

# Pooling designs associated with unitary space and ratio efficiency comparison

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**Abstract** Let  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$  be a  $(2\nu + \delta)$ -dimensional unitary space of  $\mathbb{F}_{q^2}$ , where  $\delta = 0$  or 1. In this paper we construct a family of inclusion matrices associated with subspaces of  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$ , and exhibit its disjunct property. Moreover, we compare the ratio efficiency of this construction with others, and find it smaller under some conditions.

**Keywords** Pooling designs ·  $d^e$ -disjunct matrix · Unitary space · Totally isotropic subspaces · Non-isotropic subspaces

## 1 Introduction

The basic problem of group testing is to identify the set of defective items in a large population of items. Suppose we have  $n$  items to be tested and that there are at most  $r$  defective items among them. Each *test* (or *pool*) is (or contains) a subset of items. We assume some testing mechanism exists which if applied to an arbitrary subset of the population gives a *negative outcome* if the subset contains no positive and *positive outcome* otherwise. Objectives of group testing vary from minimizing the number of tests, limiting number of pools, limiting pool sizes to tolerating a few errors. It is conceivable that these objectives are often contradicting, thus testing strategies are application dependent. A group testing algorithm is *non-adaptive* if all tests must be specified without knowing the outcomes of other tests. A non-adaptive testing

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algorithm is useful in many areas such as DNA library screening (see Du et al. 2006; D'yachkov et al. 2005; Ngo and Du 1999, for examples).

A group testing algorithm is *error tolerant* if it can detect some errors in test outcomes. A mathematical model of error-tolerance designs is an  $d^e$ -disjunct matrix. Designing good error-tolerant pooling design is a central problem in the area of non-adaptive group testing (see Du and Hwang 2006, for example). To date, there are relatively few papers addressing the problem of designing and analyzing error-tolerant pooling designs (see Aigner 1996; Balding and Torney 1996; Knill et al. 1998; Ngo 2008; Ngo and Du 2002, for examples)

A binary matrix  $M$  is said to be  $d^e$ -disjunct if given any  $d + 1$  columns of  $M$  with one designated, there are  $e + 1$  rows with a 1 in the designated column and 0 in each of the other  $d$  columns. An  $d^0$ -disjunct matrix is said to be  $d$ -disjunct. D'yachkov et al. (2007) proposed the concept of fully  $d^e$ -disjunct matrices. An  $d^e$ -disjunct matrix is *fully  $d^e$ -disjunct* if it is not  $c^b$ -disjunct whenever  $c > d$  or  $b > e$ .

The constructions of  $d^e$ -disjunct matrices were given by many authors. (see D'yachkov et al. 2005, 2007; Huang and Weng 2004; Macula 1996, 1997; Ngo and Du 2002; Zhang et al. 2007, for examples). In this paper we construct a family of inclusion matrices associated with subspaces in the unitary space  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$ , exhibit its disjunct property, and compare the ratio efficiency of this construction with others, such as in D'yachkov et al. (2005) and Zhang et al. (2007). We find it smaller under some conditions.

## 2 The unitary space

In this section we will first introduce the concepts of unitary space, and then introduce some counting formulas in the unitary space.

Let  $\mathbb{F}_{q^2}$  be a finite field with  $q^2$  elements, where  $q$  is a power of a prime.  $\mathbb{F}_{q^2}$  has an involutive automorphism  $a \mapsto \bar{a} = a^q$ , whose fixed field is  $\mathbb{F}_q$ . Let  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$  ( $\delta = 0$  or 1) be the  $(2\nu + \delta)$ -dimensional row vector space over  $\mathbb{F}_{q^2}$  and let

$$H_\delta = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & I^{(\delta)} \end{pmatrix}.$$

