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Constructing error-correcting pooling designs with symplectic space

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Abstract We construct a family of error-correcting pooling designs with the incidence matrix of two types of subspaces of symplectic spaces over finite fields. We show that the new construction gives better ratio of efficiency compared with previously known three constructions associated with subsets of a set, its analogue over a vector space, and the dual spaces of a symplectic space.

Keywords Pooling designs · d^e-disjunct matrix · Symplectic space · Totally isotropic subspaces · Non-isotropic subspaces

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1 Introduction

Given a set of n items with some defectives, the *group testing* problem is asking to identify all defectives with the minimum number of tests each of which is on a subset of items, called a *pool*, and the test-outcome is *negative* if the pool does not contain any defective and *positive* if the pool contains a defective. A *pooling design* is a grouping testing algorithm of special type, also called *nonadaptive group testing*, in which all pools are given at the beginning of the algorithm so that no test-outcome of one pool can effect the determination of another pool. The pooling design has many applications in molecular biology, such as DNA library screening, nonunique probe selection, gene detection, etc. (Du and Hwang 2006; Du et al. 2006; D'yachkov et al. 2005).

A pooling design can be represented by a binary matrix whose columns are indexed with items and rows are indexed with pools; an entry at cell (i, j) is 1 if the ith pool contains the jth item, and 0, otherwise. Consider a sample with defectives $j_1, j_2, \dots, j_d$. If we look each column j as the set of all pools that contain item j, then the test-outcome is the union of columns $j_1, j_2, \dots, j_d$. A pooling design that can identify up to d defectives must have different test-outcomes for all possible sets of at most d defectives, that is, for all subsets of at most d columns, their unions are different. Such a binary matrix is said to be $\bar{d}$*-separable*. For a $\bar{d}$-separable matrix, decoding from a test-outcome to determine all defectives is not efficient. The best known algorithm runs in time $O(n^d)$ where n is the total number of items and if $NP \neq P$, there does not exist a decoding algorithm with running time polynomially with respect to d.

To have an efficient decoding method, one usually construct pooling designs with a little stronger property that each column cannot be contained by the union of other d columns. Such a binary matrix is said ton be d*-disjunct*. With d-disjunct pooling design, decoding is very simple. Remove all items in negative pools. The remaining items are all defectives.

In practice, test-outcomes may contain errors. To make pooling design error tolerant, one introduced the concept of d^e*-disjunct* matrix (Macula 1996). A binary matrix M is said to be d^e*-disjunct* if given any $d + 1$ columns of M with one designated, there are $e + 1$ rows with a 1 in the designated column and 0 in each of the other d columns. The d^0-disjunctness is actually the d-disjunctness. D'yachkov et al. (2007) proposed the concept of fully d^e-disjunct matrices. An d^e-disjunct matrix is *fully* d^e*-disjunct* if it is not c^b-disjunct whenever $c > d$ or $b > e$.

There are several constructions of d^e-disjunct matrices in the literature (Balding and Torney 1996; Erdös et al. 1985; Guo 2009; Huang and Weng 2004; Li et al. 2009; Macula 1997; Nan and Guo 2009; Ngo and Du 2002; Zhang et al. 2008, 2009). In this paper we present a new construction associated with subspaces in $\mathbb{F}_q^{(2\nu)}$. We show that the ratio between the number of pools and the number of items for our new construction is better than those in D'yachkov et al. (2005), Macula (1996) and Zhang et al. (2008). We find it smaller under some conditions.

2 The symplectic space

In this section we will first introduce the concepts of symplectic space, and then introduce some counting formulas in the symplectic space.

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a prime power, let $\mathbb{F}_q^{(2\nu)}$ be the 2ν-dimensional row vector space over $\mathbb{F}_q$ and let

$$K = \begin{pmatrix} 0 & I^{(\nu)} \\ -I^{(\nu)} & 0 \end{pmatrix}.$$

The *symplectic group* of degree 2ν over $\mathbb{F}_q$, denoted by $Sp_{2\nu}(\mathbb{F}_q)$, consists of all $2\nu \times 2\nu$ nonsingular matrices T over $\mathbb{F}_q$ satisfying $TKT^T = K$. There is an action of $Sp_{2\nu}(\mathbb{F}_q)$ on $\mathbb{F}_q^{(2\nu)}$ defined as follows:

$$\begin{aligned} \mathbb{F}_q^{(2\nu)} \times Sp_{2\nu}(\mathbb{F}_q) &\longrightarrow \mathbb{F}_q^{(2\nu)}, \\ ((x_1, x_2, \ldots, x_{2\nu}), T) &\longmapsto (x_1, x_2, \ldots, x_{2\nu})T. \end{aligned}$$

The vector space $\mathbb{F}_q^{(2\nu)}$ together with the above group action of the symplectic group $Sp_{2\nu}(\mathbb{F}_q)$, is called 2ν-dimensional *symplectic space* over $\mathbb{F}_q$.

