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The arrangement of subspaces in the orthogonal spaces and tighter analysis of an error-tolerant pooling design

Geng-Sheng Zhang · Yu-Qin Yang

Abstract In this paper, we construct a d^z-disjunct matrix with the orthogonal spaces over finite fields of odd characteristic. We consider the arrangement problem of d $(m-1, 2(s-1), s-1)$-subspaces and the tighter bounds for an error-tolerant pooling design. Moreover, we give the tighter analysis of our construction by the results of the arrangement problem. Additionally, by comparing our construction with the previous construction out of vector spaces, we find that our construction is better under some conditions.

Keywords Orthogonal space · d^z-disjunct · Arrangement problem · Tighter analysis · Test efficiency

1 Introduction

The basic problem of group testing is to identify the set of positive (defective) objects in a large population of items. There are two kinds of group testing algorithms, adaptive algorithm and non-adaptive algorithm. A group testing algorithm is non-adaptive (NGT) if all tests must be specified without knowing the outcomes of other tests.

Designing a good error-tolerant pooling design is a central problem in the area of non-adaptive group testing. It is known that a mathematical model of this problem is a d^z-disjunct matrix, and that its capability of error-tolerant and error-correction is determined by z. So it is significant to address the value of z, i.e., the problem of

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G.-S. Zhang (✉) · Y.-Q. Yang
Department of Mathematics, Hebei Normal University, Shijiazhuang, 050016, People's Republic of China
e-mail: gshzhang@heinfo.net

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designing and analyzing the error-tolerant capability of a pooling design (see Du and Hwang 2006).

To describe this problem, we need a few definitions. A (0, 1)-matrix is d-disjunct if no union of any d columns covers another column. A d-disjunct matrix is d^z-disjunct if a column has at least $z+1$ 1-entries not covered by the union of any other d columns. A d-disjunct matrix is d^0-disjunct. A d^z-disjunct can detect $z-1$ errors and correct $\lfloor \frac{z-1}{2} \rfloor$ errors (see D'yachkov et al. 2005; Huang and Weng 2004). The error-correcting capability is doubled by the addition of at most d confirmatory and guaranteed tests as compared to the number of tests required by, and error correcting capability of, the purely nonadaptive case in D'yachkov et al. (2007).

To date, some constructions of d^z-disjunct matrix have been given. For example, Ngo and Du further extended the construction to some geometric structures like simplicity complexes and some graph properties like matchings in Ngo and Du (2002). Macula et al. (2004) constructed a α-almost d-disjunct matrix on the group testing for complexes problem. Fu and Hwang (2006) constructed with t-packings. A.G. D'yachkov et al. constructed with subspaces of $GF(q)$ in D'yachkov et al. (2005), and so on. Ngo and Du (2002) introduced a non-adaptive pooling design based on finite vector spaces, which was d-disjunct. D'yachkov et al. (2005) found that this construction in Ngo and Du (2002) can tolerate a lot of errors and the bound was tight for $d \leq q+1$. Ngo (2008) gave the tighter analysis for $d > q+1$ by addressing the hyperplane arrangement problem.

If we consider the type of the subspaces rather than only the dimension of them as we do before, we think that we can get some better conclusions about the pooling design constructed by the orthogonal spaces. It is known that, when the number of the detective items is fixed, the small the number of the tests is, the better a pooling design is. In order to chose the number of the tests effectively, we address the peak of the number of the tests, i.e., the number of the rows in the d^z-disjunct matrix. In addition, We call the ratio between the number of the rows and the number of columns in the d^z-disjunct matrix by test efficiency. We denote it by R. According to the definition of R, we know that the less R is, the better the construction is. So we give some parameters which have effect on R. we compare the test efficiency of our design with the other designs such as in D'yachkov et al. (2005), and find our design is more effective under certain conditions.

We know that the dimension of the intersection of two m-dimensional subspaces is at most $m-1$. In order to discuss the tighter analysis of an error-tolerance pooling design, in general vector spaces, we only need to discuss the $(m-1)$-dimensional subspace of a m-dimensional subspace, i.e., the hyperplane arrangement problem in Ngo (2008). However, in the orthogonal space, let P be a $(m, 2s, s)$-subspace. Then, there are two kinds of $(m-1)$-dimensional subspace of P which are $(m-1, 2s, s)$-subspace and $(m-1, 2(s-1), s-1)$-subspace. Hence, for an arrangement problem, we need to consider not only the dimension of a subspace but also the type of the subspace. Therefore it is more complex in the orthogonal space.

At the beginning, we plan to consider two subclasses of each type, one is $(m, 2s, s)$, the other is $(r, 2(s-i), s-i)$, $i = 0, 1, 2, \ldots, s$. When $i = 0$, the two subspaces have the same type, it has been studied in Ngo and Du (2002) in fact. Note that when $i = 2, 3, \ldots, s$, the intersection of the two subspaces has many cases, which it makes difficult to discuss the containing relation about the two subspaces, so we only discuss the case when $i = 1$ in the rest of this paper.

In this paper, we construct a d^z-disjunct matrix based on the orthogonal space over finite field of odd characteristic, and consider the arrangement problem of d $(m-1, 2(s-1), s-1)$-subspaces motivated by the need to give tighter bounds for an error-tolerant pooling design in Sects. 3 and 4. Moreover, we give the tighter analysis of our construction by the results of the arrangement problem in Sect. 5. Additionally, in Sect. 6, we give the comparison of test efficiency in our construction and in D'yachkov et al. (2005).

2 Preliminary

In order to construct a d^z-disjunct matrix with the orthogonal space over finite fields of odd characteristic, we introduce several basic conceptions and notes of orthogonal geometry. For more results of the orthogonal geometry, the reader can refer to the 6-th chapter of Wan (2002).

$\mathbb{F}_q$ is a finite field of odd characteristic with q elements, where q is a prime or a prime power, w is a fixed non-square element of $\mathbb{F}_q^*$. Assume

$$S_{2\nu} = \begin{pmatrix} 0 & I^{(\nu)} \\ I^{(\nu)} & 0 \end{pmatrix}, \qquad S_{2\nu+1,1} = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & 1 \end{pmatrix},$$

$$S_{2\nu+1,z} = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & w \end{pmatrix}, \qquad S_{2\nu+2} = \begin{pmatrix} 0 & I^{(\nu)} & & \\ I^{(\nu)} & 0 & & \\ & & 1 & \\ & & & -w \end{pmatrix}.$$

We introduce the notation $S_{2\nu+\delta,\Delta}$, where ν is its index and Δ denotes its definite part, i.e.,

$$\Delta = \begin{cases} \emptyset, & \text{if } \delta = 0, \\ (1) \text{ or } (w), & \text{if } \delta = 1, \\ \begin{pmatrix} 1 & \\ & -w \end{pmatrix}, & \text{if } \delta = 2. \end{cases}$$

The orthogonal group of degree $2\nu + \delta$ with respect to $S_{2\nu+\delta,\Delta}$ over $\mathbb{F}_q$ will be denoted by $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$, which consists of all $(2\nu + \delta) \times (2\nu + \delta)$ matrices T over $\mathbb{F}_q$ satisfying $T S_{2\nu+\delta,\Delta} T^T = S_{2\nu+\delta,\Delta}$. The vector space $\mathbb{F}_q^{(2\nu+\delta)}$ together with the right multiplication action of $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$ is called the $(2\nu + \delta)$-dimensional orthogonal space over $\mathbb{F}_q$ with respect to $S_{2\nu+\delta,\Delta}$. Clearly, since $O_{2\nu+1,1}(\mathbb{F}_q)$ and $O_{2\nu+1,z}(\mathbb{F}_q)$ are isomorphic, actually only three types of orthogonal groups, $O_{2\nu}(\mathbb{F}_q)$, $O_{2\nu+1,1}(\mathbb{F}_q)$, $O_{2\nu+2,1}(\mathbb{F}_q)$, need to be considered.