The *unitary group* of degree  $2\nu + \delta$  over  $\mathbb{F}_{q^2}$ , denoted by  $U_{2\nu+\delta}(\mathbb{F}_{q^2})$ , consists of all  $(2\nu + \delta) \times (2\nu + \delta)$  nonsingular matrices  $T$  over  $\mathbb{F}_{q^2}$  satisfying  $TH_\delta\bar{T}^T = H_\delta$ , where  $\bar{T}$  denotes the matrix obtained from  $T$  by replacing each entry in  $T$  by its image under the involutive automorphism  $a \mapsto \bar{a}$ . There is an action of  $U_{2\nu+\delta}(\mathbb{F}_{q^2})$  on  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$  defined as follows:

$$\begin{aligned} \mathbb{F}_{q^2}^{(2\nu+\delta)} \times U_{2\nu+\delta}(\mathbb{F}_{q^2}) &\longrightarrow \mathbb{F}_{q^2}^{(2\nu+\delta)}, \\ ((x_1, x_2, \dots, x_{2\nu+\delta}), T) &\longmapsto (x_1, x_2, \dots, x_{2\nu+\delta})T. \end{aligned}$$

The vector space  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$  together with the above group action of the unitary group  $U_{2\nu+\delta}(\mathbb{F}_{q^2})$ , is called  $(2\nu + \delta)$ -dimensional *unitary space* over  $\mathbb{F}_{q^2}$ .

Let  $P$  be an  $m$ -dimensional subspace of  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$ , denote also by  $P$  an  $m \times (2\nu + \delta)$  matrix of rank  $m$  whose rows span the subspace  $P$ . An  $m$ -dimensional subspace  $P$  is said to be of *type*  $(m, r)$  if  $PH_\delta \overline{P}^T$  is of rank  $r$ . In particular, subspaces of type  $(m, 0)$  are called  $m$ -dimensional *totally isotropic subspaces*, and subspaces of type  $(m, m)$  are called  $m$ -dimensional *non-isotropic subspaces*. By Wan (2002, Theorem 5.7) we know that subspaces of type  $(m, r)$  exist if and only if  $2r \leq 2m \leq 2\nu + \delta + r$ . Denote by  $\mathcal{M}(m, r; 2\nu + \delta)$  the set of all subspaces of  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$  of type  $(m, r)$ . Then we have

**Proposition 2.1** (Wan 2002, Theorem 5.19) *Let  $2r \leq 2m \leq 2\nu + \delta + r$ . Then*

$$|\mathcal{M}(m, r; 2\nu + \delta)| = q^{r(2\nu+\delta+r-2m)} \frac{\prod_{l=2\nu+\delta+r-2m+1}^{2\nu+\delta} (q^l - (-1)^l)}{\prod_{l=1}^r (q^l - (-1)^l) \prod_{l=1}^{m-r} (q^{2l} - 1)}.$$

Denote by  $\mathcal{M}(m_1, r_1; m, r; 2\nu + \delta)$  the set of all subspaces of type  $(m_1, r_1)$  contained in a given subspace of type  $(m, r)$ , and denote by  $\mathcal{M}'(m_1, r_1; m, r; 2\nu + \delta)$  the set of all subspaces of type  $(m, r)$  containing a given subspace of type  $(m_1, r_1)$ . Then we have

**Proposition 2.2** (Wan 2002, Theorems 5.27 and 5.36)  $\mathcal{M}(m_1, r_1; m, r; 2\nu + \delta)$  (resp.  $\mathcal{M}'(m_1, r_1; m, r; 2\nu + \delta)$ ) is non-empty if and only if

$$2s \leq 2m \leq 2\nu + \delta + r \quad \text{and} \quad \max\{0, (2m_1 - r - r_1)/2\} \leq \min\{m - r, m_1 - r_1\}. \quad (1)$$

**Proposition 2.3** (Wan 2002, Theorem 5.28) *Suppose that (1) holds. Then*

$$\begin{aligned} & |\mathcal{M}(m_1, r_1; m, r; 2\nu + \delta)| \\ &= \sum_{k=\max\{0, \lfloor (2m_1-r-r_1+1)/2 \rfloor\}}^{\min\{m-r, m_1-r_1\}} q^{r_1(r+r_1-2m_1+2k)+2(m_1-k)(m-r-k)} \\ & \quad \times \frac{\prod_{l=r+r_1-2m_1+2k+1}^r (q^l - (-1)^l) \prod_{l=m-r-k+1}^{m-r} (q^{2l} - 1)}{\prod_{l=1}^{r_1} (q^l - (-1)^l) \prod_{l=1}^{m_1-r_1-k} (q^{2l} - 1) \prod_{l=1}^k (q^{2l} - 1)}. \end{aligned}$$

