J Comb Optim (2009) 17: 339–345
DOI 10.1007/s10878-007-9115-z

Two new error-correcting pooling designs from d-bounded distance-regular graphs

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Published online: 14 November 2007
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Abstract Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$. In this paper, we construct two new classes of error-correcting pooling designs from the posets consisting of the subspaces of Γ.

Keywords Pooling designs · s^e-disjunct matrix · Distance-regular graph · Strongly closed subgraphs

1 Introduction

The basic problem of group testing is to identify the set of defective items in a large population of items. Suppose we have n items to be tested and that there are at most r defective items among them. Each *test* (or *pool*) is (or contains) a subset of items. We assume some testing mechanism exists which if applied to an arbitrary subset of the population gives a *negative outcome* if the subset contains no positive and *positive outcome* otherwise. Objectives of group testing vary from minimizing the number of tests, limiting number of pools, limiting pool sizes to tolerating a few errors. It is conceivable that these objectives are often contradicting, thus testing strategies are application dependent. A group testing algorithm is *non-adaptive* if all tests must be specified without knowing the outcomes of other tests. A non-adaptive testing algorithm is useful in many areas such as DNA library screening.

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A group testing algorithm is *error tolerant* if it can detect some errors in test outcomes. A mathematical model of error-tolerance designs is an s^e-*disjunct* matrix.

A binary matrix M is said to be s^e-*disjunct* if given any $s+1$ columns of M with one designated, there are $e+1$ rows with a 1 in the designated column and 0 in each of the other s columns. An s^0-disjunct matrix is said to be s-*disjunct*. D'yachkov et al. (2007) proposed the concept of fully s^e-disjunct matrices. An s^e-disjunct matrix is *fully* s^e-*disjunct* if it is not c^b-disjunct whenever $c>s$ or $b>e$.

The constructions of s^e-disjunct matrices were given by many authors (see D'yachkov et al. 2005, 2007; Huang and Weng 2004; Macula 1996, 1997; Ngo and Du 2002). Let Γ be a d-bounded distance-regular graph with diameter $d\geq 2$. In this paper, we construct two new classes of error-correcting pooling designs from the posets consisting of the subspaces of Γ.

2 Distance-regular graphs

In this section we shall first introduce the concepts of distance-regular graphs, and then introduce our main results.

Let $\Gamma=(X,R)$ denote a finite undirected graph without loops or multiple edges, with vertex set X and edge set R. Suppose that Γ is a connected regular graph. For vertices u and v in X, let $\partial(u,v)$ denote the *distance* between u and v. The maximum value of the distance function in Γ is called the *diameter* of Γ, denoted by $d=d(\Gamma)$. For all $u\in X$ and for all integers i $(0\leq i\leq d)$, set

$$\Gamma_i(u):=\{v\mid v\in X,\partial(u,v)=i\}.$$

Γ is said to be *distance-regular* whenever for all integers h,i,j $(0\leq h,i,j\leq d)$ and for all $u,v\in X$ with $\partial(u,v)=h$, the number

$$p_{ij}^h:=|\Gamma_i(u)\cap\Gamma_j(v)| \tag{1}$$

is independent of u,v. The constants p_{ij}^h $(0\leq h,i,j\leq d)$ are known as the *intersection numbers* of Γ. For convenience, set $c_i:=p_{i-1,1}^i$ $(1\leq i\leq d)$, $a_i:=p_{i1}^i$ $(0\leq i\leq d)$, $b_i:=p_{i+1,1}^i$ $(0\leq i\leq d-1)$, $k_i:=p_{ii}^0$ $(0\leq i\leq d)$, and put $c_0:=0$, $b_d:=0$, $k:=k_1$. Note that $c_1=1$, $a_0=0$, and

$$k=c_i+a_i+b_1\quad(0\leq i\leq d), \tag{2}$$

$$k_i=\frac{b_0b_1\cdots b_{i-1}}{c_1c_2\cdots c_i}\quad(0\leq i\leq d), \tag{3}$$

$$|X|=1+k_1+\cdots+k_d. \tag{4}$$

From now on, we assume that Γ is distance-regular with diameter d. The reader is referred to (Brouwer et al. 1989) for general theory of distance-regular graphs.

Let Δ be a subset of X. Recall that a subgraph induced on Δ of Γ is said to be *strongly closed* if $C(u,v)\cup A(u,v)\subseteq\Delta$ for every pair of vertices $u,v\in\Delta$. Suzuki

(1995) determined all the types of strongly closed subgraphs of a distance-regular graph.