Let P be an m-dimensional subspace of $\mathbb{F}_q^{(2\nu)}$, denote also by P an $m \times 2\nu$ matrix of rank m whose rows span the subspace P. An m-dimensional subspace P is said to be of *type* (m, s) if PKP^t is of rank $2s$. In particular, subspaces of type $(m, 0)$ are called m-dimensional *totally isotropic subspaces*, and subspaces of type $(2s, s)$ are called $2s$-dimensional *non-isotropic subspaces*. By (Wan 2002, Theorem 3.22) we know that subspaces of type (m, s) exist if and only if $2s \le m \le \nu + s$. Denote by $\mathcal{M}(m, s; 2\nu)$ the set of all subspaces of $\mathbb{F}_q^{(2\nu)}$ of type (m, s). Then we have

Proposition 2.1 (Wan 2002, Theorem 3.18) *Let* $2s \le m \le \nu + s$. *Then*

$$|\mathcal{M}(m, s; 2\nu)| = q^{2s(\nu+s-m)} \frac{\prod_{l=\nu+s-m+1}^{\nu}(q^{2l} - 1)}{\prod_{l=1}^{s}(q^{2l} - 1)\prod_{l=1}^{m-2s}(q^l - 1)}.$$

Denote by $\mathcal{M}(m_1, s_1; m, s; 2\nu)$ the set of all subspaces of type (m_1, s_1) contained in a given subspace of type (m, s), and denote by $\mathcal{M}'(m_1, s_1; m, s; 2\nu)$ the set of all subspaces of type (m, s) containing a given subspace of type (m_1, s_1). Then we have

Proposition 2.2 (Wan 2002, Theorems 3.26 and 3.37) $\mathcal{M}(m_1, s_1; m, s; 2\nu)$ (*resp.* $\mathcal{M}'(m_1, s_1; m, s; 2\nu)$) *is non-empty if and only if*

$$2s \le m \le \nu + s \text{ and } \max\{0, m_1 - s - s_1\} \le \min\{m - 2s, m_1 - 2s_1\}. \quad (1)$$

To deal with the disjunct property in Sect. 3, the following counting results are included for later reference.

Proposition 2.3 (Wan 2002, Theorem 3.27) *Suppose that* (1) *holds. Then*

$$|\mathcal{M}(m_1, s_1; m, s; 2\nu)| = \sum_{k=\max\{0, m_1-s-s_1\}}^{\min\{m-2s, m_1-2s_1\}} q^{2s_1(s+s_1-m_1+k)+(m_1-k)(m-2s-k)} \times \frac{\prod_{l=s+s_1-m_1+k+1}^{s}(q^{2l}-1)\prod_{l=m-2s-k+1}^{m-2s}(q^l-1)}{\prod_{l=1}^{s_1}(q^{2l}-1)\prod_{l=1}^{m_1-2s_1-k}(q^l-1)\prod_{l=1}^{k}(q^l-1)}.$$

Proposition 2.4 (Wan 2002, Theorem 3.38) *Suppose that* (1) *holds. Then*

$$|\mathcal{M}'(m_1, s_1; m, s; 2\nu)| = \sum_{k=\max\{0, m_1-s-s_1\}}^{\min\{m-2s, m_1-2s_1\}} q^{2(\nu+s-m)(s+s_1-m_1+k)+(2\nu-m-k)(m_1-2s_1-k)} \times \frac{\prod_{l=s+s_1-m_1+k+1}^{\nu+s_1-m_1}(q^{2l}-1)\prod_{l=m_1-2s_1-k+1}^{m_1-2s_1}(q^l-1)}{\prod_{l=1}^{\nu+s-m}(q^{2l}-1)\prod_{l=1}^{m-2s-k}(q^l-1)\prod_{l=1}^{k}(q^l-1)}.$$

