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A sharp upper bound on the largest Laplacian eigenvalue of weighted graphs[☆]

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Abstract

We consider weighted graphs, where the edge weights are positive definite matrices. The Laplacian of the graph is defined in the usual way. We obtain an upper bound on the largest eigenvalue of the Laplacian and characterize graphs for which the bound is attained. The classical bound of Anderson and Morley, for the largest eigenvalue of the Laplacian of an unweighted graph follows as a special case.

AMS classification: 05C50

Keywords: Weighted graph; Laplacian matrix; Largest eigenvalue; Upper bound

1. Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. A simple graph has no loops or multiple edges and therefore its

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edge set consists of distinct pairs. A weighted graph is a graph in which each edge is assigned a weight, which is usually a positive number. An unweighted graph, or simply a graph, is thus a weighted graph with each of the edges bearing weight 1.

In this paper we consider weighted graphs, where the edge weights are positive definite matrices. All weight matrices will be assumed to have the same size.

We now introduce some notation. Let G be a weighted graph on n vertices. Denote by w_{ij} the positive definite weight matrix of order p of the edge ij. We write $i \sim j$ if vertices i and j are adjacent. Let $w_i = \sum_{j:j\sim i} w_{ij}$, and we think of w_i as the weight matrix of the vertex i.

The Laplacian matrix of a graph G is denoted by $L(G)$ and is defined as $L(G) = (l_{ij})$, where

$$l_{ij} = \begin{cases} w_i & \text{if } i = j, \\ -w_{ij} & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, using the notation introduced earlier, $L(G)$ is a square matrix of order np. For any symmetric matrix A, let $\lambda_1(A)$ denote the largest eigenvalue of A. We set $\lambda_1 = \lambda_1(L(G))$.

Upper and lower bounds for the largest Laplacian eigenvalue for unweighted graphs have been investigated to a great extent in the literature [1–10,12]. For most of these bounds, Pan [11] has characterized the graphs which achieve the upper bound of the largest Laplacian eigenvalue. The main result of this paper, contained in Section 2, gives an upper bound on the largest Laplacian eigenvalue for weighted graphs, where the edge weights are positive definite matrices. We also characterize graphs for which equality holds in the upper bound. The results clearly generalize the known results for unweighted graphs. Some related results are proved in Section 3.

Let $G = (V, E)$. If V is the disjoint union of two nonempty sets V_1 and V_2 such that every vertex i in V_1 has the same largest eigenvalue $\lambda_1(w_i)$ and every vertex j in V_2 has the same largest eigenvalue $\lambda_1(w_j)$, then G will be called a semiregular graph. (Occasionally we might say explicitly that G is a $(\lambda_1(w_i), \lambda_1(w_j))$-semiregular graph.)

2. Main result

In this section we find an upper bound on the largest Laplacian eigenvalue and characterize the graphs for which the largest Laplacian eigenvalue is equal to the upper bound. For this we need the following Lemmas.

Lemma 2.1 (Rayleigh-Ritz [13]). *If A is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ then for any $\bar{x} \in R^n$ $(\bar{x} \neq \bar{0})$,*

$$\bar{x}^{\mathrm{T}} A \bar{x} \geqslant \lambda_n \bar{x}^{\mathrm{T}} \bar{x}. \tag{1}$$

Equality holds if and only if $\bar{x}$ is an eigenvector of A corresponding to the least eigenvalue λ_n.

The following is a consequence of the Cauchy–Schwarz inequality.

Lemma 2.2. *If A is a symmetric positive definite $n \times n$ matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, then for any $\bar{x} \in R^n$ $(\bar{x} \neq \bar{0})$, $\bar{y} \in R^n$ $(\bar{y} \neq \bar{0})$*

$$|\bar{x}^{\mathrm{T}} A \bar{y}| \leqslant \lambda_1 \sqrt{\bar{x}^{\mathrm{T}}\bar{x}}\sqrt{\bar{y}^{\mathrm{T}}\bar{y}}. \tag{2}$$

Equality holds if and only if $\bar{x}$ is an eigenvector of A corresponding to the largest eigenvalue λ_1 and $\bar{y} = \alpha\bar{x}$ for some $\alpha \in R$.

