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Two constructions of new error-correcting pooling designs from orthogonal spaces over a finite field of characteristic 2

Zengti Li · Suogang Gao · Hongjie Du · Feng Zou · Weili Wu

Abstract In this paper, we construct two classes of $t \times n, s^e$-disjunct matrix with subspaces in orthogonal space $\mathbb{F}_q^{(2\nu+1)}$ of characteristic 2 and exhibit their disjunct properties. We also prove that the test efficiency t/n of constructions II is smaller than that of D'yachkov et al. (J. Comput. Biol. 12:1129–1136, 2005).

Keywords Group testing algorithm · s^e-disjunct matrix · Pooling design · Orthogonal space

1 Introduction

The basic problem of group testing is to identify the set of defective items in a large population of items. Suppose we have n items to be tested and that there are at most d

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Z. Li
Department of Mathematics, Langfang Normal College, Langfang 065000, China

S. Gao (✉)
Mathematics and Information Science College, Hebei Normal University, Shijiazhuang 050016, China
e-mail: sggao@hebtu.edu.cn

H. Du · F. Zou · W. Wu
Department of Computer Science, University of Texas at Dallas, Richardson, TX 75080, USA

W. Wu
e-mail: weiliwu@utdallas.edu

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defective items among them. Each *test* (or *pool*) is (or contains) a subset of items. We assume some testing mechanism exists which if applied to an arbitrary subset of the population gives a *negative outcome* if the subset contains no positive and *positive outcome* otherwise. Objectives of group testing vary from minimizing the number of tests, limiting number of pools, limiting pool sizes to tolerating a few errors. It is conceivable that these objectives are often contradicting, thus testing strategies are application dependent. A group testing algorithm is *non-adaptive* if all tests must be specified without knowing the outcomes of other tests. A non-adaptive testing algorithm is useful in many areas such as DNA library screening (Du and Hwang 2006; Ngo and Zu 2000).

A group testing algorithm is *error tolerant* if it can detect some errors in test outcomes. A mathematical model of error-tolerance designs is an s^e*-disjunct* matrix.

A binary matrix M is said to be s^e*-disjunct* if given any $s+1$ columns of M with one designated, there are e rows with a 1 in the designated column and 0 in each of the other s columns. An s^1-disjunct matrix is said to be s*-disjunct*. D'yachkov et al. (2007) proposed the concept of fully s^e-disjunct matrices. An s^e-disjunct matrix is *fully* s^e*-disjunct* if it is not $d^{e'}$-disjunct whenever $d > s$ or $e' > e$.

Macula (1996) proposed a novel way of constructing s-disjunct matrices using the containment relation in a structure. Park et al. (2003) proposed another way with simplicial complex. Ngo and Du (2002) extended this construction to some graph properties, including matching.

Huang and Weng (2004) gave a comprehensive treatment of construction of d-disjunct matrices by using of pooling spaces, which is a significant and important addition to the general theory.

D'yachkov et al. (2005) claimed that the "containment matrix" method has opened a new door for constructing s-disjunct matrices from many mathematical structures.

In this paper, we construct two classes s^e-disjunct matrix with subspaces in orthogonal space $\mathbb{F}_q^{(2\nu+1)}$ with characteristic 2 and exhibit their disjunct properties. Given some fixed items, our goal is to detect the positive items. For a pooling design, the less the number of tests is, the better the pooling design is. It is known that the *test efficiency* is the ratio between the number of rows and the number of columns in the s^e-disjunct matrix, denoted by t/n (Huang and Hwang 2006; Zhang et al. 2008). We will give some discussions on the ratio t/n of construction II and compare it with others, such as in D'yachkov et al. (2005).

2 Orthogonal space of characteristic 2

Let $\mathbb{F}_q$ be a finite field of characteristic 2. Denote by $\mathcal{K}_n$ the set of all $n \times n$ alternate matrices over F_q. Two $n \times n$ matrices A and B over F_q are said to be *congruent* mod $\mathcal{K}_n$, denoted $A \equiv B(\text{mod } \mathcal{K}_n)$, if $A - B \in \mathcal{K}_n$. Clearly, $\equiv$ is an equivalence relation on the set of all $n \times n$ matrices. Let $[A]$ denote the equivalence class containing A. Two matrix classes $[A]$ and $[B]$ are said to be *cogredient* if there is a nonsingular $n \times n$ matrix Q over F_q such that $[QAQ^T] \equiv [B]$.

