloring with some modifications. For simplicity, only the case of two colors is considered: $\mathcal{S}_2 = \{0, 1\}$. The results hold for all $\mathcal{S}_p$, $p \geq 2$. The unit square lattice is still denoted by $\mathbb{Z}_{2\times 2}$. For $m, n \geq 2$, denote the set of all local patterns with colored edges on $\mathbb{Z}_{m\times n}$ over $\mathcal{S}_2$ by $\Sigma^e_{m\times n}$.

Fist, the ordering matrices $\mathbf{X}^e_n$ and $\mathbf{Y}^e_n$, $n \geq 2$, are introduced to arrange systematically all local patterns in $\Sigma^e_{2\times n}$ and $\Sigma^e_{n\times 2}$.

For $n = 2$, the horizontal ordering matrix $\mathbf{X}^e_2 = \sum_{j=1}^{4} \mathbf{X}^e_{2;j}$ and vertical ordering matrix $\mathbf{Y}^e_2 = \sum_{j=1}^{4} \mathbf{Y}^e_{2;j}$ are defined as

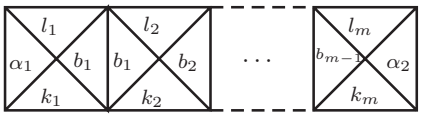
$$\mathbf{X}^e_{2;j} = \begin{bmatrix} \begin{smallmatrix}u_2\\0\ \ 0\\u_1\end{smallmatrix} & \begin{smallmatrix}u_2\\0\ \ 1\\u_1\end{smallmatrix} \\ \begin{smallmatrix}u_2\\1\ \ 0\\u_1\end{smallmatrix} & \begin{smallmatrix}u_2\\1\ \ 1\\u_1\end{smallmatrix} \end{bmatrix} \quad \text{and} \quad \mathbf{Y}^e_{2;j} = \begin{bmatrix} \begin{smallmatrix}0\\u_1\ \ u_2\\0\end{smallmatrix} & \begin{smallmatrix}1\\u_1\ \ u_2\\0\end{smallmatrix} \\ \begin{smallmatrix}0\\u_1\ \ u_2\\1\end{smallmatrix} & \begin{smallmatrix}1\\u_1\ \ u_2\\1\end{smallmatrix} \end{bmatrix},$$

where $u_1, u_2 \in \{0, 1\}$ with $j = \psi(u_1, u_2)$. For $n \geq 3$, the higher order ordering matrix $\mathbf{X}^e_n = \sum_{j=1}^{4} \mathbf{X}^e_{n;j}$ can be recursively defined recursively by

$$\left\{ \begin{array}{l} \mathbf{X}^e_{n;1} = \left(\mathbf{X}^e_{2;1} \otimes \mathbf{X}^e_{n-1;1}\right) + \left(\mathbf{X}^e_{2;2} \otimes \mathbf{X}^e_{n-1;3}\right), \\ \mathbf{X}^e_{n;2} = \left(\mathbf{X}^e_{2;1} \otimes \mathbf{X}^e_{n-1;2}\right) + \left(\mathbf{X}^e_{2;2} \otimes \mathbf{X}^e_{n-1;4}\right), \\ \mathbf{X}^e_{n;3} = \left(\mathbf{X}^e_{2;3} \otimes \mathbf{X}^e_{n-1;1}\right) + \left(\mathbf{X}^e_{2;4} \otimes \mathbf{X}^e_{n-1;3}\right), \\ \mathbf{X}^e_{n;4} = \left(\mathbf{X}^e_{2;3} \otimes \mathbf{X}^e_{n-1;2}\right) + \left(\mathbf{X}^e_{2;4} \otimes \mathbf{X}^e_{n-1;4}\right). \end{array} \right. \tag{6.1}$$

Similarly, $\mathbf{Y}_n^e = \sum_{j=1}^{4} \mathbf{Y}_{n;j}^e$, $n \geq 3$, can be defined recursively as above. Notably, comparing (6.1) with (2.6), the discussion of edge coloring is very different from that of corner coloring. Accordingly, the formulae for the transition matrices and connecting operators must be changed.

Now, for $m \geq 2$ and $1 \leq \alpha \leq 4$, the connecting ordering matrix $\mathbf{S}_{m;\alpha}^e = \left[\left(\mathbf{S}_{m;\alpha}^e\right)_{k,l}\right]_{2^m \times 2^m}$ is defined, where $\left(\mathbf{S}_{m;\alpha}^e\right)_{k,l}$ is the set of all local patterns of the form,



with $b_j \in \{0,1\}$, $1 \leq j \leq m-1$, $\alpha = \psi(\alpha_1, \alpha_2)$, $k = \psi(k_1, \ldots, k_m)$ and $l = \psi(l_1, \ldots, l_m)$. Notably,

$$\mathbf{S}_{m;\alpha}^e = \mathbf{Y}_{m+1;\alpha}^e \tag{6.2}$$

for all $m \geq 2$, $1 \leq \alpha \leq 4$.