**Proposition 2.4** (Wan 2002, Theorem 3.37) *Suppose that (1) holds. Then*

$$\begin{aligned} & |\mathcal{M}'(m_1, r_1; m, r; 2\nu + \delta)| \\ &= \sum_{k=\max\{0, \lfloor (2m_1-r-r_1+1)/2 \rfloor\}}^{\min\{m-r, m_1-r_1\}} q^{(2\nu+\delta-2m+r)(r+r_1-2m_1+2k)+2(2\nu+\delta-m-k)(m_1-r_1-k)} \\ & \quad \times \frac{\prod_{l=r+r_1-2m_1+2k+1}^{2\nu+\delta+r_1-2m_1} (q^l - (-1)^l) \prod_{l=m_1-r_1-k+1}^{m_1-r_1} (q^{2l} - 1)}{\prod_{l=1}^{2\nu+\delta+r-2m} (q^l - (-1)^l) \prod_{l=1}^{m-r-k} (q^{2l} - 1) \prod_{l=1}^k (q^{2l} - 1)}. \end{aligned}$$

### 3 The construction

In this section, we construct a family of inclusion matrices associated with subspaces of  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$ , and exhibit its disjoint property.

Given integers  $1 \leq r \leq \lfloor m/2 \rfloor$  and  $m < 2\nu + \delta$ ,  $\mathcal{M}(r, 0; m, m; 2\nu + \delta)$  is non-empty by Proposition 2.2, so we give the following definition:

**Definition 3.1** Given integers  $1 \leq r \leq \lfloor m/2 \rfloor$  and  $m < 2\nu + \delta$ . Let  $M(r, m; 2\nu + \delta)$  be the binary matrix whose rows (resp. columns) are indexed by  $\mathcal{M}(r, 0; 2\nu + \delta)$  (resp.  $\mathcal{M}(m, m; 2\nu + \delta)$ ). We also order elements of these sets lexicographically.  $M(r, m; 2\nu + \delta)$  has a 1 in row  $i$  and column  $j$  if and only if the  $i$ -th subspace of  $\mathcal{M}(r, 0; 2\nu + \delta)$  is a subspace of the  $j$ -th subspace of  $\mathcal{M}(m, m; 2\nu + \delta)$ .

By Propositions 2.1, 2.3 and 2.4,  $M(r, m; 2\nu + \delta)$  is a  $|\mathcal{M}(r, 0; 2\nu + \delta)| \times |\mathcal{M}(m, m; 2\nu + \delta)|$  matrix, whose constant row (resp. column) weight is  $|\mathcal{M}'(r, 0; m, m; 2\nu + \delta)|$  (resp.  $|\mathcal{M}(r, 0; m, m; 2\nu + \delta)|$ ).

**Theorem 3.2** Let  $1 \leq r \leq \lfloor m/2 \rfloor$  and  $m < 2\nu + \delta$ , and let  $\beta = |\mathcal{M}(r, 0; m, m; 2\nu + \delta)|$ ,  $\gamma = |\mathcal{M}(r, 0; m - 1, m - 1; 2\nu + \delta)|$ ,  $\xi = |\mathcal{M}(r, 0; m - 1, m - 2; 2\nu + \delta)|$ ,  $\eta = |\mathcal{M}(r, 0; m - 2, m - 2; 2\nu + \delta)|$ ,  $\zeta = |\mathcal{M}(r, 0; m - 2, m - 3; 2\nu + \delta)|$ ,  $\rho = |\mathcal{M}(r, 0; m - 2, m - 4; 2\nu + \delta)|$  and  $\alpha = \max\{\gamma - \eta, \gamma - \zeta, \gamma - \rho, \xi - \eta, \xi - \zeta, \xi - \rho\}$ . Then the following (i)–(iii) hold:

- (i) For  $m \geq 4$ . If  $1 \leq d \leq \lfloor \frac{\beta - \max\{\gamma, \xi\} - 1}{\alpha} \rfloor + 1$ , then  $M(r, m; 2\nu + \delta)$  is  $d^e$ -disjunct, where  $e = \beta - \max\{\gamma, \xi\} - (d - 1)\alpha - 1$ . In particular, if  $1 \leq d \leq \min\{\lfloor \frac{\beta - \max\{\gamma, \xi\} - 1}{\alpha} \rfloor + 1, q^2 + 1\}$ , then exist  $d + 1$  distinct columns of  $M(r, m; 2\nu + \delta)$ , i.e.,  $d + 1$  distinct  $m$ -dimensional non-isotropic subspaces of  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$ , such that the  $d + 1$  subspaces contain same  $(m - 2)$ -dimensional subspace  $P$  and the number of  $r$ -dimensional totally isotropic subspaces contained in  $P$  is equal to  $\min\{\eta, \zeta, \rho\}$ .
- (ii) For  $m = 3$ . If  $1 \leq d \leq q^2 - q$ , then  $M(r, m; 2\nu + \delta)$  is fully  $d^e$ -disjunct, where  $e = q^3 - d(q + 1)$ .
- (iii) For  $m = 2$ . If  $1 \leq d \leq q$ , then  $M(1, 2; 2\nu + \delta)$  is fully  $d^e$ -disjunct, where  $e = q - d$ .

*Proof* (i) Let  $C, C_1, C_2, \dots, C_d$  be  $d + 1$  distinct columns of  $M(r, m; 2\nu + \delta)$ . To obtain the maximum numbers of subspaces of  $\mathcal{M}(r, 0; 2\nu)$  in

$$C \cap \bigcup_{i=1}^d C_i = \bigcup_{i=1}^d (C \cap C_i),$$

we may assume that  $\dim(C \cap C_i) = m - 1$  and  $\dim(C \cap C_i \cap C_j) = m - 2$  for any two distinct  $i$  and  $j$ , where  $1 \leq i, j \leq d$ . So we have that  $C \cap C_i$  is an  $(m - 1)$ -dimensional subspace and contained in  $C$ , and  $C \cap C_i \cap C_j$  is an  $(m - 2)$ -dimensional subspace and contained in  $C \cap C_i$ . Suppose that  $C \cap C_i$  is a subspace of type  $(m - 1, r')$  of  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$ .

Since  $C$  is an  $m$ -dimensional non-isotropic subspace of  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$  and  $C \cap C_i \subseteq C$ ,  $\mathcal{M}(m-1, r'; m, m; 2\nu+\delta) \neq \emptyset$ . By Proposition 2.2,  $r' = m-2$  or  $m-1$ . It follows that  $C \cap C_i$  is a subspace of type  $(m-1, m-1)$  or type  $(m-1, m-2)$  of  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$ . Since  $C \cap C_i \cap C_j \subseteq C \cap C_i$ , there are two cases to be considered.

*Case 1:*  $C \cap C_i$  is of type  $(m-1, m-1)$ . Suppose that  $C \cap C_i \cap C_j$  is of type  $(m-2, r_1)$  of  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$ . Then  $\mathcal{M}(m-2, r_1; m-1, m-1; 2\nu+\delta) \neq \emptyset$ . By Proposition 2.2,  $r_1 = m-3$  or  $m-2$ . It follows that  $C \cap C_i \cap C_j$  is of type  $(m-2, m-3)$  or type  $(m-2, m-2)$  of  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$ .

*Case 2:*  $C \cap C_i$  is of type  $(m-1, m-2)$ . Suppose that  $C \cap C_i \cap C_j$  is of type  $(m-2, r_2)$  of  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$ . Then  $\mathcal{M}(m-2, r_2; m-1, m-2; 2\nu+\delta) \neq \emptyset$ . By Proposition 2.2,  $r_2 = m-4, m-3$  or  $m-2$ . It follows that  $C \cap C_i \cap C_j$  is of type  $(m-2, m-4), (m-2, m-3)$  or type  $(m-2, m-2)$  of  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$ .