Let A and B be two $n \times n$ matrices over $\mathbb{F}_q$. If there is an $n \times n$ nonsingular matrix Q over $\mathbb{F}_q$ such that $QAQ^T = B$, we say that A is cogredient to B. Let $M(m, 2s, s)$ represent following form,

$$M(m, 2s, s) = \begin{pmatrix} 0 & I^{(s)} & \\ I^{(s)} & 0 & \\ & & 0^{(m-2s)} \end{pmatrix}.$$

An m-dimensional vector subspace P of $\mathbb{F}_q^{(2\nu+\delta)}$ is said to be of type $(m, 2s, s)$, if $PS_{2\nu+\delta,\Delta}P^T$ is cogredient to $M(m, 2s, s)$, denoted by $(m, 2s, s)$-subspace. We denote by $\mathcal{M}(m, 2s, s; 2\nu + \delta, \Delta)$ the set of subspaces of type $(m, 2s, s)$ in $\mathbb{F}_q^{(2\nu+\delta)}$. Let $N(m, 2s, s; 2\nu + \delta, \Delta) = |\mathcal{M}(m, 2s, s; 2\nu + \delta, \Delta)|$.

Let P be a fixed subspace of type $(m, 2s, s)$ and Q be a fixed subspace of type$(m_2, 2s_2, s_2)$ in P. Denote by $\mathcal{M}(m_1, 2s_1, s_1; m, 2s, s; 2\nu + \delta, \Delta)$ the set of subspaces of type $(m_1, 2s_1, s_1)$ contained in P. Let $N(m_1, 2s_1, s_1; m, 2s, s; 2\nu+\delta, \Delta) = |\mathcal{M}(m_1, 2s_1, s_1; m, 2s, s; 2\nu+\delta, \Delta)|$. Denote by $\mathcal{M}(m_1, 2s_1, s_1; m_2, 2s_2, s_2; m, 2s, s; 2\nu+\delta, \Delta)$ the set of subspaces of type $(m_1, 2s_1, s_1)$ contained in Q. Let $N(m_1, 2s_1, s_1; m_2, 2s_2, s_2; m, 2s, s; 2\nu + \delta, \Delta) = |\mathcal{M}(m_1, 2s_1, s_1; m_2, 2s_2, s_2; m, 2s, s; 2\nu + \delta, \Delta)|$.

Let $\begin{bmatrix} n \\ m \end{bmatrix}_q$ be the Gaussian coefficient, i.e., $\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{\prod_{i=n-m+1}^{n}(q^i-1)}{\prod_{i=1}^{m}(q^i-1)}$.

Additionally, two Anzahl theorems which are used in this paper are as follows. According to Theorems 6.33 and 6.35 in Wan (2002) respectively, we can obtain them directly.

Lemma 2.1 *Let k be an integer with* $\max\{0, -s - s_1 + m_1\} \leq k \leq \min\{m - 2s, m_1 - 2s_1\}$. *Then*

$$\begin{aligned} &N(m_1, 2s_1, s_1; m, 2s, s; 2\nu + \delta, \Delta) \\ &\quad = \sum_k q^{2s_1(s+s_1-m_1+k)+(m_1-k)(m-2s-k)} \\ &\qquad \times \frac{\prod_{i=s+s_1-m_1+k+1}^{s}(q^i - 1)(q^{i-1} + 1) \prod_{i=m-2s-k+1}^{m-2s}(q^i - 1)}{\prod_{i=1}^{s_1}(q^i - 1) \prod_{i=0}^{s_1-1}(q^i + 1) \prod_{i=1}^{m_1-2s_1-k}(q^i - 1) \prod_{i=1}^{k}(q^i - 1)}. \end{aligned}$$

Lemma 2.2 *If $\mathcal{M}(m_1, 2s_1, s_1, k_1; m, 2s, s, k; 2\nu + \delta + l, \Delta)$ is non-empty, then*

$$\begin{aligned} &N(m_1, 2s_1, s_1, k_1; m, 2s, s, k; 2\nu + \delta + l, \Delta) \\ &\quad = N(m_1 - k_1, 2s_1, s_1; 2s + (m - k - 2s), 0) \times N(k_1, k)q^{(m_1-k_1)(k-k_1)}. \end{aligned}$$

$N(m_1 - k_1, 2s_1, s_1; 2s + (m - k - 2s), 0)$ and $N(k_1, k)$ are given in Corollary 6.31 *and Theorem* 1.7 *in* Wan (2002).

3 The construction and motivation

In this section, we will construct a (0, 1)-matrix with the orthogonal space over $\mathbb{F}_q$ of odd characteristic, and give the reason why we consider the arrangement problem of d $(m-1, 2(s-1), s-1)$-subspaces of a $(m, 2s, s)$-subspace in the orthogonal space.

Definition 3.1 Let $A(m, 2s, s; r, 2(s-1), s-1; 2\nu+\delta)$ be the (0, 1)-matrix whose rows are indexed by $(r, 2(s-1), s-1)$-subspaces and columns are indexed by $(m, 2s, s)$-subspaces in the orthogonal space over $\mathbb{F}_q$. $A(m, 2s, s; r, 2(s-1), s-1; 2\nu+\delta)$ has a 1 in row R and column C if and only if R is a subspace of C.

For any $(m, 2s, s)$-subspace P in the orthogonal space over $\mathbb{F}_q$, let $\overline{P}$ denote the set of all $(r, 2(s-1), s-1)$-subspaces of P. We assert that

$$\overline{P} \cap \overline{Q} = \overline{P \cap Q}.$$

This is because a $(r, 2(s-1), s-1)$-subspace contained in P and Q must be contained in $P \cap Q$, and that a $(r, 2(s-1), s-1)$-subspace contained in $P \cap Q$ must be contained in P and Q.

In order to prove $A(m, 2s, s; r, 2(s-1), s-1; 2\nu+\delta)$ is d^z-disjunct, we only need to find z, $z = \min\{|\overline{C_0} \setminus \bigcup_{i=1}^{d} \overline{C_i}|\}$, for any $d+1$ different $(m, 2s, s)$-subspaces $C_0, C_1, \ldots, C_d$ in the orthogonal space over $\mathbb{F}_q$.

For any $i \in \{1, 2, \ldots, d\}$, let $H_i = C_i \cap C_0$. Then

$$\left|\overline{C_0} \setminus \bigcup_{i=1}^{d} \overline{C_i}\right| = \left|\overline{C_0} \setminus \bigcup_{i=1}^{d} (\overline{C_i} \cap \overline{C_0})\right| = \left|\overline{C_0} \setminus \bigcup_{i=1}^{d} \overline{H_i}\right|.$$

To minimize $|\overline{C_0} \setminus \bigcup_{i=1}^{d} \overline{C_i}|$, we should assume all H_i are $(m-1, 2(s-1), s-1)$-subspaces of C_0. However considering that there are two kinds of $(m-1)$-dimensional subspaces of C_0, which are $(m-1, 2s, s)$-subspaces and $(m-1, 2(s-1), s-1)$-subspaces, we give the lemmas as follows to explain why we chose the $(m-1, 2(s-1), s-1)$-subspaces rather than the $(m-1, 2s, s)$-subspaces.