Now, given a basic set $\mathcal{B}_e \subset \Sigma_{2\times 2}^e$, the horizontal transition matrix $\mathbb{H}_2^e = \sum_{j=1}^{4} \mathbb{H}_{2;j}^e$ is defined by $\mathbb{H}_{2;j}^e = \left[h_{2;j;s,t}^e\right]_{2\times 2}$, where

$$\begin{cases} h_{2;j;s,t}^e = 1 & \text{if } \left(\mathbf{X}_{2;j}^e\right)_{s,t} \in \mathcal{B}_e, \\ \qquad\;\; = 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

As in (6.1), for $n \geq 3$, $\mathbb{H}_n^e = \sum_{j=1}^{4} \mathbb{H}_{n;j}^e$ can be defined recursively. The vertical transition matrix $\mathbb{V}_2^e = \sum_{j=1}^{4} \mathbb{V}_{2;j}^e$ is defined analogously.

Given $\mathcal{B}_e \subset \Sigma_{2\times 2}^e$, for $m \geq 2$ and $1 \leq \alpha \leq 4$, the connecting operator $S_{m;\alpha}^e = \left[\left(S_{m;\alpha}^e\right)_{k,l}\right]_{2^m \times 2^m}$ can be defined, where $\left(S_{m;\alpha}^e\right)_{k,l}$ is the cardinal number of all $\mathcal{B}_e$-admissible local patterns in $\left(\mathbf{S}_{m;\alpha}^e\right)_{k,l}$. Furthermore, from (6.2),

$$S_{m;\alpha}^e = \mathbb{V}_{m+1;\alpha}^e \tag{6.4}$$

for $m \geq 2$ and $1 \leq \alpha \leq 4$. Similarly, for $\mathbb{V}_2^e$, the connecting operator is denoted by $W_{m;\alpha}^e$, $m \geq 2$ and $1 \leq \alpha \leq 4$.

For $m, n \geq 2$, $\mathbb{H}_{m,n+1}^e \equiv \left(\mathbb{H}_{n+1}^e\right)^m$ can be expressed in terms of $\mathbb{H}_{m,n}^e$ and $S_{m;\alpha}^e$ as follows. First, for $n \geq 2$, let

$$\bar{H}_{n;1} = \mathbb{H}_{n;1}^e + \mathbb{H}_{n;2}^e \quad \text{and} \quad \bar{H}_{n;2} = \mathbb{H}_{n;3}^e + \mathbb{H}_{n;4}^e. \tag{6.5}$$

Then, for $m \geq 2$ and $1 \leq l \leq 2^m$, define

$$\bar{H}_{m,n}^{(l)} = \bar{H}_{n;l_1} \bar{H}_{n;l_2} \cdots \bar{H}_{n;l_m}, \tag{6.6}$$

where $l_j \in \{1, 2\}$, $1 \leq j \leq m$, with $l = \psi\left(l_1 - 1, l_2 - 1, \cdots, l_m - 1\right)$.

For $m \geq 2$, let

$$\mathbb{H}^e_{m,n+1} = \left[H^e_{m,n+1;\beta}\right]_{1\le\beta\le 4} = \begin{bmatrix} H^e_{m,n+1;1} & H^e_{m,n+1;2} \\ H^e_{m,n+1;3} & H^e_{m,n+1;4} \end{bmatrix}. \tag{6.7}$$

Then, for $1 \le q \le n-1$, apply (6.7) q times to decompose $\mathbb{H}^e_{m,n+1} = \left[H^e_{m,n+1;\beta_1;\beta_2;\ldots;\beta_q}\right]_{1\le\beta_j\le 4, 1\le j\le q}$ into 4^q-many $2^{n-q}\times 2^{n-q}$ submatrices $H^e_{m,n+1;\beta_1;\beta_2;\ldots;\beta_q}$. The results that hold for $\mathbb{H}^e_n$ are also valid for $\mathbb{V}^e_n$.

As in Proposition 2.4, $H^e_{m,n+1;\beta_1;\beta_2;\ldots;\beta_q}$ can be expressed as the product of q-many $S^e_{m;\beta}$ and $\bar{H}^{(l)}_{m,n-q+1}$.

Theorem 6.1. *For any $m, n \ge 2$ and $1 \le q \le n-1$,*

$$H^e_{m,n+1;\beta_1;\beta_2;\ldots;\beta_q} = \sum_{k,l=1}^{2^m} \left(S^e_{m;\beta_1} S^e_{m;\beta_2} \cdots S^e_{m;\beta_q}\right)_{k,l} \bar{H}^{(l)}_{m,n-q+1} \tag{6.8}$$

and

$$V^e_{m,n+1;\beta_1;\beta_2;\ldots;\beta_q} = \sum_{k,l=1}^{2^m} \left(W^e_{m;\beta_1} W^e_{m;\beta_2} \cdots W^e_{m;\beta_q}\right)_{k,l} \bar{V}^{(l)}_{m,n-q+1}. \tag{6.9}$$

Notably, in edge coloring, to color a lattice $R \subset \mathbb{Z}^2$ is to color the horizonal and vertical edges that connect the vertices in R. The definitions of rectangle-extendability, crisscross-extendability and corner-extendable conditions $C(1) \sim C(4)$ for edge coloring are similar to those for corner coloring. Furthermore, the following theorem is obtained as Theorem 3.14. The details are omitted for brevity.