In both cases,  $C \cap C_i \cap C_j$  is of type  $(m-2, m-4)$ , type  $(m-2, m-3)$  or type  $(m-2, m-2)$  of  $\mathbb{F}_{q^2}^{(2\nu+\delta)}$ . Note that  $\beta = |\mathcal{M}(r, 0; m, m; 2\nu+\delta)|$ ,  $\gamma = |\mathcal{M}(r, 0; m-1, m-1; 2\nu+\delta)|$ ,  $\xi = |\mathcal{M}(r, 0; m-1, m-2; 2\nu+\delta)|$ ,  $\eta = |\mathcal{M}(r, 0; m-2, m-2; 2\nu+\delta)|$ ,  $\zeta = |\mathcal{M}(r, 0; m-2, m-3; 2\nu+\delta)|$ ,  $\rho = |\mathcal{M}(r, 0; m-2, m-4; 2\nu+\delta)|$  and  $\alpha = \max\{\gamma - \eta, \gamma - \zeta, \gamma - \rho, \xi - \eta, \xi - \zeta, \xi - \rho\}$ . Therefore the subspaces of  $C$  not covered by  $C_1, C_2, \dots, C_d$  is at least

$$\begin{aligned} & \beta - d \times \max\{\gamma, \xi\} + (d-1) \times \min\{\eta, \zeta, \rho\} \\ & = \beta - \max\{\gamma, \xi\} - (d-1)\alpha. \end{aligned}$$

Hence  $e = \beta - \max\{\gamma, \xi\} - (d-1)\alpha - 1$ . Since  $e \geq 0$ , we obtain

$$d \leq \left\lfloor \frac{\beta - \max\{\gamma, \xi\} - 1}{\alpha} \right\rfloor + 1.$$

For  $C \cap C_1$ , by Proposition 2.3 and  $m \geq 4$ ,

$$|\mathcal{M}(m-2, m-2; m-1, m-1; 2\nu+\delta)| = \frac{q^{m-2}(q^{m-1} - (-1)^{m-1})}{q+1} \geq 1,$$

$$\begin{aligned} |\mathcal{M}(m-2, m-3; m-1, m-1; 2\nu+\delta)| &= \frac{(q^{m-2} - (-1)^{m-2})(q^{m-1} - (-1)^{m-1})}{q^2-1} \\ &\geq 1, \end{aligned}$$

$$|\mathcal{M}(m-2, m-2; m-1, m-2; 2\nu+\delta)| = q^{2(m-2)} \geq 1,$$

$$|\mathcal{M}(m-2, m-3; m-1, m-2; 2\nu+\delta)| = \frac{q^{m-3}(q^{m-2} - (-1)^{m-2})}{q+1} \geq 1,$$

$$\begin{aligned} |\mathcal{M}(m-2, m-4; m-1, m-2; 2\nu+\delta)| &= \frac{(q^{m-2} - (-1)^{m-2})(q^{m-3} - (-1)^{m-3})}{q^2-1} \\ &\geq 1. \end{aligned}$$

Hence exists an  $(m - 2)$ -dimensional subspace contained in  $C \cap C_1$ , denoted by  $P$ , such that the number of  $r$ -dimensional totally isotropic subspaces contained in  $P$  is equal to  $\min\{\eta, \zeta, \rho\}$ . By Wan (2002, Corollary 1.9), the number of  $(m - 1)$ -dimensional subspaces containing  $P$  and contained in  $C$  is equal to  $q^2 + 1$ , and each of these subspaces is of type  $(m - 1, m - 1)$  or type  $(m - 1, m - 2)$ . For  $1 \leq d \leq \min\{\lfloor \frac{\beta - \max\{\gamma, \xi\} - 1}{\alpha} \rfloor + 1, q^2 + 1\}$ , we choose  $d$  distinct  $(m - 1)$ -dimensional subspaces between  $\bar{P}$  and  $C$ , say  $P_i$  ( $1 \leq i \leq d$ ). By Proposition 2.4 and  $m < 2\nu + \delta$ ,

$$|\mathcal{M}'(m - 1, m - 1; m, m; 2\nu + \delta)| = \frac{q^{2\nu + \delta - m}(q^{2\nu + \delta - m + 1} - (-1)^{2\nu + \delta - m + 1})}{q + 1} \geq 2,$$

$$|\mathcal{M}'(m - 1, m - 2; m, m; 2\nu + \delta)| = q^{2(2\nu + \delta - m)} \geq 2.$$

So for each  $P_i$ , we can choose an  $m$ -dimensional non-isotropic subspace  $C_i$  such that  $C \cap C_i = P_i$ . Hence  $P \subseteq C \cap C_1 \cap \dots \cap C_d$ .