Let

$$G_{2\nu+\delta,\,\Delta}=\begin{pmatrix}0 & I^{(\nu)} & \\ & 0 & \\ & & \Delta\end{pmatrix},\quad \text{where } \Delta=\begin{cases}\emptyset, & \text{if } \delta=0,\\ (1), & \text{if } \delta=1,\\ \begin{pmatrix}\alpha & 1\\ & \alpha\end{pmatrix}, & \text{if } \delta=2,\end{cases}$$

where α is a fixed element of F_q such that $\alpha \notin \{x^2+x|x\in F_q\}$.

The orthogonal group of degree $2\nu+\delta$ over F_q with respect to $G_{2\nu+\delta,\,\Delta}$, denoted by $O_{2\nu+\delta,\,\Delta}(F_q)$, consists of all $(2\nu+\delta)\times(2\nu+\delta)$ matrices T over F_q satisfying $[TG_{2\nu+\delta,\,\Delta}T^T]\equiv[G_{2\nu+\delta,\,\Delta}]$.

There is an action of $O_{2\nu+\delta}(\mathbb{F}_q)$ on $\mathbb{F}_q^{(2\nu+\delta)}$ defined by

$$\begin{aligned}\mathbb{F}_q^{(2\nu+\delta)}\times O_{2\nu+\delta,\Delta}(\mathbb{F}_q) &\longrightarrow \mathbb{F}_q^{(2\nu+\delta)}\\ ((x_1,x_2,\ldots,x_{2\nu+\delta}),T) &\longmapsto (x_1,x_2,\ldots,x_{2\nu+\delta})T.\end{aligned}$$

The vector space $\mathbb{F}_q^{(2\nu+\delta)}$ together with the above group action of the orthogonal group $O_{2\nu+\delta}(\mathbb{F}_q)$, is called the $(2\nu+\delta)$-*dimensional orthogonal space over* $\mathbb{F}_q$ of characteristic 2.

Let P be an m-dimensional subspace of $\mathbb{F}_q^{(2\nu+\delta)}$. $PG_{2\nu+\delta}P'$ is *cogredient* to one of the following three forms

$$M(m,2r,r)=\begin{pmatrix}0 & I^{(r)} & \\ & 0 & \\ & & 0^{(m-2r)}\end{pmatrix},$$

$$M(m,2r+1,r)=\begin{pmatrix}0 & I^{(r)} & & \\ & 0 & & \\ & & 0^{(m-2r-1)} & \\ & & & 1\end{pmatrix},$$

and

$$M(m,2r+2,r)=\begin{pmatrix}0 & I^{(r)} & & & \\ & 0 & & & \\ & & \alpha & 1 & \\ & & & \alpha & \\ & & & & 0^{(m-2r-2)}\end{pmatrix}.$$

We say that P is a *subspace of type* $(m,2r+\gamma,r,\Gamma)$, where

$$\Gamma=\begin{cases}1 \text{ or } 0, & \text{if } e_{2\nu+1}\in P \text{ or not, respectively, in case } \delta=\gamma=1,\\ \emptyset & \text{and may be omitted, in all other cases,}\end{cases}$$

if $PG_{2\nu+\delta}P'$ is *cogredient* to $M(m,2r+\gamma,r)$. In particular, subspaces of type $(m,0,0)$ are called m-dimensional *totally singular subspaces.* The subspaces of type $(m,2r+1,r,1)$ exist if and only if $2r+1\le m\le \nu+r+1$. The subspace of type $(m,2r+1,r,1)$, which contains subspaces of type $(m_1,2r+1,r,1)$, exists if and only if $2r+1 \leq m_1 < m \leq \nu+r+1$. From Wan (2002), the number of subspaces of type $(m, 2r+1, r, 1)$ in $\mathbb{F}_q^{(2\nu+1)}$, denoted by $N(m, 2r+1, r, 1; 2\nu+1)$, is given by

$$N(m, 2r+1, r, 1; 2\nu+1) = q^{2r(\nu+r-m+1)} \frac{\prod_{i=\nu+r-m+2}^{\nu}(q^{2i}-1)}{\prod_{i=1}^{r}(q^{2i}-1)\prod_{i=1}^{m-2r-1}(q^i-1)}. \quad (1)$$