Theorem 6.2. *If*

(i) *$\mathcal{B}_e \subset \Sigma^e_{2\times 2}$ is crisscross-extendable,*
(ii) *$\mathcal{B}_e$ satisfies three of corner-extendable conditions $C(i)$, $1 \le i \le 4$,*

then $\mathbb{H}^e_n(\mathcal{B}_e)$ and $\mathbb{V}^e_n(\mathcal{B}_e)$ are primitive for all $n \ge 2$ if and only if $\Sigma(\mathcal{B}_e)$ is mixing.

The non-degeneracy of $\mathbb{H}^e_2$ and $\mathbb{V}^e_2$ is defined as follows.

Definition 6.3. *An $\mathbb{H}^e_2$ ($\mathbb{V}^e_2$) is non-degenerated if both $\bar{H}_{2;1}$ and $\bar{H}_{2;2}$ ($\bar{V}_{2;1}$ and $\bar{V}_{2;2}$) are non-compressible.*

That if both $\mathbb{H}^e_2(\mathcal{B}_e)$ and $\mathbb{V}^e_2(\mathcal{B}_e)$ are non-degenerated, then $\mathcal{B}_e$ is crisscross-extendable and satisfies corner filling conditions $C(1)$, $C(2)$ and $C(4)$, can be easily confirmed.

Definition 6.4.

(i) *For $q \ge 1$, a finite sequence $\overline{\beta}_q = \beta_1\beta_2\cdots\beta_q$ is called a diagonal sequence with length q if $\beta_j \in \{1, 4\}$ for $1 \le j \le q$.*
(ii) *A diagonal sequence $\overline{\beta}_q = \beta_1\beta_2\cdots\beta_q$ is called an S_e-invariant diagonal sequence of order (m, q) if there exist $m \ge 2$ and an invariant index set $\mathcal{K} \subseteq \{1, 2, \cdots, 2^m\}$ such that*

$$\sum_{k\in\mathcal{K}} \left(S^e_{m;\beta_1} S^e_{m;\beta_2} \cdots S^e_{m;\beta_q}\right)_{k,l} \ge 1 \tag{6.10}$$

for all $l \in \mathcal{K}$.

(iii) *A diagonal sequence* $\overline{\beta}_q = \beta_1\beta_2\cdots\beta_q$ *is called a* W_e*-invariant diagonal sequence of order* (m,q) *if there exist* $m \geq 2$ *and an invariant index set* $\mathcal{K} \subseteq \{1,2,\cdots,2^m\}$ *such that*

$$\sum_{k\in\mathcal{K}} \left(W^e_{m;\beta_1} W^e_{m;\beta_2} \cdots W^e_{m;\beta_q}\right)_{k,l} \geq 1 \tag{6.11}$$

for all $l \in \mathcal{K}$.

As Theorem 4.2, the following theorem provides a finitely checkable sufficient condition for the primitivity of $\mathbb{H}^e_n$ ($\mathbb{V}^e_n$), $n \geq 2$, and then for mixing of $\Sigma(\mathcal{B}_e)$.

Theorem 6.5. *Given* $\mathcal{B}_e \subset \Sigma^e_{2\times 2}$, *if*

(i) $\mathbb{H}^e_2$ *is non-degenerated,*
(ii) *there exists an* S_e*-invariant diagonal sequence* $\overline{\beta}_{\bar{q}} = \bar{\beta}_1\bar{\beta}_2\cdots\bar{\beta}_{\bar{q}}$ *of order* $(\bar{m},\bar{q})$ *with its invariant index set* $\mathcal{K}$,
(iii) $\sum_{l\in\mathcal{K}} \bar{H}^{(l)}_{\bar{m},n}$ *is primitive for* $2 \leq n \leq \bar{q}+1$,

then $\mathbb{H}^e_n$ *is primitive for all* $n \geq 2$. *Similarly, if*

(i)' $\mathbb{V}^e_2$ *is non-degenerated,*
(ii)' *there exists a* W_e*-invariant diagonal cycle* $\overline{\beta}_{\bar{q}} = \bar{\beta}_1\bar{\beta}_2\cdots\bar{\beta}_{\bar{q}}$ *of order* $(\bar{m},\bar{q})$ *with its invariant index set* $\mathcal{K}$,
(iii)' $\sum_{l\in\mathcal{K}} \bar{V}^{(l)}_{\bar{m},n}$ *is primitive for* $2 \leq n \leq \bar{q}+1$,

then $\mathbb{V}^e_n$ *is primitive for all* $n \geq 2$. *Furthermore, if (i)∼(iii) and (i)'∼(iii)' hold, then* $\Sigma(\mathcal{B}_e)$ *is mixing.*

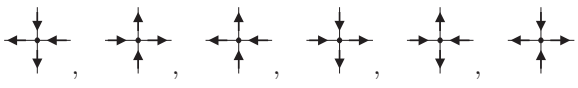
Remark 6.6.