(ii) If  $m = 3$ , then  $r = 1$ . By Proposition 2.2,  $\eta = \rho = 0$  and  $\zeta = 1$ . It follows that  $\alpha = \max\{\gamma, \xi\}$ . By Proposition 2.3,  $\beta = |\mathcal{M}(1, 0; 3, 3; 2\nu + \delta)| = q^3 + 1$ ,  $\gamma = |\mathcal{M}(1, 0; 2, 2; 2\nu + \delta)| = q + 1$ ,  $\xi = |\mathcal{M}(1, 0; 2, 1; 2\nu + \delta)| = 1$ . So  $\alpha = q + 1$ ,  $\lfloor (\beta - 1)/(\max\{\gamma, \xi\}) \rfloor = \lfloor q^3/(q + 1) \rfloor = q^2 - q$  and  $e = \beta - d \times \max\{\gamma, \xi\} - 1 = q^3 - d(q + 1)$ .

Let  $C, C_1, C_2, \dots, C_d$  be  $d + 1$  distinct columns of  $M(1, 2; 2\nu + \delta)$ . Then we may assume that  $\dim(C \cap C_i) = 2$  and  $\dim(C \cap C_i \cap C_j) = 1$  for any two distinct  $i$  and  $j$ , where  $1 \leq i, j \leq d$ . Now we show that the maximum of  $C \cap \bigcup_{i=1}^d C_i$  can be obtained. For 2-dimensional subspace  $C \cap C_1$ , exists a 1-dimensional non-isotropic subspace, denoted by  $P$ , contained in  $C \cap C_1$ . Indeed, the proof is similar to that of (i), and will be omitted. By Proposition 2.4,  $|\mathcal{M}'(1, 1; 2, 2; 3)| = q^2 - q$ ,  $|\mathcal{M}'(1, 1; 2, 1; 3)| = q + 1$ . So there are  $q^2 - q$  (resp.  $q + 1$ ) subspaces of type  $(2, 2)$  (resp. type  $(2, 1)$ ) containing  $P$  and contained in  $C$ . We distinguish the following two cases:

Case 1:  $q \neq 2$ . Then  $q^2 - q > q + 1$ . For  $1 \leq d \leq q^2 - q$ . Since  $\gamma = q + 1 > 1 = \xi$ , we choose  $d$  distinct subspaces of type  $(2, 2)$  between  $P$  and  $C$ , say  $P_i$  ( $1 \leq i \leq d$ ). By Proposition 2.4 and  $3 < 2\nu + \delta$ ,

$$|\mathcal{M}'(2, 2; 3, 3; 2\nu + \delta)| = \frac{q^{2\nu + \delta - 3}(q^{2\nu + \delta - 2} - (-1)^{2\nu + \delta - 2})}{q + 1} \geq 2.$$

So for each  $P_i$ , we can choose a 3-dimensional non-isotropic subspace  $C_i$  such that  $C \cap C_i = P_i$ . Hence each pair of  $C_i$  and  $C_j$  overlaps at the same subspace  $P$ .

Case 2:  $q = 2$ . Then  $2 = q^2 - q < q + 1 = 3$ . For  $1 \leq d \leq 2$ . The proof is similar to that of Case 2, and will be omitted.

(iii) If  $m = 2$ , then  $r = 1$ . The proof is similar to that of (ii), and will be omitted.  $\square$

In order to explain the pooling design plainly, we give an example.

*Example 3.3* Choose  $q = 2, m = 4, r = 2$  and  $\nu = 4$ . If  $1 \leq d \leq 8$ , then  $M(2, 4; 9)$  is a  $d^e$ -disjunct, where  $e = 26 - 3d$ . In particular,  $M(2, 4; 9)$  is a  $8^2$ -disjunct.

The following theorem tells us how to choose  $m$  so that the test to item is minimized.