By Lemma 2.2, the following two lemmas can be easily obtained.

Lemma 3.2 *If q is a prime or a prime power with $q > 2$, $2s-1 \le r \le m-3$, then*

$$\begin{aligned} &N(r, 2(s-1), s-1; m-1, 2s, s; m, 2s, s; 2\nu+\delta, \Delta) \\ &\quad = \frac{q^{2(s-1)(m-r-2)}(q^s-1)(q^{s-1}+1)}{2(q-1)} \cdot \frac{\prod_{i=m-r-1}^{m-2s-1}(q^i-1)}{\prod_{i=1}^{r-2s+1}(q^i-1)} \\ &\qquad \times \left[2q^{m-r-2} + \frac{q^{m-r-2}-1}{q^{r-2s+2}-1}\right]. \end{aligned}$$

Lemma 3.3 *If q is a prime or a prime power with $q > 2$, $2s-1 \le r \le m-2$, then*

$$N(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta)$$

$$= q^{2(s-1)(m-r-1)} \cdot \frac{\prod_{i=m-r}^{m-2s}(q^i-1)}{\prod_{i=1}^{r-2s+1}(q^i-1)} \left[q^{m-r-1} + \frac{q^{m-r-1}-1}{q^{r-2s+2}-1} \right].$$

Lemma 3.4 *If q is a prime or a prime power with $q > 2$, $2s-1 \leq r \leq m-3$, then*

$$N(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta)$$
$$> N(r, 2(s-1), s-1; m-1, 2s, s; m, 2s, s; 2\nu+\delta, \Delta).$$

Proof According to Lemma 3.2 and Lemma 3.3, we have that

$$\frac{N(r, 2(s-1), s-1; m-1, 2s, s; m, 2s, s; 2\nu+\delta, \Delta)}{N(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta)}$$
$$= \frac{q^{2(s-1)(m-r-2)}(q^s-1)(q^{s-1}+1)}{2(q-1)} \cdot \frac{\prod_{i=m-r-1}^{m-2s-1}(q^i-1)}{\prod_{i=1}^{r-2s+1}(q^i-1)}$$
$$\times \left[2q^{m-r-2} + \frac{q^{m-r-2}-1}{q^{r-2s+2}-1} \right]$$
$$/q^{2(s-1)(m-r-1)} \cdot \frac{\prod_{i=m-r}^{m-2s}(q^i-1)}{\prod_{i=1}^{r-2s+1}(q^i-1)} \left[q^{m-r-1} + \frac{q^{m-r-1}-1}{q^{r-2s+2}-1} \right]$$
$$= \frac{(q^s-1)(q^{s-1}+1)}{2q^{2(s-1)}(q-1)} \cdot \frac{q^{m-r-1}-1}{q^{m-2s}-1} \cdot \frac{2q^{m-2s}+q^{m-r-2}-1}{q^{m-2s+1}+q^{m-r-1}-1}.$$

In order to discuss conveniently, we denote $\frac{(q^s-1)(q^{s-1}+1)}{2q^{2(s-1)}(q-1)}$ by a, $\frac{q^{m-r-1}-1}{q^{m-2s}-1}$ by b, and $\frac{2q^{m-2s}+q^{m-r-2}-1}{q^{m-2s+1}+q^{m-r-1}-1}$ by c. Obviously, for c, because $q > 2$, we have $c < 1$; for b, because $r+1 \geq 2s$, we have $b \leq 1$; for a, when $q > 2$, by computing, we know that $2q^{2(s-1)} \cdot (q-1) - (q^s-1)(q^{s-1}+1) > 0$. So $a < 1$. i.e., $abc < 1$.

Hence,

$$\frac{N(r, 2(s-1), s-1; m-1, 2s, s; m, 2s, s; 2\nu+\delta, \Delta)}{N(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta)} < 1,$$

i.e.,

$$N(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta)$$
$$> N(r, 2(s-1), s-1; m-1, 2s, s; m, 2s, s; 2\nu+\delta, \Delta). \qquad \square$$

By Lemma 2.1, we get the following two lemmas directly.

Lemma 3.5 *If q is a prime or a prime power, then*

$$N(m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta) = \frac{(q^s-1)(q^{s-1}+1)}{q-1}.$$

Lemma 3.6 *If q is a prime or a prime power, then*

$$N(m-2, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta) = \frac{q^{2(s-1)}(q^s-1)(q^{s-1}+1)}{q-1} \cdot \left[\frac{q(q^{m-2s}-1)}{q-1} + \frac{1}{2}\right].$$

As has been mentioned above, if we want $|\overline{C_0} \setminus \bigcup_{i=1}^{d} \overline{C_i}|$ to be minimized, i.e., $|\bigcup_{i=1}^{d} \overline{H_i}|$ to be maximized, we can assume that all H_i are $(m-1, 2(s-1), s-1)$-subspaces of C_0. This is because, given d $(m-1, 2(s-1), s-1)$-subspaces of C_0, we can take the span of each of them with a vector $v \notin C_0$ to reconstruct C_i. For the group testing problem, we will address the case when $d > \frac{(q^s-1)(q^{s-1}+1)}{q-1}$ in Sect. 5. In Sect. 4, we only consider the case when $d \leq \frac{(q^s-1)(q^{s-1}+1)}{q-1}$. The above discussion motivates the following problem.

Problem 1 Given a $(m, 2s, s)$-subspace C_0 in the orthogonal space over $\mathbb{F}_q$, and an integer d such that $1 \leq d \leq \frac{(q^s-1)(q^{s-1}+1)}{q-1}$, find $(m-1, 2(s-1), s-1)$-subspace H_i $(i = 1, 2, \ldots, d)$ of C_0 that maximizes $|\bigcup_{i=1}^{d} \overline{H_i}|$. At least, find good upper bounds for the quantity.

4 The packing arrangement and tighter bounds

In order to solve the Problem 1, i.e., the arrangement problem of d $(m-1, 2(s-1), s-1)$-subspaces, we need some lemmas and formulas. By inclusion-exclusion principle, we have

$$\begin{aligned}
\left|\bigcup_{i=1}^{d} \overline{H_i}\right| &= \sum_{i=1}^{d} (-1)^{i-1} \sum_{\substack{T \subseteq \{1,\ldots,d\} \\ |T|=i}} \bigcap_{t\in T} \overline{H_t} \\
&= \sum_{i=1}^{d} (-1)^{i-1} \sum_{\substack{T \subseteq \{1,\ldots,d\} \\ |T|=i}} \overline{\bigcap_{t\in T} H_t} \\
&= \sum_{i=1}^{d} (-1)^{i-1} \sum_{\substack{T \subseteq \{1,\ldots,d\} \\ |T|=i}} N\Bigg(r, 2(s-1), s-1; \bigcap_{t\in T} H_t, 2(s-1), s-1; \\
&\qquad m, 2s, s; 2\nu+\delta, \Delta\Bigg).
\end{aligned}$$

But it is difficult to obtain $\sum_{\substack{T\subseteq\{1,\ldots,d\}\\|T|=i}} N(r, 2(s-1), s-1; \bigcap_{t\in T} H_t, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta)$, so we consider it inductively.