Theorem 2.3. *Let G be a simple connected weighted graph. Then*

$$\lambda_1 \leqslant \max_{i \sim j} \left\{ \lambda_1 \left(\sum_{k:k\sim i} w_{ik} \right) + \sum_{k:k\sim j} \lambda_1(w_{jk}) \right\}, \tag{3}$$

where w_{ij} is the positive definite weight matrix of order p of the edge ij. Moreover equality holds in (3) if and only if

(i) *G is a bipartite semiregular graph*;
(ii) *w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j.*

Proof. Let $M(G)$ be the block diagonal matrix diag $(\gamma_1 I_{p,p}, \gamma_2 I_{p,p}, \ldots, \gamma_n I_{p,p})$ where $\gamma_i = \sum_{k:k\sim i} \lambda_1(w_{ik})$, $i = 1, 2, \ldots, n$.

Let $\bar{\mathbf{X}} = (\bar{x}_1^{\mathrm{T}}, \bar{x}_2^{\mathrm{T}}, \ldots, \bar{x}_n^{\mathrm{T}})^{\mathrm{T}}$ be an eigenvector corresponding to the largest eigenvalue λ_1 of $M(G)^{-1}L(G)M(G)$. We assume that $\bar{x}_i$ is the vector component of $\bar{\mathbf{X}}$ such that $\bar{x}_i^{\mathrm{T}}\bar{x}_i = \max_{j\in V}\{\bar{x}_j^{\mathrm{T}}\bar{x}_j\}$. Since $\bar{\mathbf{X}}$ is nonzero, so is $\bar{x}_i$.

The (i, j)th block of $M(G)^{-1}L(G)M(G)$ is

$$\begin{cases} w_i & \text{if } i = j, \\ -\frac{\gamma_j}{\gamma_i} w_{ij} & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\{M(G)^{-1}L(G)M(G)\}\bar{\mathbf{X}} = \lambda_1\bar{\mathbf{X}}. \tag{4}$$

From the ith equation of (4), we have

$$\lambda_1\bar{x}_i = w_i\bar{x}_i - \sum_{j:j\sim i} \frac{\gamma_j w_{ij}}{\gamma_i}\bar{x}_j, \tag{5}$$

i.e.,

$$(\lambda_1 I_{p,p} - w_i)\bar{x}_i = -\sum_{j:j\sim i} \frac{\gamma_j w_{ij}}{\gamma_i}\bar{x}_j,$$

i.e.,

$$\begin{aligned}\bar{x}_i^{\mathrm{T}}(\lambda_1 I_{p,p} - w_i)\bar{x}_i &= -\sum_{j:j\sim i} \frac{\gamma_j}{\gamma_i}\bar{x}_i^{\mathrm{T}} w_{ij}\bar{x}_j \\ &\leqslant \sum_{j:j\sim i} \frac{\gamma_j}{\gamma_i}|\bar{x}_i^{\mathrm{T}} w_{ij}\bar{x}_j| \qquad (6)\\ &\leqslant \sum_{j:j\sim i} \frac{\gamma_j}{\gamma_i}\lambda_1(w_{ij})\sqrt{\bar{x}_i^{\mathrm{T}}\bar{x}_i}\sqrt{\bar{x}_j^{\mathrm{T}}\bar{x}_j}, \quad \text{by (2)} \qquad (7)\\ &\leqslant \bar{x}_i^{\mathrm{T}}\bar{x}_i \sum_{j:j\sim i} \frac{\gamma_j}{\gamma_i}\lambda_1(w_{ij}), \ \text{ as } \bar{x}_i^{\mathrm{T}}\bar{x}_i \geqslant \bar{x}_j^{\mathrm{T}}\bar{x}_j, \quad \text{for all } j. \qquad (8)\end{aligned}$$