**Theorem 3.4** For  $1 \leq m < 2\nu + \delta$ , the sequence  $|\mathcal{M}(m, m; 2\nu + \delta)|$  is unimodal and gets its peak at  $m = \nu$  or  $\nu + \delta$ .

*Proof* By Proposition 2.1 we have

$$\frac{|\mathcal{M}(m_2, m_2; 2\nu + \delta)|}{|\mathcal{M}(m_1, m_1; 2\nu + \delta)|} = q^{(m_2 - m_1)(2\nu + \delta - m_2 - m_1)} \frac{\prod_{i=2\nu + \delta - m_2 + 1}^{2\nu + \delta - m_1} (q^i - (-1)^i)}{\prod_{i=m_1 + 1}^{m_2} (q^i - (-1)^i)}.$$

If  $\nu + \delta \leq m_1 < m_2 < 2\nu + \delta$ , then  $m_1 + m_2 - (2\nu + \delta) > 0$ . So

$$\frac{q^{2\nu + \delta - m_2 + j} - (-1)^{2\nu + \delta - m_2 + j}}{q^{m_1 + j} - (-1)^{m_1 + j}} < 1,$$

where  $1 \leq j \leq m_2 - m_1$ . It follows that  $|\mathcal{M}(m_2, m_2; 2\nu + \delta)| < |\mathcal{M}(m_1, m_1; 2\nu + \delta)|$ . If  $1 \leq m_1 < m_2 \leq \nu$ , then  $m_1 + m_2 - (2\nu + \delta) < 0$ . So

$$\frac{q^{2\nu + \delta - m_2 + j} - (-1)^{2\nu + \delta - m_2 + j}}{q^{m_1 + j} - (-1)^{m_1 + j}} > 1,$$

where  $1 \leq j \leq m_2 - m_1$ . It follows that  $|\mathcal{M}(m_2, m_2; 2\nu + \delta)| > |\mathcal{M}(m_1, m_1; 2\nu + \delta)|$ . □

### 4 Comparison of test efficiency

Erdős et al. (1985) give a formula that  $t(d, n) > d(1 + o(1)) \ln n$ , where  $t(d, n)$  denotes the minimum number of rows for a  $d$ -disjunct matrix with  $n$  columns. To take  $t / \ln n$  as a measure of the construction of  $d^e$ -disjunct matrix is meaningful.

From  $t(d, n) > d(1 + o(1)) \ln n$ , we know that the smaller the value of  $t / \ln n$  is, the better the pooling design is. Since  $t / \ln n$  can be converted to  $t / n$  under some conditions, we can take  $t / n$  as a measure of the construction is, where  $t$  denotes the number of tests, i.e., the number of rows of inclusion matrix,  $n$  denotes the number of detected items, i.e., the number of columns of inclusion matrix.

**Definition 4.1** We call the ratio  $t / n$  test efficiency, where  $t$  denotes the number of tests,  $n$  denotes the number of detected items.

Now we give the comparison of test efficiency.

In this paper, assume that the test efficiency is  $t / n$ , then

$$\frac{t}{n} = \frac{|\mathcal{M}(r, 0; 2\nu + \delta)|}{|\mathcal{M}(m, m; 2\nu + \delta)|} = \frac{\prod_{i=1}^m (q^i - (-1)^i)}{q^{m(2\nu + \delta - m)} \prod_{i=2\nu + \delta - m + 1}^{2\nu + \delta - 2r} (q^i - (-1)^i) \prod_{i=1}^r (q^{2i} - 1)}.$$

D'yachkov et al. (2005) constructed with subspaces of  $\mathbb{F}_{q^2}$ , where  $q$  is a prime power. Each of the columns (resp. rows) is labeled by an  $k$  (resp.  $d$ )-dimensional

subspace of  $\mathbb{F}_{q^2}^{(s)}$ , where  $d < k < s$ ,  $m_{ij} = 1$  if and only if the label of row  $i$  is contained in the label of column  $j$ . In order to compare with  $t/n$ , let  $s = 2v + \delta$ ,  $d = r$  and  $k = m$ . Assume that the test efficiency is  $t_1/n_1$ , then

$$\frac{t_1}{n_1} = \frac{\prod_{i=r+1}^m (q^{2i} - 1)}{\prod_{i=2v+\delta-m+1}^{2v+\delta-r} (q^{2i} - 1)}.$$