Let $N(m_1, 0, 0; m, 0, 0; 2\nu+\delta)$ denote the number of subspaces of type $(m_1, 0, 0)$ contained in a given subspace of type $(m, 0, 0)$. From Wan (2002),

$$N(m_1, 0, 0; m, 0, 0; 2\nu+\delta) = \frac{\prod_{i=m-m_1+1}^{m}(q^i-1)}{\prod_{i=1}^{m_1}(q^i-1)}. \quad (2)$$

Let $N(m_1, 2r+1, r, 1; m, 2r+1, r, 1; 2\nu+1)$ denote the number of subspaces of type $(m_1, 2r+1, r, 1)$ contained in a given subspace of type $(m, 2r+1, r, 1)$ in $\mathbb{F}_q^{(2\nu+1)}$. From Wan (2002),

$$\begin{aligned} &N(m_1, 2r+1, r, 1; m, 2r+1, r, 1; 2\nu+1) \\ &\quad = q^{2r(m-m_1)} \frac{\prod_{i=m-m_1+1}^{m-2r-1}(q^i-1)}{\prod_{i=1}^{m_1-2r-1}(q^i-1)}. \end{aligned} \quad (3)$$

Lemma 2.1 *Let $\mathbb{F}_q^{(2\nu+1)}$ denote the $2\nu+1$-dimensional orthogonal space over a finite field $\mathbb{F}_q$ of characteristic 2, with $2r+1 \leq m_0 \leq i \leq m \leq \nu+r+1$. Fix a subspace W_0 of type $(m_0, 2r+1, r, 1)$ in $\mathbb{F}_q^{(2\nu+1)}$, and a subspace W of type $(m_0, 2r+1, r, 1)$ in $\mathbb{F}_q^{(2\nu+1)}$ such that $W_0 \subset W$. Then the number of subspaces A of type $(i, 2r+1, r, 1)$ in $\mathbb{F}_q^{(2\nu+1)}$, where $W_0 \subset A \subset W$, is $N(i-m_0, 0, 0; m-m_0, 0, 0; 2(\nu+r+1-m_0))$.*

Proof Let $\sigma = \nu+r+1-m_0$. Since the orthogonal group $O_{2\nu+\delta}(\mathbb{F}_q)$ acts transitively on each set of subspaces of the same type, we may assume that W has the matrix representation of the form

$$W = \begin{array}{c} \begin{matrix} r & m_0-2r-1 & \sigma & r & m_0-2r-1 & \sigma & 1 \end{matrix} \\ \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & W_1 & 0 & 0 & W_2 & 0 \end{pmatrix} \end{array} \begin{matrix} r \\ r \\ 1 \\ m_0-2r-1 \\ m-m_0, \end{matrix}$$

where (W_1, W_2) is a subspace of type $(m-m_0, 0, 0)$ in $\mathbb{F}_q^{2(\nu+r+1-m_0)}$. By (2), the number of subspaces A of type $(i, 2r+1, r, 1)$, where $W_0 \subset A \subset W$, is $N(i-m_0, 0, 0; m-m_0, 0, 0; 2(\nu+r+1-m_0))$. □

3 Construction I

Definition 3.1 For $2r+1 \leq d_0 < d < k \leq \nu+r+1$, assume that P_0 is a fixed subspace of type $(d_0, 2r+1, r, 1)$ in $\mathbb{F}_q^{(2\nu+1)}$. Let M be a binary matrix whose columns (rows) are indexed by all subspaces of type $(k, 2r+1, r, 1)$ containing P_0 (all subspaces of type $(d, 2r+1, r, 1)$ containing P_0) in $\mathbb{F}_q^{(2\nu+1)}$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise. This matrix is denoted by $M_1(\nu, d, k)$.

Theorem 3.1 *Suppose* $2r+1 \le d_0 < d < k \le \nu + r + 1$ *and set* $b = \frac{q(q^{k-d_0-1}-1)}{q^{k-d}-1}$. *Then* $M_1(\nu, d, k)$ *is* s^e*-disjunct for* $1 \le s \le b$ *and*

$$
\begin{aligned}
e &= q^{k-d} N(d-d_0-1, 0, 0; k-d_0-1, 0, 0; 2(\nu+r+1-d_0)) \\
&\quad - (s-1) q^{k-d-1} N(d-d_0-1, 0, 0; k-d_0-2, 0, 0; 2(\nu+r+1-d_0)).
\end{aligned}
$$

Proof Let $C, C_1, \ldots, C_s$ be $s+1$ distinct columns of $M_1(\nu, d, k)$. To obtain the maximum numbers of subspaces of type $(d, 2r+1, r, 1)$ which contain P_0 in

$$
C \cap \bigcup_{i=1}^{s} C_i = \bigcup_{i=1}^{s} (C \cap C_i),
$$

we may assume that each $C \cap C_i (1 \le i \le s)$ is a subspace of type $(k-1, 2r+1, r, 1)$.