Theorem 2.5 *For $1 \le m < \nu$, the sequence $|\mathcal{M}(2m, m; 2\nu)|$ is unimodal and gets its peak at $m = \lfloor \frac{\nu}{2} \rfloor$ or $\lfloor \frac{\nu+1}{2} \rfloor$.*

Proof By Proposition 2.1 we have

$$\frac{|\mathcal{M}(2m_2, m_2; 2\nu)|}{|\mathcal{M}(2m_1, m_1; 2\nu)|} = q^{2(m_2-m_1)(\nu-m_2-m_1)} \frac{(q^{2(\nu-m_2+1)}-1)\cdots(q^{2(\nu-m_1)}-1)}{(q^{2(m_1+1)}-1)\cdots(q^{2m_2}-1)}.$$

If $\lfloor \frac{\nu+1}{2} \rfloor \le m_1 < m_2 < \nu$, then $|\mathcal{M}(2m_2, m_2; 2\nu)| < |\mathcal{M}(2m_1, m_1; 2\nu)|$. If $1 \le m_1 < m_2 \le \lfloor \frac{\nu}{2} \rfloor$, then $|\mathcal{M}(2m_2, m_2; 2\nu)| > |\mathcal{M}(2m_1, m_1; 2\nu)|$. □

3 The construction

In this section, we construct a family of inclusion matrices associated with subspaces of $\mathbb{F}_q^{(2\nu)}$, and exhibit its disjunct property.

Definition 3.1 Given integers $1 \le r \le m < \nu$. Let $M(r, 2m; 2\nu)$ be the binary matrix whose rows (resp. columns) are indexed by $\mathcal{M}(r, 0; 2\nu)$ (resp. $\mathcal{M}(2m, m; 2\nu)$). We also order elements of these sets lexicographically. $M(r, 2m; 2\nu)$ has a 1 in row i and column j if and only if the i-th subspace of $\mathcal{M}(r, 0; 2\nu)$ is a subspace of the j-th subspace of $\mathcal{M}(2m, m; 2\nu)$.

By Propositions 2.1, 2.3 and 2.4, $M(r, 2m; 2\nu)$ is a $|\mathcal{M}(r, 0; 2\nu)| \times |\mathcal{M}(2m, m; 2\nu)|$ matrix, whose constant row (resp. column) weight is $|\mathcal{M}'(r, 0; 2m, m; 2\nu)|$ (resp. $|\mathcal{M}(r, 0; 2m, m; 2\nu)|$). Theorem 2.5 tells us how to choose m so that the test to item is minimized.

Theorem 3.2 *Let* $1 \le r \le m < \nu$, *and let* $\beta = |\mathcal{M}(r, 0; 2m, m; 2\nu)|$, $\gamma = |\mathcal{M}(r, 0; 2m-1, m-1; 2\nu)|$, $\delta = |\mathcal{M}(r, 0; 2m-2, m-1; 2\nu)|$, $\xi = |\mathcal{M}(r, 0; 2m-2, m-2; 2\nu)|$ *and* $\alpha = \max\{\gamma - \delta, \gamma - \xi\}$. *Then the following* (i)–(iii) *hold*:

(i) *For* $m \ge 2$ *and* $m \ge r+1$, *if* $1 \le d \le \lfloor \frac{\beta-\gamma-1}{\alpha} \rfloor + 1$, *then* $M(r, 2m; 2\nu)$ *is* d^e*-disjunct, where* $e = \beta - \gamma - (d-1)\alpha - 1$. *In particular, if* $1 \le d \le \min\{\lfloor \frac{\beta-\gamma-1}{\alpha} \rfloor + 1, q+1\}$, *then* $M(r, 2m; 2\nu)$ *is fully* d^e*-disjunct.*

(ii) *For* $m \ge 2$ *and* $m = r$, *if* $1 \le d \le \lfloor \frac{\beta-1}{\gamma} \rfloor$, *then* $M(r, 2m; 2\nu)$ *is* d^e*-disjunct, where* $e = \beta - d\gamma - 1$. *In particular, if* $1 \le d \le \min\{\lfloor \frac{\beta-1}{\gamma} \rfloor, q+1\}$, *then* $M(r, 2m; 2\nu)$ *is fully* d^e*-disjunct.*

(iii) *For* $m = 1$, *if* $1 \le d \le q$, *then* $M(1, 2; 2\nu)$ *is fully* d^e*-disjunct, where* $e = q - d$.