Zhang et al. (2007) construct a  $d^z$ -disjunct matrix with subspaces in a dual space of the unitary space  $\mathbb{F}_{q^2}^{(2s+\delta)}$ , where  $q$  is a prime power. Each of the columns (resp. rows) is labeled by subspaces of type  $(k, 0)$  (resp. subspaces of type  $(d, 0)$ ) which are contained in  $P_0^\perp$  and containing  $P_0$ , where  $m_0 < d < k < s$  and  $P_0$  is a given subspace of type  $(m_0, 0)$ ,  $m_{ij} = 1$  if and only if  $i$  is contained in  $j$ . In the same way, let  $s - m_0 = v$ ,  $d - m_0 = r$  and  $k - m_0 = m$ . Assume that the test efficiency is  $t_2/n_2$ , then

$$\frac{t_2}{n_2} = \frac{\prod_{i=r+1}^m (q^{2i} - 1)}{\prod_{i=2v+\delta-2m+1}^{2v+\delta-2r} (q^i - (-1)^i)}.$$

**Theorem 4.2** *If  $r^2 + 2r - 2m \geq 0$ , then  $t/n < t_1/n_1$ .*

*Proof* If  $r^2 + 2r - 2m \geq 0$ , then we have

$$\begin{aligned} \frac{t/n}{t_1/n_1} &= \frac{\prod_{i=1}^m (q^i - (-1)^i) \prod_{i=2v+\delta-m+1}^{2v+\delta-r} (q^{2i} - 1)}{q^{m(2v+\delta-m)} \prod_{i=2v+\delta-m+1}^{2v+\delta-2r} (q^i - (-1)^i) \prod_{i=1}^m (q^{2i} - 1)} \\ &= \frac{\prod_{i=2v+\delta-2r+1}^{2v+\delta-r} (q^{2i} - 1) \prod_{i=2v+\delta-m+1}^{2v+\delta-2r} (q^i + (-1)^i)}{q^{m(2v+\delta-m)} \prod_{i=1}^m (q^i + (-1)^i)} \\ &< \frac{\prod_{i=2v+\delta-2r+1}^{2v+\delta-r} q^{2i} \prod_{i=2v+\delta-m+1}^{2v+\delta-2r} q^{i+1}}{q^{m(2v+\delta-m)} \prod_{i=1}^m q^{i-1}} \\ &= \frac{1}{q^{r^2+r-2m}} \\ &\leq 1. \end{aligned}$$

□

**Theorem 4.3**  *$t/n < t_2/n_2$ .*

*Proof*

$$\begin{aligned} \frac{t/n}{t_2/n_2} &= \frac{|\mathcal{M}(r, 0; 2v + \delta)| |\mathcal{M}(m, 0; 2v + \delta)|}{|\mathcal{M}(m, m; 2v + \delta)| |\mathcal{M}(r, 0; 2v + \delta)|} \\ &= \frac{\prod_{i=2v+\delta-2m+1}^{2v+\delta-m} (q^i - (-1)^i)}{q^{m(2v+\delta-m)} \prod_{i=1}^m (q^i + (-1)^i)} \\ &< \frac{\prod_{i=2v+\delta-2m+1}^{2v+\delta-m} q^{i+1}}{q^{m(2v+\delta-m)} \prod_{i=1}^m q^{i-1}} \end{aligned}$$



$$\begin{aligned}
 &= \frac{1}{q^{m(2v+\delta+m-2)/2}} \\
 &\leq 1. \quad \square
 \end{aligned}$$

*Example 4.4* Choose  $q = 2, m = 4, r = 2, v = 4$  and  $\delta = 1$ . Then

$$\frac{t}{n} = \frac{27}{2^{20}}, \quad \frac{t_1}{n_1} = \frac{51}{13(2^{14} - 1)}, \quad \frac{t_2}{n_2} = \frac{119}{99}.$$

Clearly,  $\frac{t}{n} < \frac{t_1}{n_1} < \frac{t_2}{n_2}$ .

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