Then each $C \cap C_i$ covers $N(d-d_0, 0, 0; k-d_0-1, 0, 0; 2(\nu+r+1-d_0))$ subspaces of type $(d, 2r+1, r, 1)$ containing P_0 from Lemma 2.1. However, the coverage of each pair of C_i and C_j overlaps at a subspace of type $(k-2, 2r+1, r, 1)$ containing P_0, where $1 \le i, j \le s$. Therefore, from Lemma 2.1 only C_1 covers the full $N(d-d_0, 0, 0; k-d_0-1, 0, 0; 2(\nu+r+1-d_0))$ subspaces of type $(d, 2r+1, r, 1)$ containing P_0, while each of $C_2, \ldots, C_s$ can cover a maximum of $N(d-d_0, 0, 0; k-d_0-1, 0, 0; 2(\nu+r+1-d_0)) - N(d-d_0, 0, 0; k-d_0-2, 0, 0; 2(\nu+r+1-d_0))$ subspaces of type $(d, 2r+1, r, 1)$ not covered by C_1. By (2), the subspaces of type $(d, 2r+1, r, 1)$ of C not covered by $C_1, C_2, \ldots, C_s$ is at least

$$
\begin{aligned}
e &= N(d-d_0, 0, 0; k-d_0, 0, 0; 2(\nu+r+1-d_0)) \\
&\quad - N(d-d_0, 0, 0; k-d_0-1, 0, 0; 2(\nu+r+1-d_0)) \\
&\quad - (s-1)(N(d-d_0, 0, 0; k-d_0-1, 0, 0; 2(\nu+r+1-d_0)) \\
&\quad - N(d-d_0, 0, 0; k-d_0-2, 0, 0; 2(\nu+r+1-d_0))) \\
&= q^{k-d} N(d-d_0-1, 0, 0; k-d_0-1, 0, 0; 2(\nu+r+1-d_0)) \\
&\quad - (s-1) q^{k-d-1} N(d-d_0-1, 0, 0; k-d_0-2, 0, 0; 2(\nu+r+1-d_0)).
\end{aligned}
$$

Note that $\frac{N(d-d_0-1,0,0;k-d_0-1,0,0;2(\nu+r+1-d_0))}{N(d-d_0-1,0,0;k-d_0-2,0,0;2(\nu+r+1-d_0))} = \frac{q^{k-d_0-1}-1}{q^{k-d}-1}$, by (2). Since $e > 0$,

$$
s < \frac{q(q^{k-d_0-1}-1)}{q^{k-d}-1} + 1.
$$

Set

$$
b = \frac{q(q^{k-d_0-1}-1)}{q^{k-d}-1}.
$$

Then $1 \le s \le b$. □

For example, if $q = 2, d_0 = 3, d = 5, k = 6$, then $1 \le s \le b = \frac{2(2^2-1)}{2-1} = 6$. Setting $s = 3$, we have

$$\begin{aligned} e &= 2N(1,0,0;2,0,0;2(\nu+r-2)) - 2N(1,0,0;1,0,0;2(\nu+r-2)) \\ &= 2 \times \frac{2^2-1}{2-1} - 2 \\ &= 4. \end{aligned}$$

Therefore, $M_1(\nu, 5, 6)$ is 3^4-*disjunct*. □

Corollary 3.2 *Suppose that* $2r + 1 \le d_0 < d < k \le \nu + r + 1$ *and* $1 \le s \le \min\{b, q + 1\}$. *Then* $M_1(\nu, d, k)$ *is not* s^{e+1}-*disjunct, where* b *and* e *are as in Theorem* 3.1.