Proof (i) Let $C, C_1, C_2, \ldots, C_d$ be $d+1$ distinct columns of $M(r, 2m; 2\nu)$. To obtain the maximum numbers of subspaces of $\mathcal{M}(r, 0; 2\nu)$ in

$$C \cap \bigcup_{i=1}^{d} C_i = \bigcup_{i=1}^{d} (C \cap C_i),$$

we may assume that $\dim(C \cap C_i) = 2m-1$ and $\dim(C \cap C_i \cap C_j) = \dim((C \cap C_i) \cap (C \cap C_j)) = 2m-2$ for any two distinct i and j, where $1 \le i, j \le d$. Since C is a $2m$-dimensional non-isotropic subspace of $\mathbb{F}_q^{(2\nu)}$, $C \cap C_i$ (resp. $C \cap C_i \cap C_j$) is a subspace of type $(2m-1, m-1)$ (resp. type $(2m-2, m-1)$ or type $(2m-2, m-2)$) of $\mathbb{F}_q^{(2\nu)}$ by Proposition 2.2. By Proposition 2.2, $\delta > 0$ and $\xi > 0$. By Proposition 2.3, the number of subspaces of C not covered by $C_1, C_2, \ldots, C_d$ is at least

$$\beta - d\gamma + (d-1) \times \min\{\delta, \xi\} = \beta - \gamma - (d-1)\alpha.$$

Hence, we may take $e = \beta - \gamma - (d-1)\alpha - 1$ under the assumption that d. Since $e \ge 0$, we obtain

$$d \le \left\lfloor \frac{\beta - \gamma - 1}{\alpha} \right\rfloor + 1.$$

Now we show that the maximal dimension of $C \cap \bigcup_{i=1}^{d} C_i$ is achieved by an explicit construction. For $C \cap C_1$, by Proposition 2.3, $|\mathcal{M}(2m-2, m-1; 2m-1, m-1; 2\nu)| \ge 1$ and $|\mathcal{M}(2m-2, m-2; 2m-1, m-1; 2\nu)| \ge 1$. Hence there exists a $(2m-2)$-dimensional subspace contained in $C \cap C_1$, denoted by P, such that the number of r-dimensional totally isotropic subspaces contained in P is equal to $\min\{\delta, \xi\}$. By (Wan 2002, Corollary 1.9), the number of $(2m-1)$-dimensional subspaces containing P and contained in C is equal to $q+1$, and each of these subspaces is a subspace of type $(2m-1, m-1)$. For $1 \le d \le \min\{\lfloor \frac{\beta-\gamma-1}{\alpha} \rfloor + 1, q+1\}$, we choose d distinct $(2m-1)$-dimensional subspaces between P and C_0, say P_i $(1 \le i \le d)$. Since $|\mathcal{M}'(2m-1, m-1; 2m, m; 2\nu)| \ge 2$ by Proposition 2.4, for each P_i, we can choose a $2m$-dimensional non-isotropic subspace C_i such that $C \cap C_i = P_i$. Hence, each pair of C_i and C_j overlap at the same subspace P. Therefore, (i) is proved.

For (ii), $\delta = 0$ by Proposition 2.2, and then $\alpha = \gamma$. For (iii), if $m = 1$, then $r = 1$. Both cases can be proved similar to that of case (i), and will be omitted. □

4 Comparison of test efficiency

Erdös et al. (1985) give a formula that $t(d,n) > d(1+o(1))\ln n$, where $t(d,n)$ denotes the minimum number of rows for a d-disjunct matrix with n columns. To take $t/\ln n$ as a measure of the construction of d^e-disjunct matrix is meaningful.

From $t(d,n) > d(1+o(1))\ln n$, we know that the smaller the value of $t/\ln n$ is, the better the pooling design is. Since $t/\ln n$ can be converted to t/n under some conditions, we can take t/n as a measure of the construction is, where t denotes the number of tests, i.e., the number of rows of inclusion matrix, n denotes the number of detected items, i.e., the number of columns of inclusion matrix.