(i) *As above, the method of primitive commutative cycles for corner coloring in Section 4 is also valid for edge coloring. For brevity, the detailed statements of primitive commutative cycles for edge coloring are omitted. The results concerning strong specification in Section 5 apply to the edge coloring problem and detailed statements are also omitted here.*

(ii) *It is known edge-coloring problem can convert into corner-coloring problem, but the number of symbols will become large in general. In practice, many problems are more convenient to be studied by transforming into edge-coloring problems; see Examples 6.7 and 6.8. Hence, the methods for edge coloring are discussed in this section.*

The following six-vertex model (or ice-type model) is used to illustrate Theorem 6.5.

Example 6.7. *The rule of the six-vertex model is that the number of arrows that point inwards at each vertex is two, such that*

$\mathcal{B}_6 =$ 

For ease of computing, the rightward and upward arrows in each pattern in $\mathcal{B}$ *are replaced by the digit (color)* 1 *and the leftward and downward arrows in each pattern*

are replaced by the digit (color) 0*. Then, the six-vertex model can be transformed into an edge-coloring problem with the colors in* $\mathcal{S}_2 = \{0, 1\}$*. Indeed,*

$$\mathcal{B}_{e;6} = \Big\{ \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 1 & & 1 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \end{smallmatrix} \Big\}.$$

Clearly,

$$\mathbb{H}^e_{2;1} = \mathbb{H}^e_{2;4} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbb{H}^e_{2;2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \textit{and} \quad \mathbb{H}^e_{2;3} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

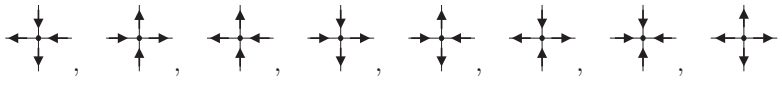
Also, $\mathbb{V}^e_{2;j} = \mathbb{H}^e_{2;j}$*,* $1 \le j \le 4$*. From (6.4),*

$$\mathbb{S}^e_{2;1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is easily verified. Then, $\overline{\beta}_1 = 1$ *is an* S_e*-invariant diagonal sequence of order* $(2, 1)$ *with index set* $\mathcal{K} = \{1, 2, 3, 4\}$*. Clearly,* $\sum_{l=1}^{4} \bar{H}^{(l)}_{2,2}$ *is primitive. Hence, by Theorem 6.5,* $\mathbb{H}^e_n$ *is primitive for all* $n \ge 2$*. Since* $\mathbb{V}^e_{2;j} = \mathbb{H}^e_{2;j}$*,* $1 \le j \le 4$*,* $\mathbb{V}^e_n$ *is also primitive for all* $n \ge 2$*. Therefore, the six-vertex model is mixing. However, it does not have strong specification.*

The following well-known eight-vertex model is shown to have strong specification.

Example 6.8. *The rule of the eight-vertex model is that the number of arrows that point inwards at each vertex is even, such that*

$\mathcal{B}_8 =$ 

As in Example 6.7, the basic set $\mathcal{B}_8$ *of the eight-vertex model can be transformed as follows;*

$$\mathcal{B}_{e;8} = \Big\{ \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix} \Big\}.$$

Indeed, $\mathcal{B}_{e;8}$ *is the set of all tiles in* $\Sigma^e_{2\times 2}$ *that have even sums of digits on their four edges.*

It can be verified that for any $k \ge 2$*, the following admissible pattern* U *on* $\mathcal{A}_{M\times N}$*,* $M, N \ge 2k - 3$ *, can be extended to* $\mathcal{A}_{(M+4-2k)\times(N+4-2k);k}((k-2, k-2))$ *but* U *can not extend to* $\mathbb{Z}_{(M+4)\times(N+4)}((-2, -2))$*. Therefore,* $\mathcal{B}_{e;8}$ *does not satisfy* $(HFC)_k$ *for all* $k \ge 2$*.*

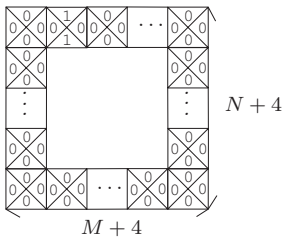
$U =$ 

Figure 6.1.

Before proving strong specification of $\mathcal{B}_{e;8}$, some notation must be defined. A lattice $R_{sc} = \bigcup_{(i,j)\in\mathcal{I}} \mathbb{Z}_{2\times 2}((i,j))$, $\mathcal{I} \subset \mathbb{Z}^2$, is called a simple-curve lattice if each of its unit square lattices has exactly two shared edges that connect to two other unit square lattices of R_{sc}. Clearly, two kinds of simple-curve lattices exist. They are (i) bounded simple-curve lattices and (ii) unbounded simple-curve lattices.

(i) A bounded simple-curve lattice. *(ii) An unbounded simple-curve lattice.*

Figure 6.2.

Now, the following facts can be easily verified and the proofs are omitted for brevity.