For any two $(m_1, 2s_1, s_1)$-subspaces P and Q in the orthogonal space over $\mathbb{F}_q$, we know that

$$\dim(P) + \dim(Q) = \dim(\mathrm{span}(P \cup Q)) + \dim(P \cap Q).$$

Additionally, suppose C to be a $(m, 2s, s)$-subspace in the orthogonal space over $\mathbb{F}_q$, X to be a $(m-1, 2(s-1), s-1)$-subspace of C, and Y to be a $(m_1, 2s_1, s_1)$-subspace of C and $Y \subset C$. Assume $X \cap Y$ to be a $(r, 2s_2, s_2)$-subspace. Then the relation between X and Y is as follows: $Y \subseteq X$ or $m_1 = r+1, s_1 = s_2$ or $s_2 + 1$.

This conclusion leads to the following important lemma.

Lemma 4.1 *Let $H_1, \ldots, H_x$ be x $(m-1, 2(s-1), s-1)$-subspaces of a $(m, 2s, s)$-subspace C in the orthogonal space over $\mathbb{F}_q$, whose intersection is I, i.e., $I = H_1 \cap \cdots \cap H_x$. Let H be any $(m-1, 2(s-1), s-1)$-subspace of C not containing I, and set $Y_i = H \cap H_i$, $i \in \{1, 2, \ldots, x\}$. Then, for any subset $S \subseteq \{1, 2, \ldots, x\}$, we have*

$$\dim\left(\bigcap_{i\in S} H_i\right) = \dim\left(\bigcap_{i\in s} Y_i\right) + 1.$$

If $\bigcap_{i\in S} H_i$ is a $(m_i, 2s_i, s_i)$-subspace, then $\bigcap_{i\in S} Y_i$ is a $(m_i - 1, 2s_i, s_i)$-subspace or a $(m_i - 1, 2(s_i - 1), s_i - 1)$-subspace.

Proof For any S, since $\bigcap_{i\in S} H_i \nsubseteq H$, we have

$$\begin{aligned}\dim\left(\bigcap_{i\in S} H_i\right) &= \dim\left(H \cap \left(\bigcap_{i\in S} H_i\right)\right) + 1\\ &= \dim\left(\bigcap_{i\in S} (H \cap H_i)\right) + 1\\ &= \dim\left(\bigcap_{i\in S} Y_i\right) + 1.\end{aligned}$$

i.e.,

$$\dim\left(\bigcap_{i\in S} H_i\right) - \dim\left(\bigcap_{i\in S} Y_i\right) = 1.$$

Therefore, if $\bigcap_{i\in S} H_i$ is a $(m_i, 2s_i, s_i)$-subspace, then $\bigcap_{i\in S} Y_i$ is a $(m_i - 1, 2s_i, s_i)$-subspace or a $(m_i - 1, 2(s_i - 1), s_i - 1)$-subspace. □

Actually, Y_i contained in H is a $(m-2, 2(s-1), s-1)$-subspace or a $(m-2, 2(s-2), s-2)$-subspace, and $\dim(H) - \dim(Y_i) = 1$. This lemma tells us that the inter-relationship (in terms of dimensions of intersections) between $Y_1, \ldots, Y_x$ is the same as the inter-relationship between $H_1, \ldots, H_x$.

Consider a $(m-2, 2(s-2), s-2)$-subspace X of a $(m, 2s, s)$-subspace C in the orthogonal space over $\mathbb{F}_q$, where $s \geq 2$. Then, the number of $(m-1, 2(s-1), s-1)$-subspaces of C containing X is $\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]_q = q+1$. Let Y be a $(m-3, 2(s-3), s-3)$-subspace or a $(m-3, 2(s-2), s-2)$-subspace of C. Then the number of $(m-1, 2(s-1), s-1)$-subspaces containing Y is

$$\begin{bmatrix}3\\2\end{bmatrix}_q = \frac{\prod_{i=2}^{3}(q^i-1)}{\prod_{i=1}^{2}(q^i-1)} = q^2+q+1.$$

In the following, we will compute $|\bigcup_{i=1}^{x} \overline{H_i}|$ for all arrangement of x $(m-1, 2(s-1), s-1)$-subspaces, where $2 \leq x \leq q^2+q+1$.

Suppose C_0 to be a $(m, 2s, s)$-subspace in the orthogonal space over $\mathbb{F}_q$. Denote $\mathcal{H}$ to be any set of $(m-1, 2(s-1), s-1)$-subspaces of C_0 (at least two). Let $x(\mathcal{H})$ be the maximum number of $(m-1, 2(s-1), s-1)$-subspaces in $\mathcal{H}$ whose intersection is a $(m-2, 2(s-2), s-2)$-subspace. Note that $2 \leq x(\mathcal{H}) \leq q+1$. Also define $g(\mathcal{H}) = |\bigcup_{H\in\mathcal{H}} \overline{H}|$.

Lemma 4.2 *Suppose q is a prime or a prime power with $q > 2$, $2s-1 \leq r \leq m-3$. Assuming that the number of $(m-1, 2(s-1), s-1)$-subspaces of a $(m, 2s, s)$-subspace in the orthogonal space over $\mathbb{F}_q$ is x, then*

$$\begin{aligned} g(\mathcal{H}) &= xN(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta) \\ &\quad - kN(r, 2(s-1), s-1; m-2, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta), \end{aligned}$$

where $2 \leq x \leq \frac{(q^s-1)(q^{s-1}+1)}{q-1}$, and k is an integer with

$$0 \leq k \leq \frac{q^{2(s-1)}(q^s-1)(q^{s-1}+1)}{q-1}\left[\frac{q(q^{m-2s}-1)}{q-1}+\frac{1}{2}\right].$$

In particular, if $k=0$, for any x, $g(\mathcal{H})$ gets the top value $xN(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta)$.

Proof According to Lemma 4.1, for two $(m-1, 2(s-1), s-1)$-subspaces P_1 and P_2 of a $(m, 2s, s)$-subspace from the orthogonal space over $\mathbb{F}_q$, we know that $P_1 \cap P_2$ is a $(m-2, 2(s-2), s-2)$-subspace or a $(m-2, 2(s-1), s-1)$-subspace $(s \geq 2)$. Therefore, the intersection of any two $(m-2, 2(s-2), s-2)$-subspaces or $(m-2, 2(s-1), s-1)$-subspaces is a $(m-3, 2(s-2), s-2)$-subspace or a $(m-3, 2(s-3), s-3)$-subspace. According to inclusion-exclusion principle, we obtain $g(\mathcal{H}) = xN(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta) - kN(r, 2(s-1), s-1; m-2, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta)$.

The number of $(m-1, 2(s-1), s-1)$-subspaces, i.e., the value of x is at most $N(m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta)$, i.e., the number of $(m-1, 2(s-1), s-1)$-subspaces contained in a fixed $(m, 2s, s)$-subspace.