In this paper, assume that the test efficiency is t/n, then

$$\frac{t}{n} = \frac{|\mathcal{M}(r,0;2\nu)|}{|\mathcal{M}(2m,m;2\nu)|} = \frac{\prod_{i=1}^{m}(q^{2i}-1)}{q^{2m(\nu-m)}\prod_{i=\nu-m+1}^{\nu-r}(q^{2i}-1)\prod_{i=1}^{r}(q^{i}-1)}.$$

D'yachkov et al. (2005) constructed with subspaces of $\mathbb{F}_q$, where q is a prime power. Each of the columns (resp. rows) is labeled by an k (resp. d)-dimensional subspace of $\mathbb{F}_q^{(s)}$, where $d < k < s$, $m_{ij} = 1$ if and only if the label of row i is contained in the label of column j. In order to compare with t/n, let $s = 2\nu$, $d = r$ and $k = 2m$. Assume that the test efficiency is t_1/n_1, then

$$\frac{t_1}{n_1} = \frac{\prod_{i=r+1}^{2m}(q^{i}-1)}{\prod_{i=2\nu-2m+1}^{2\nu-r}(q^{i}-1)}.$$

Macula (1996) proposed a way of constructing d-disjunct matrix which uses the containment relation in a structure. More specifically, let $S = \{1, 2, \ldots, s\}$ be the base set, then each of the columns (resp. rows) is labeled by a k (resp. d) subset of S, where $d < k < s$, $m_{ij} = 1$ if and only if the label of row i is contained in the label of column j . In the same way, let $s = 2\nu$, $d = r$ and $k = 2m$. Assume that the test efficiency is t_2/n_2, then

$$\frac{t_2}{n_2} = \frac{(2m)\cdots(r+1)}{(2\nu-r)\cdots(2\nu-2m+1)}.$$

Zhang et al. (2008) constructed a d^z-disjunct matrix with subspaces in a dual space of the symplectic space $\mathbb{F}_q^{(2s)}$, where q is a prime power. Each of the columns (resp. rows) is labeled by subspaces of type $(k,0)$ (resp. subspaces of type $(d,0)$) which are contained in $P_0^{\perp}$ and containing P_0, where $m_0 < d < k < s$ and P_0 is a given subspace of type $(m_0,0)$, $m_{ij} = 1$ if and only if i is contained in j. In the same way, let $s - m_0 = \nu$, $d - m_0 = r$ and $k - m_0 = 2m$. Assume that the test efficiency is t_3/n_3, then

$$\frac{t_3}{n_3} = \frac{\prod_{i=r+1}^{2m}(q^{i}-1)}{\prod_{i=\nu-2m+1}^{\nu-r}(q^{2i}-1)}.$$

Theorem 4.1 *If* $2\nu - 3m \le 0$ *and* $r^2 + r - 4m \ge 0$, *then* $\frac{t/n}{t_1/n_1} < \frac{1}{q^{(r^2+r-4m)/2}}$. *If* $2\nu - 3m > 0$ *and* $r^2 + r - 6m \ge 0$, *then* $\frac{t/n}{t_1/n_1} < \frac{1}{q^{(r^2+r-6m)/2}}$.

Proof If $2\nu - 3m \leq 0$ and $r^2 + r - 4m \geq 0$, then we have

$$\begin{aligned}
\frac{t/n}{t_1/n_1} &= \frac{\prod_{i=1}^{m}(q^i+1)\prod_{i=2\nu-2m+1}^{2\nu-r}(q^i-1)}{q^{2m(\nu-m)}\prod_{i=\nu-m+1}^{\nu-r}(q^{2i}-1)\prod_{i=m+1}^{2m}(q^i-1)} \\
&= \frac{\prod_{i=1}^{m}(q^i+1)q^{m(3m-2\nu)}\prod_{i=2\nu-2m+1}^{2\nu-m}(q^i-1)\prod_{i=2\nu-m+1}^{2\nu-r}(q^i-1)}{q^{2m(\nu-m)+m(3m-2\nu)}\prod_{i=\nu-m+1}^{\nu-r}(q^{2i}-1)\prod_{i=r+1}^{2m}(q^i-1)} \\
&< \frac{\prod_{i=1}^{m}(q^i+1)\prod_{i=2\nu-m+1}^{2\nu-r}(q^i-1)}{q^{m^2}\prod_{i=\nu-m+1}^{\nu-r}(q^{2i}-1)} \\
&< \frac{\prod_{i=1}^{m}q^{i+1}\prod_{i=2\nu-m+1}^{2\nu-r}q^i}{q^{m^2}\prod_{i=\nu-m+1}^{\nu-r}q^{2i-1}} \\
&= \frac{1}{q^{(r^2+r-4m)/2}}.
\end{aligned}$$

If $2\nu - 3m > 0$ and $r^2 + r - 6m \geq 0$, then we have

$$\begin{aligned}
\frac{t/n}{t_1/n_1} &< \frac{\prod_{i=1}^{m}q^{i+1}\prod_{i=2\nu-2m+1}^{2\nu-r}q^i}{q^{2m(\nu-m)}\prod_{i=\nu-m+1}^{\nu-r}q^{2i-1}\prod_{i=m+1}^{2m}q^{i-1}} \\
&= \frac{1}{q^{(r^2+r-6m)/2}}.
\end{aligned}$$

□

Example 4.2 Let $m = r = 4$, $\nu = 6$ and $q = 2$. Then $\frac{t/n}{t_1/n_1} < \frac{1}{4}$.