Fact 1. *Suppose R_{sc} is unbounded. If the non-shared edges on R_{sc} are colored with 0 and 1, then there exists exactly one pattern $U \in \Sigma_{R_{sc}}(\mathcal{B}_{e;8})$ in which colors on the non-shared edges are as assumed.*

Fact 2. *Suppose R_{sc} is bounded and its non-shared edges are colored with 0 and 1. Let $\mathcal{E}_i(R_{sc})$ be the set of all interior non-shared edges and $\mathcal{E}_o(R_{sc})$ be the set of all exterior non-shared edges.*

Given a global pattern $W \in \Sigma(\mathcal{B}_{e;8})$, for any bounded simple-curve lattice R_{sc}, both sums of the digits (colors) of $W\mid_{R_{sc}}$ on $\mathcal{E}_i(R_{sc})$ and $\mathcal{E}_o(R_{sc})$ are even.

Conversely, if both sums of digits on $\mathcal{E}_i(R_{sc})$ and on $\mathcal{E}_o(R_{sc})$ are even, then there exists a pattern $U \in \Sigma_{R_{sc}}(\mathcal{B}_{e;8})$ in which the colors of its non-shared edges are as assumed.

Let two allowable patterns $U_1 \in \Pi_{R_1}(W_1)$ and $U_2 \in \Pi_{R_2}(W_2)$ with $d(R_1, R_2) \geq 2$ where $W_1, W_2 \in \Sigma(\mathcal{B}_{e;8})$ and $R_1, R_2 \subset \mathbb{Z}^2$. The Jordan curve theorem can be used to verify the existence of $\widetilde{R}_i \supseteq R_i$, $i \in \{1,2\}$, satisfting the following;

(i) *$\widetilde{R}_1 \bigcup \widetilde{R}_2 = \mathbb{Z}^2$ and $\widetilde{R}_1 \bigcap \widetilde{R}_2 = \emptyset$,*

(ii) *the union of all $\mathbb{Z}_{2\times 2}((i,j))$ with $\mathbb{Z}_{2\times 2}((i,j)) \bigcap \widetilde{R}_1 \neq \emptyset$ and $\mathbb{Z}_{2\times 2}((i,j)) \bigcap \widetilde{R}_2 \neq \emptyset$, can be represented as a union of simple-curve lattices that do not overlap each other.*

Then, a global pattern $W \in \Sigma(\mathcal{B}_{e;8})$ with $W\mid_{R_1} = U_1$ and $W\mid_{R_2} = U_2$ can be constructed by the following steps.

Step 1. *If $\mathbb{Z}_{2\times 2}((i,j)) \subset \widetilde{R}_k$, $k \in \{1,2\}$, color $W\mid_{\mathbb{Z}_{2\times 2}((i,j))} = W_k\mid_{\mathbb{Z}_{2\times 2}((i,j))}$.*

Step 2. *From Facts 1 and 2 above, the unit square lattices that remain after Step 1 can be tiled with the tiles in $\mathcal{B}_{e;8}$.*

Therefore, $\Sigma(\mathcal{B}_{e;8})$ has strong specification.

Remark 6.9. *By converting edge coloring into corner coloring, Example 6.8 is an example for strong specification $\not\Rightarrow$ (HFC)$_k$.*

Appendix A.

In this Appendix, $(\mathrm{HFC})_k$, UFP and corner gluing are expressed in terms of $\mathbb{H}_n$ and $\mathbb{S}_m$, which are very useful in verification by using numerical computation.

First, $(\mathrm{HFC})_k$ can be expressed in terms of the horizontal transition matrices $\mathbb{H}_n$ and the connecting operators $S_{m;\alpha,\beta}$ and $W_{m;\alpha,\beta}$.

Theorem A.1. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, for $k \geq 2$, $\mathcal{B}$ satisfies $(HFC)_k$ with size (M,N) if and only if for $1 \leq i,j \leq p^M$ and $1 \leq \alpha_l \leq p^2$, $1 \leq l \leq N+4$, if*

(i) (a) $\sum_{q=1}^{p^M} \left(S_{M+1;\alpha_1,\alpha_2}\right)_{q,i} \geq 1,$ (b) $\sum_{q=1}^{p^M} \left(S_{M+1;\alpha_{N+3},\alpha_{N+4}}\right)_{j,q} \geq 1,$

(c) $\sum_{q=1}^{p^{N+4}} \left(\mathbb{H}_{N+4}\right)_{q,s} \geq 1,$ (d) $\sum_{q=1}^{p^{N+4}} \left(\mathbb{H}_{N+4}\right)_{t,q} \geq 1,$

where

$$\begin{cases} s = \psi\left(\alpha_{1,1}, \alpha_{2,1}, \cdots, \alpha_{N+4,1}\right) \text{ and } t = \psi\left(\alpha_{1,2}, \alpha_{2,2}, \cdots, \alpha_{N+4,2}\right) \\ \alpha_{n,1}, \alpha_{n,2} \in \mathcal{S}_p \text{ such that } \psi\left(\alpha_{n,1}, \alpha_{n,2}\right) = \alpha_n, 1 \leq n \leq N+4, \end{cases}$$