Proof Let C be a subspace of type $(k, 2r + 1, r, 1)$ containing P_0, and E be a fixed subspace of type $(k - 2, 2r + 1, r, 1)$ containing P_0 and contained in C. By Lemma 2.1, we obtain the number of subspaces of type $(k - 1, 2r + 1, r, 1)$ containing E and contained in C is

$$N(1, 0, 0; 2, 0, 0; 2(\nu + r - k + 3)) = q + 1.$$

For $1 \le s \le \min\{b, q + 1\}$, we choose s distinct subspaces of type $(k - 1, 2r + 1, r, 1)$ containing E and contained in C, and denote these subspaces by Q_i $(1 \le i \le s)$. For each Q_i, we choose a subspace C_i of type $(k, 2r + 1, r, 1)$ such that $C \cap C_i = Q_i$, where $1 \le i \le s$. Hence each pair of C_i and C_j overlaps at the same subspace E of type $(k - 2, 2r + 1, r, 1)$, where $1 \le i, j \le s$. By Theorem 3.1, it follows that the corollary hold. □

Corollary 3.3 *Suppose that* $d = d_0 + 1$ *and* $1 \le s \le q$. *Then* $M_1(\nu, d, k)$ *is* s^e-*disjunct, but it is not* s^{e+1}-*disjunct, where* $e = q^{k-d_0-2}(q - s + 1)$.

Proof Setting $d = d_0 + 1$ in the e formula of Theorem 3.1, we obtain

$$e = q^{k-d_0-2}(q - s + 1).$$

The second statement follows directly from Corollary 3.2. □

4 Construction II

Definition 4.1 For $3 \le 2r + 1 \le d < k \le \nu + r + 1$, let M be a binary matrix whose columns (rows) are indexed by all subspaces of type $(k, 2r + 1, r, 1)$ (all subspaces of type $(d, 2r + 1, r, 1)$) in $\mathbb{F}_q^{(2\nu+1)}$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise. This matrix is denoted by $M_2(\nu, d, k)$.

Theorem 4.1 *Suppose* $3 \le 2r + 1 \le d - 1 < k - 2 \le \nu + r - 1$. *If* $1 \le s \le q^{2r}$, *then* $M_2(\nu, d, k)$ *is* s^e-*disjunct, where* $e = q^{(k-d-1)(d-1)+2r}$.

Proof Let $C, C_1, \ldots, C_s$ be $s+1$ distinct columns of $M_2(\nu, d, k)$. To obtain the maximum number of subspaces of type $(d, 2r+1, r, 1)$ in

$$C \cap \bigcup_{i=1}^{s} C_i = \bigcup_{i=1}^{s} (C \cap C_i),$$

we may assume that each $C \cap C_i$ is a subspace of type $(k-1, 2r+1, r, 1)$, where $1 \leq i \leq s$. By (3), the number of the subspaces of type $(d, 2r+1, r, 1)$ of C not covered by $C_1, C_2, \ldots, C_s$ is at least

$$\begin{aligned}
&N(d, 2r+1, r, 1; k, 2r+1, r, 1; 2\nu+1) \\
&\quad - sN(d, 2r+1, r, 1; k-1, 2r+1, r, 1; 2\nu+1) \\
&= q^{2r(k-d-1)} \frac{\prod_{i=k-d+1}^{k-2r-2}(q^i-1)}{\prod_{i=1}^{d-2r-1}(q^i-1)} (q^{k-1} - q^{2r} - s(q^{k-d}-1)).
\end{aligned}$$

Since $2r+1 \leq d-1 < k-2$, we obtain

$$\begin{aligned}
&\frac{\prod_{i=k-d+1}^{k-2r-2}(q^i-1)}{\prod_{i=1}^{d-2r-1}(q^i-1)} \\
&\quad = \frac{\prod_{i=0}^{d-2r-3}(q^{i+k-d+1}-1)}{\prod_{i=0}^{d-2r-3}(q^{i+1}-1)} \frac{1}{q^{d-2r-1}-1} \\
&\quad = \prod_{i=0}^{d-2r-3} \frac{q^{i+k-d+1}-1}{q^{i+1}-1} \frac{1}{q^{d-2r-1}-1} \\
&\quad = \prod_{i=0}^{d-2r-3} q^{k-d} \frac{q^{i+1} - \frac{1}{q^{k-d}}}{q^{i+1}-1} \frac{1}{q^{d-2r-1}-1} \\
&\quad > q^{(d-2r-2)(k-d)-(d-2r-1)}.
\end{aligned}$$