Theorem 4.3 *Let $\theta = m(4\nu - 2m - 2) - 2r\nu + r(3r+1)/2$. If $2\nu - r \leq 2m$ and $\theta \geq 0$, then $\frac{t/n}{t_2/n_2} < \frac{1}{q^\theta}$. If $2\nu - r > 2m$, $\theta > 0$ and $q \geq (\frac{2\nu-2m+1}{r+1})^{(2m-r)/\theta}$, then $\frac{t/n}{t_2/n_2} < (\frac{2\nu-2m+1}{r+1})^{2m-r}/q^\theta$.*

Proof If $2\nu - r \leq 2m$ and $\theta \geq 0$, then we have

$$\begin{aligned}
\frac{t/n}{t_2/n_2} &= \frac{\prod_{i=1}^{m}(q^i+1)\prod_{i=r+1}^{m}(q^i-1)\times(2\nu-r)\cdots(2\nu-2m+1)}{q^{2m(\nu-m)}\prod_{i=\nu-m+1}^{\nu-r}(q^{2i}-1)\times(2m)\cdots(r+1)} \\
&< \frac{\prod_{i=1}^{m}q^{i+1}\prod_{i=r+1}^{m}q^i}{q^{2m(\nu-m)}\prod_{i=\nu-m+1}^{\nu-r}q^{2i-1}} \\
&= \frac{1}{q^\theta}.
\end{aligned}$$

If $2\nu - r > 2m$, $\theta > 0$ and $q \geq (\frac{2\nu-2m+1}{r+1})^{(2m-r)/\theta}$, then we have

$$\frac{t/n}{t_2/n_2} = \frac{\prod_{i=1}^{m}(q^i+1)\prod_{i=r+1}^{m}(q^i-1)\times(2\nu-r)\cdots(2\nu-2m+1)}{q^{2m(\nu-m)}\prod_{i=\nu-m+1}^{\nu-r}(q^{2i}-1)\times(2m)\cdots(r+1)}$$

$$< \frac{\prod_{i=1}^{m} q^{i+1} \prod_{i=r+1}^{m} q^{i}}{q^{2m(\nu-m)} \prod_{i=\nu-m+1}^{\nu-r} q^{2i-1}} \times \left(\frac{2\nu-2m+1}{r+1}\right)^{2m-r}$$
$$= \frac{(\frac{2\nu-2m+1}{r+1})^{2m-r}}{q^{\theta}}. \qquad \square$$

Example 4.4 Let $m = r = 3$, $\nu = 6$ and $q = 2$. Then $2\nu - r = 9 > 6 = 2m$, $\theta = 9$, $(2\nu - 2m + 1)/(r + 1) = 7/4$ and $2 \geq (7/4)^{3/9}$. Now $\frac{t/n}{t_2/n_2} < \frac{7^3}{2^{15}}$.

Theorem 4.5 *If* $m \geq 2$, *then* $\frac{t/n}{t_3/n_3} < \frac{1}{q^{m(2m-3)}}$.

Proof If $m \geq 2$, then we have

$$\frac{t/n}{t_3/n_3} = \frac{\prod_{i=1}^{m}(q^i+1) \prod_{i=2\nu-2m+1}^{\nu-m}(q^{2i}-1)}{q^{2m(\nu-m)} \prod_{i=m+1}^{2m}(q^i-1)}$$
$$< \frac{\prod_{i=1}^{m} q^{i+1} \prod_{i=2\nu-2m+1}^{\nu-m} q^{2i}}{q^{2m(\nu-m)} \prod_{i=m+1}^{2m} q^{i-1}}$$
$$= \frac{1}{q^{m(2m-3)}}. \qquad \square$$

Example 4.6 Let $m = 3$ and $q = 2$. Then $\frac{t/n}{t_3/n_3} < \frac{1}{8}$.

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