By Lemma 3.5, we know that

$$N(m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta) = \frac{(q^s-1)(q^{s-1}+1)}{q-1}.$$

Hence,

$$2 \le x \le \frac{(q^s-1)(q^{s-1}+1)}{q-1}.$$

By Lemma 3.6, we know that

$$\begin{aligned} &N(m-2, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta) \\ &\quad = \frac{q^{2(s-1)}(q^s-1)(q^{s-1}+1)}{q-1} \cdot \left\lceil \frac{q(q^{m-2s}-1)}{q-1} + \frac{1}{2} \right\rceil. \end{aligned}$$

Hence,

$$k \le \frac{q^{2(s-1)}(q^s-1)(q^{s-1}+1)}{q-1} \cdot \left\lceil \frac{q(q^{m-2s}-1)}{q-1} + \frac{1}{2} \right\rceil.$$

Obviously, when $k=0$, $g(\mathcal{H})$ gets the top value $xN(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta)$. □

We will discuss tighter bounds of the arrangement and the packing arrangement.

Firstly, we consider the simplest case when $r=1$ and $s=1$. The number of $(1,0,0)$-subspaces contained in a $(m,2,1)$-subspace from the orthogonal space over $\mathbb{F}_q$ is $N(1,0,0; m,2,1; 2\nu+\delta, \Delta)$.

According to Lemma 2.1, we know

$$N(1,0,0; m,2,1; 2\nu+\delta, \Delta) = \frac{2q^{m-1}+q^{m-2}-1}{q-1}.$$

Let C_0 be a $(m,2,1)$-subspace from the orthogonal space over $\mathbb{F}_q$, V be any $(m-2,0,0)$-subspace of C_0, and $H_1, \ldots, H_{q+1}$ be the set of all $(m-1,0,0)$-subspaces containing V. Then, by the inclusion-exclusion principle, we obtain

$$\begin{aligned} &\left| \bigcup_{i=1}^{q+1} \overline{H_i} \right| \\ &\quad = (q+1)N(1,0,0; m-1,0,0; 2\nu+\delta, \Delta) \\ &\qquad - qN(1,0,0; m-2,0,0; 2\nu+\delta, \Delta). \end{aligned}$$

According to Lemma 2.1, we know

$$N(1,0,0; m-1,0,0; 2\nu+\delta, \Delta) = \frac{q^{m-1}-1}{q-1},$$

$$N(1,0,0; m-2,0,0; 2\nu+\delta, \Delta) = \frac{q^{m-2}-1}{q-1}.$$

Hence,

$$\left| \bigcup_{i=1}^{q+1} \overline{H_i} \right| = (q+1) \cdot \frac{q^{m-1}-1}{q-1} - q \cdot \frac{q^{m-2}-1}{q-1} = \frac{q^m-1}{q-1}.$$

Assuming that $\bigcup_{i=1}^{x}\overline{H_i}$ cover $\overline{C_0}$, then $|\bigcup_{i=1}^{x}\overline{H_i}| \geq |\overline{C_0}|$, i.e.,

$$x \cdot \frac{q^{m-1}-1}{q-1} - (x-1)\cdot\frac{q^{m-2}-1}{q-1} \geq \frac{2q^{m-1}+q^{m-2}-1}{q-1}.$$

By computing, we obtain

$$x \geq \frac{2q}{q-1}.$$

Because x is an integer, it is at least $\lceil\frac{2q}{q-1}\rceil$. In order to discuss it conveniently, we denote $\lceil\frac{2q}{q-1}\rceil$ by u.

The following theorem follows immediately.

Theorem 4.3 *When* $r=1$, $s=1$, $m\geq 4$ *and* $d\geq\lceil\frac{2q}{q-1}\rceil$, $|\bigcup_{i=1}^{u}\overline{H_i}|$ *is larger than the number of* $(1,0,0)$*-subspaces contained in* C_0. *Hence, one way to obtain the maximum of* $|\bigcup_{i=1}^{d}\overline{H_i}|$ *is to find* $\lceil\frac{2q}{q-1}\rceil$ $(m-1,0,0)$*-subspaces containing a* $(m-2,0,0)$*-subspace of* C_0.

We aim for a good upper bound for $|\bigcup_{i=1}^{d}\overline{H_i}|$. Firstly, the definition of a particular arrangement is given as follows.

Definition 4.4 (Packing arrangement) Suppose $2\leq d\leq q^2+q+1$. Let C_0 be a $(m,2s,s)$-subspace, V be a $(m-2,2(s-2),s-2)$-subspace of C_0, and W be any $(m-3,2(s-3),s-3)$-subspace of V. The packing arrangement of d $(m-1,2(s-1),s-1)$-subspaces is an arrangement in which $q+1$ $(m-1,2(s-1),s-1)$-subspaces, say $H_1,\ldots,H_{q+1}$, all contain V and the rest of $(m-1,2(s-1),s-1)$-subspaces contain W.

The following theorem tells the "cost" of this arrangement.

Theorem 4.5 *Suppose q is a prime or a prime power, $q>2$, and $2s-1\leq r\leq m-3$. Let $2\leq d\leq q^2+q+1$, and let $H_1,\ldots,H_d$ be in the packing configuration. Then*

$$\left|\bigcup_{i=1}^{d}\overline{H_i}\right| = dN(r,2(s-1),s-1;m-1,2(s-1),s-1;m,2s,s;2\nu+\delta,\Delta).$$

Proof Without loss of generality, assume that $H_1,\ldots,H_{q+1}$ intersect at a $(m-2,2(s-2),s-2)$-subspace V and the rest of the $(m-1,2(s-1),s-1)$-subspaces all contain a $(m-3,2(s-3),s-3)$-subspace W, $W\subset V$. Consider any H_i, where $q+1<i\leq d$. Let $V_j=H_i\cap H_j$, for any $j\in\{1,\ldots,i-1\}$. We first show $\{V_1,\ldots,V_{q+1}\}=\{V_1,\ldots,V_{i-1}\}$. Note that all V_j contain W, by Lemma 4.1, we know that $V_1,\ldots,V_{q+1}$ are different $(m-2,2(s-2),s-2)$-subspaces of H_i. Moreover, the total number of $(m-2,2(s-2),s-2)$-subspaces in H_i containing W is

$$\begin{bmatrix} s-1-(s-3) \\ s-2-(s-3) \end{bmatrix}_q = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = q+1.$$

Therefore,

$$\{V_1, \ldots, V_{q+1}\} = \{V_1, \ldots, V_{i-1}\}.$$

Next, because V_i is a $(m-2, 2(s-2), s-2)$-subspace and $N(r, 2(s-1), s-1; m-1, 2(s-2), s-2; m, 2s, s; 2\nu+\delta, \Delta) = 0$, we have $\overline{V_i} = 0$, $(i = 1, \ldots, q+1)$.