A distance-regular graph Γ with diameter d is said to be *d-bounded*, if every strongly closed subgraph of Γ is regular, and any two vertices x and y are contained in a common strongly closed subgraph with diameter $\partial(x, y)$. For instance, the ordinary 5-gon is a 2-bounded distance-regular graph. But the ordinary 6-gon is not a 3-bounded distance-regular graph. Indeed, let $1 \sim 2 \sim 3 \sim 4 \sim 5 \sim 6 \sim 1$ be the ordinary 6-gon. Then it is clear that $1 \sim 2 \sim 3$ is strongly closed, but it is not regular. By (Weng 1997, Theorem 4.3) and (Weng 1999, Theorem 5.7), the following graphs are all *d-bounded* distance- regular graphs: (1) Hamming graph $H(d, q)$ $(d \geq 3,\ q \geq 3)$ with classical parameters $(d, b, \alpha, \beta) = (d, 1, 0, q - 1)$ and $H(d, 2)$ $(d \geq 3)$, i.e., d-cube; (2) Hermitian forms graphs $Her_{-b}(d)$ of diameter $d \geq 3$, where $b = -r$, r is a prime power and intersection numbers $c_2 > 1$, $a_1 \neq 0$; (3) Dual polar graph ${}^2A_{2d-1}(-b)$ with diameter $d \geq 3$, where $b = -r$, r is a prime power and intersection numbers $c_2 > 1$, $a_1 \neq 0$.

Weng (1997, 1998, 1999) used the term *weak-geodetically closed subgraphs* for strongly closed subgraphs, obtained many important properties when a distance-regular graph is d-bounded. A regular strongly closed subgraph of Γ is said to be a *subspace* of Γ.

Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$. Let $P(i)$ be a set of all subspaces with diameter i in Γ, where $0 \leq i \leq d$. Pick $x \in X$, let $P(x)$ be the set of all subspaces containing x in Γ and let

$$P(x, i) = \{\Delta \in P(x) \mid d(\Delta) = i\}.$$

Definition 2.1 Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$. Given integers $0 \leq m < k \leq d - 1$. Let $M(m, k; d)$ be the binary matrix whose rows (resp. columns) are indexed by $P(m)$ (resp. $P(k)$). We also order elements of these sets lexicographically. $M(m, k; d)$ has a 1 in row i and column j if and only if the i^{th} subspace of $P(m)$ is a subspace of the j^{th} subspace of $P(k)$.

Definition 2.2 Let Γ be a d-bounded distance-regular graph with diameter $d \geq 3$. Given integers $1 \leq m < k \leq d - 1$. Let $M_x(m, k; d)$ be the binary matrix whose rows (resp. columns) are indexed by $P(x, m)$ (resp. $P(x, k)$). We also order elements of these sets lexicographically. $M_x(m, k; d)$ has a 1 in row i and column j if and only if the i^{th} subspace of $P(x, m)$ is a subspace of the j^{th} subspace of $P(x, k)$.

Example 2.3 The 3-cube, with vertex set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and edge set $\{\{1, 2\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 6\}, \{3, 4\}, \{3, 7\}, \{4, 8\}, \{5, 6\}, \{5, 8\}, \{6, 7\}, \{7, 8\}\}$, is a 3-bounded distance-regular graph. Let $M(1, 2; 3)$ be the binary matrix whose rows (resp. columns) are indexed by $P(1)$ (resp. $P(2)$), where $P(1)$ (resp. $P(2)$) is a set of all subspaces with diameter 1 (resp. 2), i.e., $P(1)$ (resp. $P(2)$) is a set of all edges (resp. all 4-gons). Then

$$M(1,2;3) = \begin{pmatrix} 1&1&0&0&0&0\\ 1&0&1&0&0&0\\ 0&1&1&0&0&0\\ 1&0&0&1&0&0\\ 0&1&0&1&0&0\\ 1&0&0&0&1&0\\ 0&0&0&1&1&0\\ 0&0&1&0&1&0\\ 0&1&0&0&0&1\\ 0&1&1&0&0&0\\ 0&0&0&1&0&1\\ 0&0&0&0&1&1 \end{pmatrix}.$$

Now we state our main results.

Theorem 2.4 *Let $M(m,k;d)$ be as in Definition* 2.1. *If $1 \le s \le \lfloor \frac{N'(m,k)}{N'(m,k-1)} \rfloor - 1$, then $M(m,k;d)$ is s^e-disjunct, where $e = N'(m,k) - sN'(m,k-1) - 1$ and $N'(m,k)$, $N'(m,k-1)$ are given by Lemma* 3.4.

Theorem 2.5 *Let $M(k-1,k;d)$ be as in Definition* 2.1. *If $1 \le s \le N'(k-1,k) - 1$, then $M(k-1,k;d)$ is fully s^e-disjunct, where $e = N'(k-1,k) - s - 1$ and $N'(k-1,k)$ is given by Lemma* 3.4.