(ii) *there exist $1 \leq i', j' \leq p^M$ with $i' = \psi\left(i'_1, i'_2, \cdots, i'_M\right)$ and $j' = \psi\left(j'_1, j'_2, \cdots, j'_M\right)$ such that*

(a)
$$\left(S_{M+1;\alpha_2,\alpha_3} S_{M+1;\alpha_3,\alpha_4}, \cdots, S_{M+1;\alpha_{k-1},\alpha_k}\right)_{i,i'} \geq 1,$$

(b)
$$\left(S_{M+1;\alpha_{N_1+1},\alpha_{N_1+2}} S_{M+1;\alpha_{N_1+2},\alpha_{N_1+3}}, \cdots, S_{M+1;\alpha_{N+2},\alpha_{N+3}}\right)_{j',j} \geq 1,$$

(c)
$$\sum_{q=1}^{p^{N_2}} \left(W_{N_2+1;\beta_1,\beta_2} W_{N_2+1;\beta_2,\beta_3}, \cdots, W_{N_2+1;\beta_{k-2},\beta_{k-1}}\right)_{s',q} \geq 1,$$

(d)
$$\sum_{q=1}^{p^{N_2}} \left(W_{N_2+1;\beta'_1,\beta'_2} W_{N_2+1;\beta'_2,\beta'_3}, \cdots, W_{N_2+1;\beta'_{k-2},\beta'_{k-1}}\right)_{q,t'} \geq 1,$$

where

$$\begin{cases} N_1 = N+4-k \text{ and } N_1 = N+4-2k, \\ s' = \psi\left(\alpha_{k+1,1}, \alpha_{k+2,1}, \cdots, \alpha_{N_1,1}\right) \text{ and } t' = \psi\left(\alpha_{k+1,2}, \alpha_{k+2,2}, \cdots, \alpha_{N_1,2}\right) \\ \beta_1 = \psi(\alpha_{k,1}, \alpha_{N_1+1,1}) \text{ and } \beta_l = \psi(i'_{l-1}, j'_{l-1}), l \in \{2,3,\cdots,k-1\}, \\ \beta'_l = \psi(i'_{M-k+2+l}, j'_{M-k+2+l}), l \in \{1,2,\cdots,k-2\}, \text{ and } \beta'_{k-1} = \psi(\alpha_{k,2}, \alpha_{N_1+1,2}), \end{cases}$$

then

$$\left(S_{M+1;\alpha_2,\alpha_3} S_{M+1;\alpha_3,\alpha_4}, \cdots, S_{M+1;\alpha_{N+2},\alpha_{N+3}}\right)_{i,j} \geq 1.$$

Notably, conditions (i) (a)$\sim$(d) present that the patterns U on $\mathcal{A}_{M\times N}$ are $\mathcal{B}$-admissible as in Theorem 5.2. As in Fig. A.1, conditions (ii) (a)$\sim$(d) present that U can be extended to $\mathcal{A}_{(M+4-2k)\times(N+4-2k);k}((k-2,k-2))$.

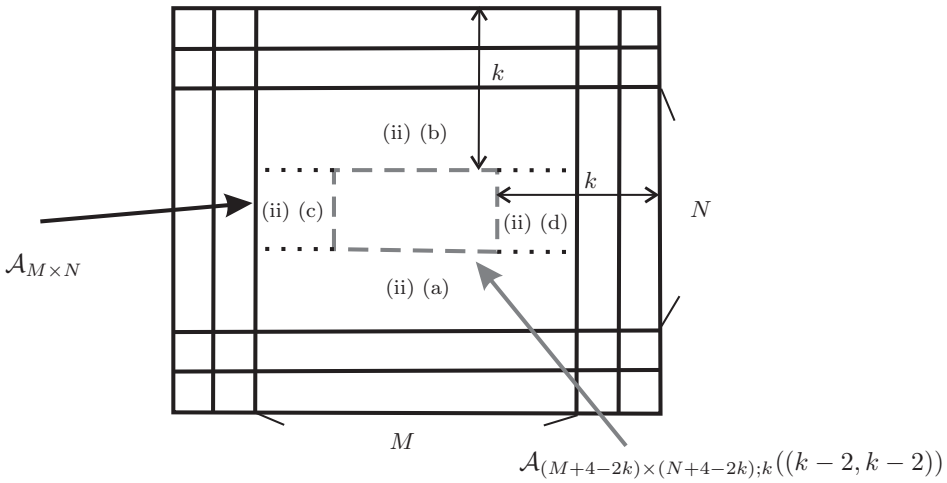


Figure A.1.

Next, UFP with rectangle-extendability can be expressed in terms of the horizontal transition matrices $\mathbb{H}_n$ and the connecting operators $S_{m;\alpha,\beta}$.