Then,

$$\begin{aligned}\left|\overline{H_i} \setminus \bigcup_{j=1}^{i-1} \overline{H_j}\right| &= \left|\overline{H_i} \setminus \bigcup_{j=1}^{i-1} (\overline{H_j} \cup \overline{H_i})\right| = \left|\overline{H_i} \setminus \bigcup_{j=1}^{i-1} \overline{V_j}\right| \\ &= \left|\overline{H_i} \setminus \bigcup_{j=1}^{q+1} \overline{V_j}\right| = |\overline{H_i}| - \left|\bigcup_{j=1}^{q+1} \overline{V_j}\right| = |\overline{H_i}| \\ &= N(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta).\end{aligned}$$

Finally,

$$\begin{aligned}\left|\bigcup_{i=1}^{d} \overline{H_i}\right| &= \left|\bigcup_{i=1}^{q+1} H_i\right| + \sum_{i=q+2}^{d} \left|\overline{H_i} \setminus \bigcup_{j=1}^{i-1} \overline{H_j}\right| \\ &= (q+1)N(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta) \\ &\quad - qN(r, 2(s-1), s-1; m-1, 2(s-2), s-2; m, 2s, s; 2\nu+\delta, \Delta) \\ &\quad + (d-q+1)N(r, 2(s-1), s-1; m-1, 2(s-1), s-1; \\ &\qquad m, 2s, s; 2\nu+\delta, \Delta) \\ &= dN(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta).\end{aligned}$$ □

Theorem 4.6 *Suppose q is a prime or a prime power, $q > 2$, $2s-1 \le r \le m-3$, $2 \le d \le t$, $t = \frac{(q^s-1)(q^{s-1}+1)}{q-1}$ and $s \ge 2$. Consider any d $(m-1, 2(s-1), s-1)$-subspaces $H_1, \ldots, H_d$ of a $(m, 2s, s)$-subspace in the orthogonal space over $\mathbb{F}_q$. Then*

$$\left|\bigcup_{i=1}^{d} \overline{H_i}\right| \le dN(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta).$$

Moreover, when $2 \le d \le q+1$, there exists an arrangement of $(m-1, 2(s-1), s-1)$-subspaces achieving the right hand side.

Proof

$$\begin{aligned}\left|\bigcup_{i=1}^{d} \overline{H_i}\right| &\le dN(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta) \\ &\quad - (d-1)N(r, 2(s-1), s-1; m-1, 2(s-2), s-2; m, 2s, s; 2\nu+\delta, \Delta) \\ &= dN(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta).\end{aligned}$$

When $2 \le d \le q+1$, let d $(m-1, 2(s-1), s-1)$-subspaces intersect at a $(m-2, 2(s-2), s-2)$-subspace. Moreover, any two $(m-1, 2(s-1), s-1)$-subspaces don't intersect at a $(m-2, 2(s-1), s-1)$-subspace. Thus, the left is equal to the right in the above formula, i.e., there exists an arrangement of $(m-1, 2(s-1), s-1)$-subspaces achieving the right hand side. □

According to Theorem 4.5 and Theorem 4.6, it is easy to obtain the following corollary.

Corollary 4.7 *Suppose q is a prime or a prime power, $q > 2$, $2s-1 \le r \le m-3$, and*

$$d \ge \max\left\{\frac{(q^s-1)(q^{s-1}+1)}{q-1}, q^2+q+1\right\}.$$

Then

$$\left|\bigcup_{i=1}^{d} \overline{H_i}\right| \le dN(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu+\delta, \Delta).$$

5 The tighter analysis of $A(m, 2s, s; r, 2(s-1), s-1; 2\nu+\delta)$

At first, we will show that $A(m, 2s, s; r, 2(s-1), s-1; 2\nu+\delta)$ is not a good design under some cases.

Theorem 5.1 *If $2s-1 \le r \le m-3$, $d \ge t$ and $t = \frac{(q^s-1)(q^{s-1}+1)}{q-1}$, then $A(m, 2s, s; r, 2(s-1), s-1; 2\nu+\delta)$ is not d-disjunct.*

Proof Consider any $(m, 2s, s)$-subspace C_0 in the orthogonal space over $\mathbb{F}_q$. Let $H_1, \ldots, H_t$ be the all $(m-1, 2(s-1), s-1)$-subspaces of C_0. Let v_i be any $(1, 0, 0)$-subspace in the orthogonal space over $\mathbb{F}_q$, and $v_i \notin C_0$, $(1 \le i \le t)$. For any $i = 1, \ldots, t$, define $C_i = \mathrm{span}\{H_i, v_i\}$. Then every C_i is a $(m, 2s, s)$-subspace in the orthogonal space over $\mathbb{F}_q$. For $i = t+1, \ldots, d$, choose arbitrarily $d-t$ $(m, 2s, s)$-subspaces C_i, which are different from $C_1, \ldots, C_t$. Since any $(r, 2(s-1), s-1)$-subspace of C_0 is contained in some H_i, $i = 1, \ldots, t$, it is also contained in C_i. Then

$$\overline{C_0} \setminus \bigcup_{i=1}^{d} \overline{C_i} = \overline{C_0} \setminus \bigcup_{i=1}^{t} \overline{C_i} = \emptyset.$$

Namely $A(m, 2s, s; r, 2(s-1), s-1; 2\nu+\delta)$ is not d-disjunct. □

Corollary 5.2 *If $m \ge 4$, $d \ge k$ and $k = \lceil \frac{2q}{q-1} \rceil$, then $A(m, 2, 1; 1, 0, 0; 2\nu+\delta)$ is not d-disjunct.*

In the following, we consider the tighter analysis of $A(m, 2s, s; r, 2(s-1), s-1; 2\nu+\delta)$ under some cases. By Theorem 4.3, we get the following corollary directly.

Corollary 5.3 *If* $2 \leq d \leq u$, $u = \lceil \frac{2q}{q-1} \rceil$, *then* $A(m, 2, 1; 1, 0, 0; 2\nu + \delta)$ *is* d^z-*disjunct, but it is not* d^{z+1}-*disjunct, where* $z = N(1, 0, 0; m, 2, 1; 2\nu + \delta, \Delta) - dN(1, 0, 0; m - 1, 0, 0; 2\nu + \delta, \Delta)$.

Theorem 5.4 *If* $2s - 1 \leq r \leq m - 3$, $2 \leq d \leq t$ *and* $t = \frac{(q^s-1)(q^{s-1}+1)}{q-1}$, *then* $A(m, 2s, s; r, 2(s-1), s-1; 2\nu + \delta)$ *is* d^z-*disjunct, where* $z = N(r, 2(s-1), s-1; m, 2s, s; 2\nu + \delta, \Delta) - dN(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu + \delta, \Delta)$. *Moreover, when* $2 \leq d \leq q^2 + q + 1$, $A(m, 2s, s; r, 2(s-1), s-1; 2\nu + \delta)$ *is not* d^{z+1}-*disjunct.*

Proof By Theorem 4.6, when $d \leq t$, it is easy to obtain that $A(m, 2s, s; r, 2(s-1), s-1; 2\nu + \delta)$ is d^z-disjunct. By Theorem 4.5, when $d \leq q^2 + q + 1$, it is easy to obtain that $A(m, 2s, s; r, 2(s-1), s-1; 2\nu + \delta)$ is not d^{z+1}-disjunct.