From (8), we get

$$\frac{\bar{x}_i^{\mathrm{T}}(\lambda_1 I_{p,p} - w_i)\bar{x}_i}{\bar{x}_i^{\mathrm{T}}\bar{x}_i} \leqslant \sum_{j:j\sim i} \frac{\gamma_j}{\gamma_i}\lambda_1(w_{ij}), \quad \text{as } \bar{x}_i^{\mathrm{T}}\bar{x}_i > 0$$

i.e.,

$$\lambda_1 - \lambda_1(w_i) \leqslant \frac{\bar{x}_i^{\mathrm{T}}(\lambda_1 I_{p,p} - w_i)\bar{x}_i}{\bar{x}_i^{\mathrm{T}}\bar{x}_i} \leqslant \sum_{j:j\sim i} \frac{\gamma_j}{\gamma_i}\lambda_1(w_{ij}), \quad \text{by (1)} \qquad (9)$$

i.e.,

$$\begin{aligned}\lambda_1 &\leqslant \lambda_1(w_i) + \frac{1}{\gamma_i}\sum_{j:j\sim i} \gamma_j\lambda_1(w_{ij}) \\ &\leqslant \lambda_1(w_i) + \frac{1}{\gamma_i}\max_{j:j\sim i}\{\gamma_j\}\sum_{j:j\sim i}\lambda_1(w_{ij}) \qquad (10)\\ &= \lambda_1(w_i) + \max_{j:j\sim i}\left\{\sum_{k:k\sim j}\lambda_1(w_{jk})\right\}, \quad \text{as } \gamma_i = \sum_{j:j\sim i}\lambda_1(w_{ij}) \\ &\leqslant \max_{i\sim j}\left\{\lambda_1\left(\sum_{k:k\sim i} w_{ik}\right) + \sum_{k:k\sim j}\lambda_1(w_{jk})\right\}.\end{aligned}$$

This completes the proof of (3). Now suppose that equality in (3) holds. Then all inequalities in the above argument must be equalities.

From equality in (8), we get

$$\bar{x}_k^{\mathrm{T}}\bar{x}_k = \bar{x}_i^{\mathrm{T}}\bar{x}_i, \quad \text{for all } k,\ k \sim i.$$

From this we get $\bar{x}_k \neq \bar{0}$ for all k, $k \sim i$ as $\bar{x}_i \neq \bar{0}$.

From equality in (7) and using Lemma 2.2, we get that $\bar{x}_i$ is an eigenvector of w_{ij} for the largest eigenvalue $\lambda_1(w_{ij})$ and for any j, $j \sim i$, $\bar{x}_j = b_{ij}\bar{x}_i$, for some b_{ij}.

For vertex j such that $j \sim i$,

$$\bar{x}_i^{\mathrm{T}} \bar{x}_i = \bar{x}_j^{\mathrm{T}} \bar{x}_j,$$

i.e.,

$$(b_{ij}^2 - 1)\bar{x}_i^{\mathrm{T}} \bar{x}_i = 0, \quad \text{as } \bar{x}_j = b_{ij}\bar{x}_i,$$

i.e.,

$$b_{ij} = \pm 1, \quad \text{as } \bar{x}_i^{\mathrm{T}} \bar{x}_i > 0.$$

Since w_{ij} is a positive definite matrix and $\bar{x}_i$ is an eigenvector of w_{ij} for the largest eigenvalue $\lambda_1(w_{ij})$, we have

$$\bar{x}_i^{\mathrm{T}} w_{ij} \bar{x}_i > 0. \tag{11}$$

From equality in (6), we have

$$\sum_{j:j\sim i} \frac{\gamma_j}{\gamma_i} |b_{ij}| \left|\bar{x}_i^{\mathrm{T}} w_{ij} \bar{x}_i\right| = -\sum_{j:j\sim i} \frac{\gamma_j}{\gamma_i} b_{ij} (\bar{x}_i^{\mathrm{T}} w_{ij} \bar{x}_i), \quad \text{by } \bar{x}_j = b_{ij}\bar{x}_i,$$

i.e.,

$$\sum_{j:j\sim i} (b_{ij} + 1) \frac{\gamma_j}{\gamma_i} (\bar{x}_i^{\mathrm{T}} w_{ij} \bar{x}_i) = 0, \quad \text{by (11) and } |b_{ij}| = 1. \tag{12}$$

Since $b_{ij} = \pm 1$, therefore from (12), we get $b_{ij} = -1$, for all j, $j \sim i$. Hence $\bar{x}_j = -\bar{x}_i$, for all j, $j \sim i$.

From equality in (9) and using Lemma 2.1, we get that $\bar{x}_i$ is an eigenvector of w_i corresponding to the largest eigenvalue $\lambda_1(w_i)$.