Theorem A.2. *Suppose $\mathcal{B}\subset\Sigma_{2\times 2}(p)$ is rectangle-extendable. $\Sigma(\mathcal{B})$ has the UFP if and only if there exist a positive integer g such that for $m,n\geq 2$ and $1\leq a,b\leq p^{m-2}$, $1\leq c,d\leq p^n$, $1\leq i,j\leq p^{m+2g+4}$ and $1\leq s,t\leq p^{n+2g+4}$ with*

$$\begin{cases} a=\psi(a_1,a_2,\cdots,a_{m-2}) & \text{and} \quad b=\psi(b_1,b_2,\cdots,b_{m-2}),\\ c=\psi(c_1,c_2,\cdots,c_n) & \text{and} \quad d=\psi(d_1,d_2,\cdots,d_n),\\ i=\psi(i_1,i_2,\cdots,i_{m+2g+4}) & \text{and} \quad j=\psi(j_1,j_2,\cdots,j_{m+2g+4}),\\ s=\psi(s_1,s_2,\cdots,s_{n+2g+4}) & \text{and} \quad t=\psi(t_1,t_2,\cdots,t_{n+2g+4}),\end{cases}$$

if

(i) (a) $\sum_{q=1}^{p^{\bar m}-4}\left(S_{\bar m-3;\alpha_1,\alpha_2}\right)_{q,i}\geq 1,$ (b) $\sum_{q=1}^{p^{\bar m-4}}\left(S_{\bar m-3;\alpha_{\bar n-1},\alpha_{\bar n}}\right)_{j,q}\geq 1,$

(c) $\sum_{q=1}^{p^{\bar n}}\left(\mathbb{H}_{\bar n}\right)_{q,s}\geq 1,$ (d) $\sum_{q=1}^{p^{\bar n}}\left(\mathbb{H}_{\bar n}\right)_{t,q}\geq 1,$

(e) $\left(S_{\bar m-3;\alpha_2,\alpha_3}S_{\bar m-3;\alpha_3,\alpha_4},\cdots,S_{\bar m-3;\alpha_{\bar n-2},\alpha_{\bar n-1}}\right)_{i,j}\geq 1,$

where

$$\begin{cases}\bar m=m+2g+4 \text{ and } \bar n=n+2g+4,\\ \alpha_l=\psi\left(s_l,t_l\right), 1\leq l\leq \bar n,\end{cases}$$

(ii) $\left(S_{m-1;\beta_1,\beta_2}S_{m-1;\beta_2,\beta_3}\cdots S_{m-1;\beta_{n-1},\beta_n}\right)_{a,b}\geq 1,$

where $\beta_l=\psi(c_l,d_l), l\in\{1,2,\cdots,n\}$,

then there exist $\xi_l,\xi_l',\eta_l,\eta_l'\in\mathcal{S}_p$, $1\leq l\leq g$, such that

(1) $\left(S_{\bar m-3;\alpha_2,\alpha_3}S_{\bar m-3;\alpha_3,\alpha_4}\cdots S_{\bar m-3;\alpha_{g+2},\alpha_{g+3}}\right)_{i,i'}\geq 1,$

(2) $\left(S_{g+1;\gamma_1,\gamma_2}S_{g+1;\gamma_2,\gamma_3}\cdots S_{g+1;\gamma_{n-1},\gamma_n}\right)_{i^{(1)},i^{(2)}}\geq 1,$

(3) $\left(S_{g+1;\gamma_1',\gamma_2'}S_{g+1;\gamma_2',\gamma_3'}\cdots S_{g+1;\gamma_{n-1}',\gamma_n'}\right)_{i^{(3)},i^{(4)}}\geq 1,$

(4) $\left(S_{\bar{m}-3;\alpha_{n+g+2},\alpha_{n+g+3}} S_{\bar{m}-3;\alpha_{n+g+3},\alpha_{n+g+4}} \cdots S_{\bar{m}-3;\alpha_{\bar{n}-2},\alpha_{\bar{n}-3}}\right)_{j',j} \geq 1,$

where

$$\begin{cases} i' = \psi(\xi_1, \xi_2, \cdots, \xi_g, c_1, a_1, a_2, \cdots, a_{m-2}, d_1, \xi'_1, \xi'_2, \cdots, \xi'_g), \\ j' = \psi(\eta_1, \eta_2, \cdots, \eta_g, c_n, b_1, b_2, \cdots, b_{m-2}, d_n, \eta'_1, \eta'_2, \cdots, \eta'_g), \\ i^{(1)} = \psi(\xi_1, \xi_2, \cdots, \xi_g) \text{ and } i^{(2)} = \psi(\eta_1, \eta_2, \cdots, \eta_g), \\ i^{(3)} = \psi(\xi'_1, \xi'_2, \cdots, \xi'_g) \text{ and } i^{(4)} = \psi(\eta'_1, \eta'_2, \cdots, \eta'_g). \end{cases}$$

For example, let $m = n = 3$ and $g = 2$; see Fig. A.2.