We will show that $z > 0$ as follows. By Lemma 3.3, we know

$$\begin{aligned} &N(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu + \delta, \Delta) \\ &\quad = q^{2(s-1)(m-r-1)} \cdot \frac{\prod_{i=m-r}^{m-2s}(q^i - 1)}{\prod_{i=1}^{r-2s+1}(q^i - 1)} \left[q^{m-r-1} + \frac{q^{m-r-1} - 1}{q^{r-2s+2} - 1} \right]. \end{aligned}$$

Additionally, according to Lemma 2.1

$$\begin{aligned} &N(r, 2(s-1), s-1; m, 2s, s; 2\nu + \delta, \Delta) \\ &\quad = q^{2(s-1)(m-r-1)} \cdot \frac{(q^s - 1)(q^{s-1} + 1)}{2(q-1)} \cdot \frac{\prod_{i=m-r}^{m-2s}(q^i - 1)}{\prod_{i=1}^{r-2s+1}(q^i - 1)} \\ &\qquad \times \left[2q^{m-r-1} + \frac{q^{m-r-1} - 1}{q^{r-2s+2} - 1} \right]. \end{aligned}$$

Then,

$$\begin{aligned} z &= N(r, 2(s-1), s-1; m, 2s, s; 2\nu + \delta, \Delta) \\ &\quad - dN(r, 2(s-1), s-1; m-1, 2(s-1), s-1; m, 2s, s; 2\nu + \delta, \Delta) \\ &= q^{2(s-1)(m-r-1)} \cdot \frac{(q^s - 1)(q^{s-1} + 1)}{2(q-1)} \cdot \frac{\prod_{i=m-r}^{m-2s}(q^i - 1)}{\prod_{i=1}^{r-2s+1}(q^i - 1)} \\ &\quad \times \left[2q^{m-r-1} + \frac{q^{m-r-1} - 1}{q^{r-2s+2} - 1} \right] \\ &\quad - d \cdot q^{2(s-1)(m-r-1)} \cdot \frac{\prod_{i=m-r}^{m-2s}(q^i - 1)}{\prod_{i=1}^{r-2s+1}(q^i - 1)} \left[q^{m-r-1} + \frac{q^{m-r-1} - 1}{q^{r-2s+2} - 1} \right]. \end{aligned}$$

By computing, we know that $z > 0$. □

6 The discussions of test efficiency

The basic problem of group testing is to identify most positive items with least tests. Therefore it is significant to discuss how to make the ratio R smaller. In our d^z-disjunct matrix, we denote the number of the rows by n_r, and the number of the columns by n_c. Then according to the construction of our pooling design, we get that $n_r = N(r, 2(s-1), s-1; 2\nu+\delta, \Delta)$ and $n_c = N(m, 2s, s; 2\nu+\delta, \Delta)$.

In Wan (2002), we know that

$$\begin{aligned} n_r &= N(r, 2(s-1), s-1; 2\nu+\delta, \Delta) \\ &= q^{2(s-1)(\nu+s-r-1)+\delta(s-1)} \frac{\prod_{i=\nu+s-r}^{\nu}(q^i-1)(q^{i+\delta-1}+1)}{\prod_{i=0}^{s-1}(q^i-1)\prod_{i=0}^{s-2}(q^i+1)\prod_{i=1}^{r-2(s-1)}(q^i-1)}, \\ n_c &= N(m, 2s, s; 2\nu+\delta, \Delta) \\ &= q^{2s(\nu+s-m)+\delta s} \frac{\prod_{i=\nu+s-m+1}^{\nu}(q^i-1)(q^{i+\delta-1}+1)}{\prod_{i=1}^{s}(q^i-1)\prod_{i=0}^{s-1}(q^i+1)\prod_{i=1}^{m-2s}(q^i-1)}. \end{aligned}$$

We address the peak of the rows in the d^z-disjunct as follows. Because n_r has two parameters, s and t, we may assume that s has been fixed at first. The following theorem tells us how to choose r so that the number of tests is not much too large.

Theorem 6.1 *When s is fixed, for $2(s-1) < r$, the sequence $n_r = N(r, 2(s-1), s-1; 2\nu+\delta, \Delta)$ is unimodal and gets its peak at $r = [\frac{2\nu+2s+\delta-3}{3}]$.*

Proof Assume $r_1 < r_2$. Then

$$\begin{aligned} \frac{n_{r_1}}{n_{r_2}} &= \frac{N(r_1, 2(s-1), s-1; 2\nu+\delta, \Delta)}{N(r_2, 2(s-1), s-1; 2\nu+\delta, \Delta)} \\ &= q^{2(s-1)(r_2-r_1)} \cdot \frac{\prod_{i=\nu+s-r_1}^{\nu}(q^i-1)(q^{i+\delta-1}+1)\prod_{i=1}^{r_2-2s+2}(q^i-1)}{\prod_{i=\nu+s-r_2}^{\nu}(q^i-1)(q^{i+\delta-1}+1)\prod_{i=1}^{r_1-2s+2}(q^i-1)} \\ &= q^{2(s-1)(r_2-r_1)} \cdot \frac{\prod_{i=r_1-2s+3}^{r_2-2s+2}(q^i-1)}{\prod_{i=\nu+s-r_2}^{\nu+s-r_1-1}(q^i-1)(q^{i+\delta-1}+1)} \\ &= q^{2(s-1)(r_2-r_1)} \cdot \prod_{i=1}^{r_2-r_1} \frac{q^{r_1-2s+2+i}-1}{(q^{\nu+s-r_2+i-1}-1)(q^{\nu+s-r_2+\delta+i-2}+1)} \\ &= q^{2(s-1)(r_2-r_1)} \cdot \frac{q^{r_1-2s+3}-1}{(q^{\nu+s-r_1}-1)(q^{\nu+s-r_1+\delta-2}+1)} \\ &\quad \cdot \frac{q^{r_1-2s+4}-1}{(q^{\nu+s-r_1-2}-1)(q^{\nu+s-r_1+\delta-3}+1)} \\ &\quad \cdots \frac{q^{r_2-2s+2}-1}{(q^{\nu+s-r_2}-1)(q^{\nu+s-r_2+\delta-1}+1)} \end{aligned}$$

$$= \frac{q^{2(s-1)(q^{r_1-2s+3}-1)}}{(q^{\nu+s-r_1-1}-1)(q^{\nu+s-r_1+\delta-2}+1)} \cdot \frac{q^{2(s-1)(q^{r_1-2s+4}-1)}}{(q^{\nu+s-r_1-2}-1)(q^{\nu+s-r_1+\delta-3}+1)}$$

$$\cdot \cdots \cdot \frac{q^{2(s-1)(q^{r_2-2s+2}-1)}}{(q^{\nu+s-r_2}-1)(q^{\nu+s-r_2+\delta-1}+1)}.$$

We know that

$$\begin{aligned}
&\frac{q^{2(s-1)(q^{r_1-2s+3}-1)}}{(q^{\nu+s-r_1-1}-1)(q^{\nu+s-r_1+\delta-2}+1)} \\
&\quad < \frac{q^{2(s-1)(q^{r_1-2s+4}-1)}}{(q^{\nu+s-r_1-2}-1)(q^{\nu+s-r_1+\delta-3}+1)} \\
&\quad < \cdots \\
&\quad < \frac{q^{2(s-1)(q^{r_2-2s+2}-1)}}{(q^{\nu+s-r_2}-1)(q^{\nu+s-r_2+\delta-1}+1)}.
\end{aligned}$$

So we have that if

$$\frac{q^{2(s-1)(q^{r_2-2s+2}-1)}}{(q^{\nu+s-r_2}-1)(q^{\nu+s-r_2+\delta-1}+1)} < 1,$$

i.e., $0 < r_1 < r_2 < \frac{2\nu+2s+\delta-1}{3}$, then $n_{r_1} < n_{r_2}$;

if

$$\frac{q^{2(s-1)(q^{r_1-2s+3}-1)}}{(q^{\nu+s-r_1-1}-1)(q^{\nu+s-r_1+\delta-2}+1)} > 1,$$

i.e., $\frac{2\nu+2s+\delta-4}{3} < r_1 < r_2 \le m$, then $n_{r_1} > n_{r_2}$.