Since $1 \leq s \leq q^{2r}$, and $2r+1 \leq d-1$, we obtain

$$\begin{aligned}
q^{k-1} - q^{2r} - s(q^{k-d}-1) &\geq q^{k-1} - q^{2r} - q^{2r}(q^{k-d}-1) \\
&= q^{k-d+2r}(q^{d-2r-1}-1) \\
&\geq q^{k-d+2r}.
\end{aligned}$$

Hence, $e = q^{(k-d-1)(d-1)+2r}$. □

For example, if $q=2, r=1, d=5, k=7$, then $1 \leq s \leq 2^2 = 4$. Setting $s=4$, we have $M_2(\nu, 5, 7)$ is 4^{64}-*disjunct*.

Theorem 4.2 *Suppose* $3 \leq 2r+1 \leq d-1 < \nu + r$. *Let* $p = \frac{q^d - q^{2r}}{q-1} - 1$. *If* $1 \leq s \leq p$, *then* $M_2(\nu, d, d+1)$ *is fully* s^e-*disjunct, where* $e = p - s$.

Proof By (3), we have $N(d,2r+1,r,1;d+1,2r+1,r,1;2\nu+1)=p+1$. It follows that we can pick $s+1$ distinct subspaces $C,C_1,\dots,C_s$ of type $(d+1,2r+1,r,1)$ such that $C\cap C_i$ and $C\cap C_j$ are two distinct subspaces of type $(d,2r+1,r,1)$, where $1\le i,j\le s$. By the principle of inclusion and exclusion, the number of subspaces of type $(d,2r+1,r,1)$ in C but not in each C_i is $p-s+1$, where $1\le i\le s$. It follows that $e\le p-s$.

On the other hand, similar to the proof of Theorem 4.1 we obtain

$$e\ge N(d,2r+1,r,1;d+1,2r+1,r,1;2\nu+1)-s-1=p-s.$$

Hence, $e=p-s$. □

The following theorem tells us how to choose k so that the test to item ratio is minimized.

Theorem 4.3 *For* $2r+1\le m\le \nu+r+1$, *the sequence* $N(m,2r+1,r,1;2\nu+1)$ *is unimodal and gets its peak at* $m=\lfloor\frac{2\nu+2r}{3}\rfloor+1$, *or* $m=\lfloor\frac{2\nu+2r}{3}\rfloor+2$.

Proof For $2r+1\le m_1<m_2\le \nu+r+1$, by (1), we have

$$\begin{aligned}
&\frac{N(m_2,2r+1,r,1;2\nu+1)}{N(m_1,2r+1,r,1;2\nu+1)}\\
&\quad=\frac{q^{2r(\nu+r-m_2+1)}}{q^{2r(\nu+r-m_1+1)}}\frac{\prod_{i=\nu+r-m_2+2}^{\nu}(q^{2i}-1)}{\prod_{i=1}^{r}(q^{2i}-1)\prod_{i=1}^{m_2-2r-1}(q^i-1)}\\
&\qquad\Big/\frac{\prod_{i=\nu+r-m_1+2}^{\nu}(q^{2i}-1)}{\prod_{i=1}^{r}(q^{2i}-1)\prod_{i=1}^{m_1-2r-1}(q^i-1)}\\
&\quad=\frac{1}{q^{2r(m_2-m_1)}}\frac{\prod_{i=\nu+r-m_2+2}^{\nu+r-m_1+1}(q^{2i}-1)}{\prod_{i=m_1-2r}^{m_2-2r-1}(q^i-1)}\\
&\quad=\prod_{i=0}^{m_2-m_1-1}\frac{q^{2(\nu+r-m_2+2+i)}-1}{q^{m_1+i}-q^{2r}}.
\end{aligned}\tag{4}$$