From equality in (10), we get

$$\sum_{k:k\sim j} \lambda_1(w_{jk}) = \sum_{k:k\sim s} \lambda_1(w_{sk}),$$

$$\text{i.e., } \gamma_j = \gamma_s, \text{ for all } j,\ s \text{ such that } j \sim i,\ s \sim i.$$

From the jth equation of (4) and the just established relation $\bar{x}_j = -\bar{x}_i$, for $j \sim i$,

$$-\lambda_1 \bar{x}_i = -w_j \bar{x}_i - \sum_{k:k\sim j} \frac{\gamma_k w_{jk}}{\gamma_j} \bar{x}_k,$$

i.e.,

$$\lambda_1 \bar{x}_i = w_j \bar{x}_i + \sum_{k:k\sim j} \frac{\gamma_k w_{jk}}{\gamma_j} \bar{x}_k.$$

Applying the same technique on the above equation as in Eq. (5), we get that

$$\bar{x}_k = \bar{x}_i, \text{ for all } k,\ k \sim j,\ j \sim i; \quad \gamma_k = \gamma_i, \text{ for all } k,\ k \sim j,\ j \sim i;$$

and $\bar{x}_i$ is a common eigenvector of w_j and w_{jk} corresponding to the largest eigenvalues $\lambda_1(w_j)$ and $\lambda_1(w_{jk})$, for all k, $k \sim j$, $j \sim i$.

For every vertex k, where $k \sim j$ and $j \sim i$, we have, using $\bar{x}_k = \bar{x}_i$,

$$\lambda_1 \bar{x}_i = w_k \bar{x}_i - \sum_{l:l\sim k} \frac{\gamma_l w_{kl}}{\gamma_k} \bar{x}_l.$$

and we proceed as before to obtain

$$\bar{x}_l = -\bar{x}_i, \quad \text{for } l \sim k, \text{ where } k \sim j, \ j \sim i.$$

Denote by N_i the neighborhood of the vertex i.

By employing similar procedures, we obtain the following:

$$\begin{aligned}
&\bar{x}_j = -\bar{x}_i, \quad \text{for all } j \in N_i,\\
&\bar{x}_k = \bar{x}_i, \quad \text{for all } k \in N_j, \text{ where } j \in N_i,\\
&\bar{x}_l = -\bar{x}_i, \quad \text{for all } l \in N_k, \text{ where } k \in N_j, \ j \in N_i,\\
&\quad \ldots
\end{aligned}$$

and so on.

Let $U = \{k : \bar{x}_k = \bar{x}_i\}$ and $W = \{k : \bar{x}_k = -\bar{x}_i\}$. So, $N_j \subseteq U$ and $N_i \subseteq W$. Further, for any vertex $r \in N_{N_i}$ (where N_{N_i} is the neighbor of neighbor set of vertex i), there exists a vertex $p \in N_i$ such that $i \sim p$ & $r \sim p$. Therefore $\bar{x}_p = -\bar{x}_i$ and $\bar{x}_r = \bar{x}_i$. Hence $N_{N_i} \subseteq U$. By a similar argument, we can show that $N_{N_j} \subseteq W$. Continuing the procedure, it is easy to see, since G is connected, that $V = U \cup W$ and that the subgraphs induced by U and W respectively are empty graphs. Hence G is bipartite. Moreover, γ_i is constant over each partite set and $\bar{x}_i$ is a common eigenvector of w_i and w_{ij} corresponding to the largest eigenvalues $\lambda_1(w_i)$ and $\lambda_1(w_{ij})$ for all i, j.

Therefore

$$\begin{aligned}
\lambda_1 \bar{x}_i &= w_i \bar{x}_i + \sum_{j:j\sim i} \frac{\gamma_j w_{ij}}{\gamma_i} \bar{x}_i\\
&= w_i \bar{x}_i + \frac{\gamma_j}{\gamma_i} w_i \bar{x}_i, \quad \text{as } \gamma_i \text{ is constant over each partite set}\\
&= \left(1 + \frac{\gamma_j}{\gamma_i}\right) w_i \bar{x}_i.
\end{aligned}$$

For $i, k \in U$,

$$\lambda_1 \bar{x}_i = \left(1 + \frac{\gamma_j}{\gamma_i}\right) w_i \bar{x}_i = \left(1 + \frac{\gamma_j}{\gamma_i}\right) w_k \bar{x}_i,$$

i.e.,

$$w_i \bar{x}_i = w_k \bar{x}_i,$$

i.e.,

$$(\lambda_1(w_i) - \lambda_1(w_k))\, \bar{x}_i = 0, \quad \text{as } \bar{x}_i \text{ is an eigenvector of } w_i \text{ corresponding to the largest eigenvalue } \lambda_1(w_i) \text{ for all } i.$$

Since $\bar{x}_i \neq \bar{0}$, therefore $\lambda_1(w_i)$ is constant for all $i \in U$. Similarly we can show that $\lambda_1(w_j)$ is constant for all $j \in W$.