So the sequence $n_r = N(r, 2(s-1), s-1; 2\nu+\delta, \Delta)$ is unimodal and gets its peak at $r = [\frac{2\nu+2s+\delta-3}{3}]$. □

In our construction, we denote $\frac{n_r}{n_c}$ by $\overline{R}$.

When $r < m-2$,

$$\overline{R} = \frac{n_r}{n_c} = q^{2sm+2r-2\nu-4s-2sr+2-\delta}(q^s-1)(q^{s-1}+1) \times \frac{\prod_{i=r-2s+3}^{m-2s}(q^i-1)}{\prod_{i=\nu+s-m+1}^{\nu+s-r-1}(q^i-1)(q^{i+\delta-1}+1)}.$$

When $r = m-2$,

$$\overline{R} = \frac{n_r}{n_c} = \frac{(q^s-1)(q^{s-1}+1)}{q^{2\nu+2+\delta-2m}(q^{\nu+s-m+1}-1)(q^{\nu+s-m+\delta}+1)}.$$

When $r = m-1$,

$$\overline{R} = \frac{n_r}{n_c} = \frac{(q^s-1)(q^{s-1}+1)}{q^{2(\nu+s-m)+\delta}(q^{m-2s+1}-1)}.$$

When $r=m$,

$$\overline{R}=\frac{n_r}{n_c}=\frac{(q^{\nu+s-m}-1)(q^{\nu+s-m+\delta-1}+1)(q^s-1)(q^{s-1}+1)}{q^{2\nu+4s+\delta-2m-2(q^{m-2s+1}-1)(q^{m-2s+2}-1)}}.$$

Because the formula of $\overline{R}$ has three parameters, in order to address m's effect on $\overline{R}$, we can assume that s and r have been fixed firstly.

Theorem 6.2 *When $r<m-2$, for the fixed s and r, the sequence $\overline{R}$ is unimodal and gets its valley at $m=[\frac{2\nu+2s+\delta}{3}]$.*

Proof Assume $m_1<m_2$, we denote $\frac{n_r}{n_c}$ by $\overline{R}_{m_1}$ ($\overline{R}_{m_2}$) when $m=m_1$ ($m=m_2$). Then

$$\begin{aligned}\frac{\overline{R}_{m_1}}{\overline{R}_{m_2}}&=\frac{\prod_{i=\nu+s-m_2+1}^{\nu+s-m_1}(q^i-1)(q^{i+\delta-1}+1)}{q^{2s(m_2-m_1)}\cdot\prod_{i=m_1-2s+1}^{m_2-2s}(q^i-1)}\\&=\frac{(q^{\nu+s-m_2+1}-1)(q^{\nu+s-m_2+\delta}+1)}{q^{m_2}-1}\cdot\frac{(q^{\nu+s-m_2+2}-1)(q^{\nu+s-m_2+\delta+1}+1)}{q^{m_2-1}-1}\\&\quad\cdots\frac{(q^{\nu+s-m_1}-1)(q^{\nu+s-m_1+\delta-1}+1)}{q^{m_1+1}-1}.\end{aligned}$$

We know that

$$\begin{aligned}&\frac{(q^{\nu+s-m_2+1}-1)(q^{\nu+s-m_2+\delta}+1)}{q^{m_2}-1}\\&\quad<\frac{(q^{\nu+s-m_2+2}-1)(q^{\nu+s-m_2+\delta+1}+1)}{q^{m_2-1}-1}\\&\quad<\cdots\\&\quad<\frac{(q^{\nu+s-m_1}-1)(q^{\nu+s-m_1+\delta-1}+1)}{q^{m_1+1}-1}.\end{aligned}$$

So we have that if

$$\frac{(q^{\nu+s-m_1}-1)(q^{\nu+s-m_1+\delta-1}+1)}{q^{m_1+1}-1}<1,$$

i.e., $[\frac{2\nu+2s+\delta}{3}]\le m_1\le m_2\le 2\nu+\delta$, then $\overline{R}_{m_1}<\overline{R}_{m_2}$;
if

$$\frac{(q^{\nu+s-m_2+1}-1)(q^{\nu+s-m_2+\delta}+1)}{q^{m_2}-1}>1,$$

i.e., $0<m_1<m_2\le[\frac{2\nu+2s+\delta}{3}]$, then $\overline{R}_{m_1}>\overline{R}_{m_2}$.

So the sequence $\overline{R}=\frac{n_r}{n_c}$ is unimodal and gets its valley at $m=[\frac{2\nu+2s+\delta}{3}]$. □

Secondly, we assume that s and m have been fixed and address r's effect on $\overline{R}$ in our construction by the similar way as we do in the above theorem.

Theorem 6.3 *When $r < m - 2$, for the fixed s and m, the sequence $\overline{R}$ is unimodal and gets its peak at $r = [\frac{2\nu+2s+\delta-3}{3}]$.*

Moreover, we find that the ratio efficiency $\overline{R}$ in this paper is smaller than that in D'yachkov et al. (2005) under some conditions.

In D'yachkov et al. (2005) constructed a d^z-disjunct matrix with subspaces of vector space. Each of the columns (rows) is labeled by a $m(r)$-dimensional subspace, $m_{ij} = 1$ if and only if the label of row i is contained in the label of column j. We denote the test efficiency in D'yachkov et al. (2005) to be $\widetilde{R}$. In order to compare $\overline{R}$ with $\widetilde{R}$, let the dimension of the vector space be $2\nu + \delta$.

Then when $r < m - 2$,

$$\widetilde{R} = \frac{\begin{bmatrix} 2\nu+\delta \\ r \end{bmatrix}_q}{\begin{bmatrix} 2\nu+\delta \\ m \end{bmatrix}_q} = \prod_{i=1}^{m-r} \frac{q^{r+i} - 1}{q^{2\nu+\delta-m+i} - 1}.$$

When $r = m - 2$,

$$\widetilde{R} = \frac{\begin{bmatrix} 2\nu+\delta \\ m-2 \end{bmatrix}_q}{\begin{bmatrix} 2\nu+\delta \\ m \end{bmatrix}_q} = \frac{(q^{m-1} - 1)(q^m - 1)}{(q^{2\nu+\delta-m+1} - 1)(q^{2\nu+\delta-m+2} - 1)}.$$

When $r = m - 1$,

$$\widetilde{R} = \frac{\begin{bmatrix} 2\nu+\delta \\ m-1 \end{bmatrix}_q}{\begin{bmatrix} 2\nu+\delta \\ m \end{bmatrix}_q} = \frac{q^{m-1} - 1}{q^{2\nu+\delta-m+1} - 1}.$$

When $r = m$,

$$\widetilde{R} = \frac{\begin{bmatrix} 2\nu+\delta \\ m \end{bmatrix}_q}{\begin{bmatrix} 2\nu+\delta \\ m \end{bmatrix}_q} = 1.$$

According to the formulas of $\overline{R}$ and $\widetilde{R}$, we get the following two theorems through computing.

Theorem 6.4 *For $r < m - 2$, when $s < r + 1 < \nu + 1$ and $2sm + \nu + r + 4 \leq 2m + 4s + 2sr$, $\overline{R} < \widetilde{R}$.*

Theorem 6.5 *For $r = m$ and $r = m - 1$, $\overline{R} < R'$.*

For $r = m - 2$, we do not find some good conclusion about the relation between $\overline{R}$ and $\widetilde{R}$.

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