Theorem 2.6 *Let $M_x(m,k;d)$ be as in Definition* 2.2. *If $1 \le s \le \lfloor \frac{N(0,m;k)}{N(0,m;k-1)} \rfloor - 1$, then $M_x(m,k;d)$ is s^e-disjunct, where $e = N(0,m;k) - sN(0,m;k-1) - 1$ and $N(0,m;k)$, $N(0,m;k-1)$ are given by Proposition* 3.3. *In particular, $M_x(k-1,k;d)$ is fully ${s_1}^{e_1}$-disjunct, where $1 \le s_1 \le N(0,k-1;k) - 1$, $e_1 = N(0,k-1;k) - s_1 - 1$ and $N(0,k-1;k)$ is given by Proposition* 3.3.

Remark. By Theorem 2.5, $M(1,2;3)$ of Example 2.3 is a fully s^{3-s}-disjunct matrix, where $1 \le s \le 3$.

3 Proof of main results

Proposition 3.1 (Weng 1999, Lemmas 4.2, 4.5) *Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a d-bounded distance-regular graph with diameter d. Then the following* (i)–(ii) *hold.*

- (i) *Let Δ be a subspace of Γ and $0 \le i \le d(\Delta)$. Then Δ is distance-regular with intersection numbers $c_i(\Delta) = c_i$, $a_i(\Delta) = a_i$, $b_i(\Delta) = b_i - b_{d(\Delta)}$.*
- (ii) *For any $x, y \in V(\Gamma)$, the subspace of diameter $\partial(x,y)$ containing x, y is unique.*

Proposition 3.2 (Weng 1997, Lemma 2.6) *Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a d-bounded distance-regular graph with diameter d. Then we have $b_i > b_{i+1}$, $0 \le i \le d-1$.*

Proposition 3.3 (Gao et al. 2007, Lemma 2.1) *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$. Suppose Δ and Δ' are strongly closed subgraphs with diameter i and $i+s+t$, respectively, and with $\Delta \subseteq \Delta'$. Then the number of the strongly closed subgraphs $\widetilde{\Delta}$ with diameter $i+s$ satisfying $\Delta \subseteq \widetilde{\Delta} \subseteq \Delta'$, denoted by $N(i, i+s; i+s+t)$, is determined by i, s and t, independently of the choice of Δ and Δ', it is*

$$\frac{(b_i - b_{i+s+t})(b_{i+1} - b_{i+s+t}) \cdots (b_{i+s-1} - b_{i+s+t})}{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})},$$

where $0 \leq i,\ s,\ t \leq d$ and $i+1 \leq i+s \leq i+s+t \leq d$.

Lemma 3.4 *Let Γ be a d-bounded distance-regular graph with diameter d, and Δ is a fixed subspace with diameter $i+s$ in the Γ. Then the number of the subspaces with diameter i in the Δ, denoted by $N'(i, i+s)$, is determined by i and s, independent of the choice of Δ and is given by*

$$\frac{(b_0 - b_{i+s})(b_1 - b_{i+s}) \cdots (b_{i-1} - b_{i+s})(1 + \sum_{l=1}^{i+s} \frac{(b_0 - b_{i+s})(b_1 - b_{i+s}) \cdots (b_{l-1} - b_{i+s})}{c_1 c_2 \cdots c_l})}{(b_0 - b_i)(b_1 - b_i) \cdots (b_{i-1} - b_i)(1 + \sum_{l=1}^{i} \frac{(b_0 - b_i)(b_1 - b_i) \cdots (b_{l-1} - b_i)}{c_1 c_2 \cdots c_l})},$$

where $0 \leq i \leq i+s \leq d$.

Proof For each $x \in V(\Delta)$, by Proposition 3.3, there are $N(0, i; i+s)$ subspaces with diameter i in Δ. Thus there are total $|V(\Delta)| N(0, i; i+s)$ such subspaces. But each of these subspaces repeats α times, where α equals the number of vertices in a subspace with diameter i. So the number of the subspaces with diameter i in Δ is $|V(\Delta)| N(0, i; i+s)/\alpha$. By Proposition 3.1 and (4),

$$|V(\Delta)| = 1 + \sum_{l=1}^{i+s} \frac{(b_0 - b_{i+s})(b_1 - b_{i+s}) \cdots (b_{l-1} - b_{i+s})}{c_1 c_2 \cdots c_l},$$

$$\alpha = 1 + \sum_{l=1}^{i} \frac{(b_0 - b_i)(b_1 - b_i) \cdots (b_{l-1} - b_i)}{c_1 c_2 \cdots c_l}.$$