Hence G is a bipartite semiregular graph.

Conversely, suppose that conditions (i)–(ii) of the Theorem hold for the graph G. We must prove that

$$\lambda_1 = \max_{i \sim j}\left\{\lambda_1\left(\sum_{k:k\sim i} w_{ik}\right) + \sum_{k:k\sim j} \lambda_1(w_{jk})\right\}.$$

Let $\bar{x}$ be a common eigenvector of w_{ij} corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j. Then

$$\begin{aligned} w_i\bar{x} &= \sum_{j:j\sim i} w_{ij}\bar{x} \\ &= \sum_{j:j\sim i} \lambda_1(w_{ij})\bar{x}. \end{aligned}$$

Thus $\sum_{j:j\sim i} \lambda_1(w_{ij})$ is an eigenvalue of w_i. So,

$$\sum_{j:j\sim i} \lambda_1(w_{ij}) \leqslant \lambda_1(w_i). \tag{13}$$

Since w_{ij}'s are positive definite matrices, we have

$$\lambda_1(w_i) \leqslant \sum_{j:j\sim i} \lambda_1(w_{ij}). \tag{14}$$

From (13) and (14), we get

$$\lambda_1(w_i) = \sum_{j:j\sim i} \lambda_1(w_{ij}). \tag{15}$$

Thus each w_i also has eigenvector $\bar{x}$ corresponding to the largest eigenvalue $\lambda_1(w_i)$.

Let U, W be the partite sets of G. Also, let $\lambda_1(w_i) = \alpha$ for $i \in U$ and $\lambda_1(w_i) = \beta$ for $i \in W$.

The following equation can be easily verified:

$$(\alpha+\beta)\begin{pmatrix} \bar{x} \\ \bar{x} \\ \cdot \\ \bar{x} \\ -\bar{x} \\ -\bar{x} \\ \cdot \\ -\bar{x} \end{pmatrix} = \left(\begin{array}{ccc|ccc} w_1 & \cdot & 0 & -\frac{\beta}{\alpha}w_{1k+1} & \cdot & -\frac{\beta}{\alpha}w_{1n} \\ 0 & \cdot & 0 & -\frac{\beta}{\alpha}w_{2k+1} & \cdot & -\frac{\beta}{\alpha}w_{2n} \\ \cdots & \cdot & \cdots & \cdots & \cdot & \cdots \\ 0 & \cdot & w_k & -\frac{\beta}{\alpha}w_{kk+1} & \cdot & -\frac{\beta}{\alpha}w_{kn} \\ \hline -\frac{\alpha}{\beta}w_{k+11} & \cdot & -\frac{\alpha}{\beta}w_{k+1k} & w_{k+1} & \cdot & 0 \\ -\frac{\alpha}{\beta}w_{k+21} & \cdot & -\frac{\alpha}{\beta}w_{k+2k} & 0 & \cdot & 0 \\ \cdots & \cdot & \cdots & \cdots & \cdot & \cdots \\ -\frac{\alpha}{\beta}w_{n1} & \cdot & -\frac{\alpha}{\beta}w_{nk} & 0 & \cdot & w_n \end{array}\right)\begin{pmatrix} \bar{x} \\ \bar{x} \\ \cdot \\ \bar{x} \\ -\bar{x} \\ -\bar{x} \\ \cdot \\ -\bar{x} \end{pmatrix}.$$

Thus $\alpha+\beta$ is an eigenvalue of $M(G)^{-1}L(G)M(G)$. So, $\alpha+\beta \leqslant \lambda_1$.