If $\lfloor\frac{2\nu+2r}{3}\rfloor+2\le m_1<m_2\le \nu+r+1$, then $\frac{2\nu+2r}{3}+1\le m_1$. It implies that

$$2m_1+m_2>3m_1\ge 2\nu+2r+3.\tag{5}$$

Since $i\le m_2-m_1-1$, by (5) we have

$$m_1+2m_2>2\nu+2r+4+(m_2-m_1-1)\ge 2\nu+2r+4+i.$$

So

$$m_1+i>2(\nu+r-m_2+2+i).$$

It follows that

$$m_1 + i - 2r - 1 \geq 2(v + r - m_2 + 2 + i) - 2r.$$

So

$$q^{2(v+r-m_2+2+i)-2r} \leq q^{m_1+i-2r-1}.$$

Thus

$$\begin{aligned} q^{2(v+r-m_2+2+i)-2r} - \frac{1}{q^{2r}} &< q^{m_1+i-2r-1} + [(q-1)q^{m_1+i-2r-1} - 1] \\ &= q^{m_1+i-2r} - 1. \end{aligned}$$

It follows that

$$\frac{q^{2(v+r-m_2+2+i)-2r} - \frac{1}{q^{2r}}}{q^{m_1+i-2r} - 1} < 1.$$

So

$$\frac{q^{2(v+r-m_2+2+i)} - 1}{q^{m_1+i} - q^{2r}} < 1.$$

From (4) we have

$$N(m_2, 2r+1, r, 1; 2v+1) < N(m_1, 2r+1, r, 1; 2v+1).$$

If $2r + 1 \leq m_1 < m_2 \leq \lfloor \frac{2v+2r}{3} \rfloor + 1$, then $m_2 \leq \frac{2v+2r}{3} + 1$. Thus

$$m_1 + 2m_2 < 3m_2 \leq 2v + 2r + 3 < 2v + 2r + 4 + i.$$

It follows that

$$m_1 + i < 2v + 2r - 2m_2 + 4 + 2i = 2(v + r - m_2 + 2 + i).$$

Thus

$$q^{m_1+i} - q^{2r} < q^{2(v+r-m_2+2+i)} - q^{2r} < q^{2(v+r-m_2+2+i)} - 1.$$

It follows that

$$\frac{q^{2(v+r-m_2+2+i)} - 1}{q^{m_1+i} - q^{2r}} > 1.$$

From (4), we have

$$N(m_2, 2r+1, r, 1; 2v+1) > N(m_1, 2r+1, r, 1; 2v+1). \qquad \square$$

Theorem 4.4 *If $d = 2r + 1$ and $k = 2r + 2$, then the test efficiency of construction II is smaller than that of* D'yachkov et al. (2005).

Proof If $d = 2r + 1$ and $k = 2r + 2$, then the disjunct matrix of construction II is $M_2(\nu, 2r+1, 2r+2)$, and the disjunct matrix of D'yachkov et al. (2005) is $M(n, 2r+2, 2r+1)$. Let $\frac{t}{n}$ be the test efficiency of $M_2(\nu, 2r+1, 2r+2)$ and let $\frac{t_1}{n_1}$ be the test efficiency of $M(n, 2r+2, 2r+1)$, respectively.

Then

$$\begin{aligned}\frac{t}{n} &= \frac{N(d, 2r+1, r, 1; 2\nu+1)}{N(k, 2r+1, r, 1; 2\nu+1)} \\ &= \frac{N(2r+1, 2r+1, r, 1; 2\nu+1)}{N(2r+2, 2r+1, r, 1; 2\nu+1)} \\ &= q^{2r(\nu+r-2r-1+1)} \frac{\prod_{i=\nu+r-2r-1+2}^{\nu}(q^{2i}-1)}{\prod_{i=1}^{r}(q^{2i}-1)\prod_{i=1}^{2r+1-2r-1}(q^i-1)} \\ &\quad \times \frac{\prod_{i=1}^{r}(q^{2i}-1)\prod_{i=1}^{2r+2-2r-1}(q^i-1)}{q^{2r(\nu+r-2r-2+1)}\prod_{i=\nu+r-2r-2+2}^{\nu}(q^{2i}-1)} \\ &= \frac{q^{2r+1}-q^{2r}}{q^{2\nu-2r}-1},\end{aligned}$$

and

$$\frac{t_1}{n_1} = \frac{\begin{bmatrix} 2\nu+1 \\ d \end{bmatrix}_q}{\begin{bmatrix} 2\nu+1 \\ k \end{bmatrix}_q} = \frac{\prod_{i=d+1}^{k}(q^i-1)}{\prod_{i=2\nu+1-k+1}^{2\nu+1-d}(q^i-1)} = \frac{q^{2r+2}-1}{q^{2\nu-2r}-1}.$$

Therefore, $\frac{t}{n} < \frac{t_1}{n_1}$. □

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