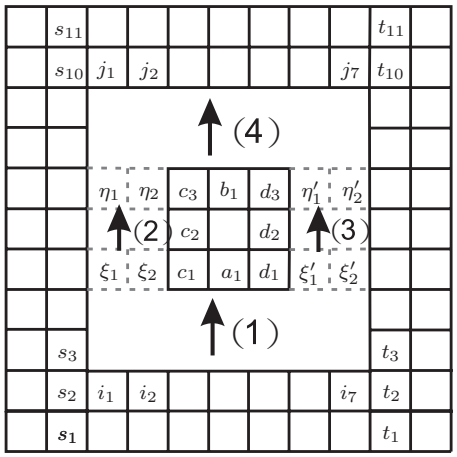


Figure A.2.

Similarly, corner gluing condition with rectangle-extendability also can be expressed in terms of the horizontal transition matrices $\mathbb{H}_n$, $\mathbb{V}_n$ and the connecting operators $S_{m;\alpha,\beta}$.

Theorem A.3. *Suppose $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ is rectangle-extendable. $\Sigma(\mathcal{B})$ is corner gluing if and only if there exist a positive integer g such that for $m, n \geq 2$ and $1 \leq a \leq p^m$, $1 \leq b \leq p^{n-1}$, $1 \leq s \leq p^{n+g+2}$ and $1 \leq t \leq p^{m+g}$ with*

$$\begin{cases} a = \psi(a_1, a_2, \cdots, a_m) & \text{and} \quad b = \psi(b_1, b_2, \cdots, b_{n-1}), \\ s = \psi(s_1, s_2, \cdots, s_{n+g+2}) & \text{and} \quad t = \psi(t_1, t_2, \cdots, t_{m+g}), \end{cases}$$

if

(i) (a) $\sum_{q=1}^{p^{n+g+2}} (\mathbb{H}_{n+g+2})_{q,s} \geq 1,$ (b) $\sum_{q=1}^{p^{m+g}} (V_{m+g+1;s_1+1,s_2+1})_{q,t} \geq 1,$

(c) *there exist $c_j \in \mathcal{S}_p$, $1 \leq j \leq n+g$, such that*

$$\sum_{q=1}^{p^{m+g-1}} \left(S_{m+g;\alpha_1,\alpha_2} S_{m+g;\alpha_2,\alpha_3}, \cdots, S_{m+g;\alpha_{n+g},\alpha_{n+g+1}}\right)_{t',q} \geq 1,$$

where

$$\begin{cases} t' = \psi(t_1, t_2, \cdots, t_{m+g-1}), \\ \alpha_1 = \psi\left(s_2, t_{m+g}\right) \text{ and } \alpha_l = \psi\left(s_{l+1}, t_{l-1}\right), 2 \leq l \leq n+g+1, \end{cases}$$

(ii) *there exist* $d_j \in \mathcal{S}_p$, $1 \le j \le n-1$, *such that*

$$\sum_{q=1}^{p^{m-2}} \left(S_{m-1;\beta_1,\beta_2} S_{m-1;\beta_2,\beta_3}, \cdots, S_{m-1;\beta_{n-1},\beta_n}\right)_{a',q} \ge 1,$$

where

$$\begin{cases} a' = \psi(a_2, a_3, \cdots, a_{m-1}), \\ \alpha_1 = \psi\left(a_1, a_m\right) \ \text{and} \ \alpha_l = \psi\left(b_{l-1}, d_{l-1}\right), 2 \le l \le n, \end{cases}$$

then there exist $\xi_l, \eta_l \in \mathcal{S}_p$, $1 \le l \le g$, *such that*

(1) $\left(S_{m+g;\gamma_1,\gamma_2} S_{m+g;\gamma_2,\gamma_3} \cdots S_{m+g;\gamma_{g+1},\gamma_{g+2}}\right)_{t',\eta'} \ge 1$,

(2) $\sum_{q=1}^{p^g} \left(S_{g+1;\delta_1,\delta_2} S_{g+1;\delta_2,\delta_3} \cdots S_{g+1;\delta_{n-1},\delta_n}\right)_{\eta,q} \ge 1$,

where

$$\begin{cases} \gamma_1 = \psi\left(s_2, t_{m+g}\right) \ \text{and} \ \gamma_{g+2} = \psi\left(s_{g+3}, a_m\right), \\ \gamma_l = \psi\left(s_{l+1}, \xi_{l-1}\right), 2 \le l \le g+2, \\ \delta_1 = \psi\left(s_{g+3}, a_1\right) \ \text{and} \ \delta_l = \psi\left(s_{g+l+2}, b_{l-1}\right), 2 \le l \le n, \\ \eta = \psi(\eta_1, \eta_2, \cdots, \eta_g) \ \text{and} \ \eta' = \psi(\eta_1, \eta_2, \cdots, \eta_g, a_1, a_2, \cdots, a_{m-1}). \end{cases}$$

For example, let $m = n = 3$ and $g = 2$; see Fig. A.3.

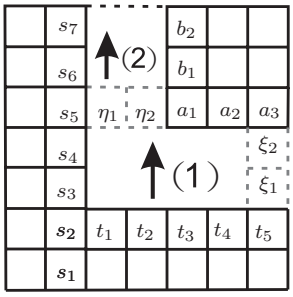


Figure A.3.

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