We have

$$\begin{aligned}\lambda_1\left(\sum_{k:k\sim i} w_{ik}\right)+\sum_{k:k\sim j}\lambda_1(w_{jk}) &= \lambda_1(w_i)+\lambda_1(w_j), \quad \text{by (15)}\\ &= \alpha+\beta \quad \text{for } i\sim j. \end{aligned} \tag{16}$$

Since

$$\begin{aligned}\lambda_1 &\leqslant \max_{i\sim j}\left\{\lambda_1\left(\sum_{k:k\sim i} w_{ik}\right)+\sum_{k:k\sim j}\lambda_1(w_{jk})\right\}\\ &= \alpha+\beta, \quad \text{by (16)}.\end{aligned}$$

Thus $\lambda_1=\alpha+\beta=\max_{i\sim j}\left\{\lambda_1\left(\sum_{k:k\sim i} w_{ik}\right)+\sum_{k:k\sim j}\lambda_1(w_{jk})\right\}$.

Hence the theorem is proved. □

3. Some related results

In this section we obtain some consequences of Theorem 2.3 and prove certain related results.

Corollary 3.1. *Let G be a simple connected weighted graph and let w_{ij} be the positive definite weight matrix of the edge ij. Then*

$$\lambda_1=\max_{i\sim j}\left\{\sum_{k:k\sim i}\lambda_1(w_{ik})+\sum_{k:k\sim j}\lambda_1(w_{jk})\right\}$$

if and only if

(i) *G is a bipartite semiregular graph*;
(ii) *w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j.*

Proof. We have

$$\begin{aligned}\lambda_1 &\leqslant \max_{i\sim j}\left\{\lambda_1\left(\sum_{k:k\sim i} w_{ik}\right)+\sum_{k:k\sim j}\lambda_1(w_{jk})\right\}, \quad \text{by (3)}\\ &\leqslant \max_{i\sim j}\left\{\sum_{k:k\sim i}\lambda_1(w_{ik})+\sum_{k:k\sim j}\lambda_1(w_{jk})\right\}, \quad \text{by (14)}.\end{aligned}$$

First we suppose that the equality holds. Then we can see that the two inequalities in the above argument must be equalities. In particular, we have

$$\lambda_1 = \max_{i\sim j}\left\{\lambda_1\left(\sum_{k:k\sim i} w_{ik}\right) + \sum_{k:k\sim j}\lambda_1(w_{jk})\right\}.$$

Using Theorem 2.3, we see that (i)–(ii) of the Corollary hold.

Conversely, suppose that (i)–(ii) all hold for the graph G. Using Theorem 2.3, we get

$$\lambda_1 = \max_{i\sim j}\left\{\lambda_1\left(\sum_{k:k\sim i} w_{ik}\right) + \sum_{k:k\sim j}\lambda_1(w_{jk})\right\}.$$

Since w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j, then

$$\lambda_1\left(\sum_{k:k\sim i} w_{ik}\right) = \sum_{k:k\sim i}\lambda_1(w_{ik}).$$

Hence we get the required result. □

Corollary 3.2. *Let G be a simple connected weighted graph and let each weight w_i be a positive number. Then*

$$\lambda_1 \leqslant \max_{i\sim j}\{w_i + w_j\},$$

with equality if and only if G is a bipartite regular graph or G is a bipartite semiregular graph.

Proof. When w_{ij}'s are positive numbers in place of matrices,

$$\lambda_1(w_{ij}) = w_{ij} \quad \text{and} \quad \lambda_1(w_i) = w_i.$$

Hence we get the required result. □

The classical inequality of [1] is an immediate consequence of the preceding result and is stated next.

Corollary 3.3 [1]. *Let G be a simple connected unweighted graph and let d_i be the degree of vertex i. Then*

$$\lambda_1 \leqslant \max_{i\sim j}\{d_i + d_j\},$$

with equality if and only if G is a bipartite regular graph or G is a bipartite semiregular graph.