So we have the desired result. □

Proof of Theorem 2.4. Let $\Delta, \Delta_1, \Delta_2, \ldots, \Delta_s$ be $s+1$ distinct columns of $M(m, k; d)$. To obtain the maximum number of subspaces of $P(m)$ in

$$\Delta \cap \bigcup_{i=1}^{s} \Delta_i = \bigcup_{i=1}^{s} (\Delta \cap \Delta_i),$$

we may assume that each $\Delta \cap \Delta_i$ is a subspace with diameter $k-1$. By Lemma 3.4, the number of the subspaces of Δ not covered by $\Delta_1, \Delta_2, \ldots, \Delta_s$ is at least

$$N'(m,k) - sN'(m,k-1)$$

Hence $e = N'(m,k-1) - sN'(m,k-1) - 1$. □

Proof of Theorem 2.5. Fix $\Delta \in P(k)$. By Lemma 3.4 and Proposition 3.2, we have the number of all subspaces with diameter $k-1$ in Δ is $N'(k-1,k)$. Suppose that there are s distinct subspaces with diameter $k-1$ in Δ, say $\Delta'_1, \Delta'_2, \ldots, \Delta'_s$. By Proposition 3.2 and Proposition 3.3, for any subspace Δ'_j, $1 \le j \le s$, we know that there exist two distinct subspaces Δ, $\Delta_j \in P(k)$ such that $\Delta'_j \subseteq \Delta$, Δ_j. By Proposition 3.1 (ii), $\Delta \cap \Delta_j = \Delta'_j$. Hence, we can pick $s+1$ distinct subspaces $\Delta, \Delta_1, \Delta_2, \ldots, \Delta_s$ of $P(k)$ such that $\Delta'_i = \Delta \cap \Delta_i$ and $\Delta'_j = \Delta \cap \Delta_j$ are two distinct subspaces of $P(k-1)$, where $1 \le i \ne j \le s$. Thus, the number of subspaces of $P(k-1)$ contained in Δ but not contained in each Δ_i is $N'(k-1,k) - s$. It follows that $e \le N'(k-1,k) - s - 1$.

On the other hand, similar to the proof Theorem 2.4 we obtain $e \ge N'(k-1,k) - s - 1$. Hence $e = N'(k-1,k) - s - 1$. It is easy to show that $M(k-1,k;d)$ is fully s^e-disjunct, where $e = N'(k-1,k) - s - 1$. □

Proof of Theorem 2.6. Similar to the proofs of Theorem 2.4 and Theorem 2.5. □

4 Examples

A distance-regular graph Γ is said to have *classical parameters* (d, b, α, β) whenever the diameter of Γ is d, and the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right), \quad 0 \le i \le d,$$

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right), \quad 0 \le i \le d,$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{i-1}.$$

Tsai (2003) gave many examples of d-bounded distance-regular graphs. We only consider Hamming graph $H(d,q)$. Change Hamming graph $H(d,q)$ $(d \ge 3,\ q \ge 3)$ for Γ in Theorems 2.4–2.6. Then the following Corollaries 4.1–4.3 hold.

Corollary 4.1 *If* $1 \le s \le \lfloor \frac{qk}{k-m} \rfloor - 1$, *then* $M(m,k;d)$ *is* s^e*-disjunct, where* $e = \binom{k}{m} q^{k-m} - s\binom{k-1}{m} q^{k-m-1} - 1$.

Corollary 4.2 *If* $1 \le s \le qk - 1$, *then* $M(k-1,k;d)$ *is fully* s^e*-disjunct, where* $e = qk - s - 1$.

Corollary 4.3 *If* $1 \le s \le \lfloor \frac{k}{k-m} \rfloor - 1$, *then* $M_x(m,k;d)$ *is* s^e*-disjunct, where* $e = \binom{k}{m} - s\binom{k-1}{m} - 1$. *In particular,* $M_x(k-1,k;d)$ *is fully* ${s_1}^{e_1}$*-disjunct, where* $1 \le s_1 \le k-1$, $e_1 = k - s_1 - 1$.

Remark If we take $s = qk - 1$ in Corollary 4.2, then $M(k-1, k; d)$ is fully $(sq-1)$-disjunct and its error correcting capability is better than that of Theorems 6 and 7 in (Ngo and Du 2002). If we take $s = k - 1$ in Corollary 4.3, then $M_x(k-1, k; d)$ is fully $(k-1)$-disjunct; this is the $M(k-1, k, d)$ of Proposition 1 in (Macula 1996).

Acknowledgements This paper is supported by Natural Science Foundation of Hebei Province, China (No. A2005000141), and the "LSAZ200702 Program" of Langfang Teachers' College.

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