Proof. For undirected graph, $w_{ij} = 1$ for $i \sim j$. Therefore $w_i = d_i$. Using Corollary 3.2 we get the required result. □

Lemma 3.4. *Let G be a $(\lambda_1(w_i), \lambda_1(w_j))$-semiregular bipartite graph of order n with first l vertices of the same largest eigenvalue $\lambda_1(w_i)$ and the remaining m vertices of the same largest eigenvalue $\lambda_1(w_j)$. Also let $\bar{x}$ be a common eigenvector of w_{ij} corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j; where $w_i = \sum_{k:k\sim i} w_{ik}$, for all i. Then $\lambda_1(w_i) + \lambda_1(w_j)$ is the largest eigenvalue of $L(G)$ and the corresponding eigenvector is*

$$\Big(\underbrace{\lambda_1(w_i)\bar{x}^{\mathrm{T}}, \lambda_1(w_i)\bar{x}^{\mathrm{T}}, \ldots, \lambda_1(w_i)\bar{x}^{\mathrm{T}}}_{l}, \underbrace{-\lambda_1(w_j)\bar{x}^{\mathrm{T}}, -\lambda_1(w_j)\bar{x}^{\mathrm{T}}, \ldots, -\lambda_1(w_j)\bar{x}^{\mathrm{T}}}_{m}\Big)^{\mathrm{T}}.$$

Proof. Since $\bar{x}$ is a common eigenvector of w_{ij} corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j; from earlier calculations (15) we have

$$\lambda_1(w_i) = \sum_{j:j\sim i} \lambda_1(w_{ij}), \quad \text{for all } i.$$

Also we have w_i has eigenvector $\bar{x}$ corresponding to the largest eigenvalue $\lambda_1(w_i)$, for all i.

From (3), we get

$$\lambda_1 \leqslant \max_{r\sim s}\{\lambda_1(w_r) + \lambda_1(w_s)\} = \lambda_1(w_i) + \lambda_1(w_j). \tag{17}$$

We can see easily that $\lambda = \lambda_1(w_i) + \lambda_1(w_j)$ satisfies

$$L(G)\mathbf{X} = \lambda\mathbf{X},$$

where

$$\mathbf{X} = \Big(\underbrace{\lambda_1(w_i)\bar{x}^{\mathrm{T}}, \lambda_1(w_i)\bar{x}^{\mathrm{T}}, \ldots, \lambda_1(w_i)\bar{x}^{\mathrm{T}}}_{l}, \underbrace{-\lambda_1(w_j)\bar{x}^{\mathrm{T}}, -\lambda_1(w_j)\bar{x}^{\mathrm{T}}, \ldots, -\lambda_1(w_j)\bar{x}^{\mathrm{T}}}_{m}\Big)^{\mathrm{T}}.$$

Thus

$$\lambda_1 \geqslant \lambda_1(w_i) + \lambda_1(w_j). \tag{18}$$

From (17) and (18), we get the required result. □

In the remainder of the paper we assume that the vertices are ordered such that $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_n$, where γ_i is defined, as before, by

$$\gamma_i = \sum_{j:j\sim i} \lambda_1(w_{ij}).$$

Theorem 3.5. *Let G be a simple connected weighted graph and let $\gamma_i = \sum_{k:k\sim i} \lambda_1(w_{ik})$. Then $\lambda_1 = \gamma_1 + \gamma_2$ $(\gamma_1 \geqslant \gamma_2)$ if and only if*

(i) *G is a star graph whose edge weights all have the same largest eigenvalue or G is a bipartite regular graph*;
(ii) *w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j.*

Proof. We have

$$\begin{aligned}\lambda_1 &\leqslant \max_{i\sim j}\left\{\lambda_1\left(\sum_{k:k\sim i} w_{ik}\right)+\sum_{k:k\sim j}\lambda_1(w_{jk})\right\}, \quad \text{by (3)}\\ &\leqslant \max_{i\sim j}\left\{\sum_{k:k\sim i}\lambda_1(w_{ik})+\sum_{k:k\sim j}\lambda_1(w_{jk})\right\}, \quad \text{by (14)}\\ &\leqslant \sum_{k:k\sim 1}\lambda_1(w_{1k})+\sum_{k:k\sim 2}\lambda_1(w_{2k})=\gamma_1+\gamma_2,\ \text{as } \gamma_1\geqslant\gamma_2\geqslant\cdots\geqslant\gamma_n.\end{aligned}$$

First we suppose that $\lambda_1=\gamma_1+\gamma_2$ holds. Then we can see that all inequalities in the above argument must be equalities. In particular, we have

$$\lambda_1=\max_{i\sim j}\left\{\sum_{k:k\sim i}\lambda_1(w_{ik})+\sum_{k:k\sim j}\lambda_1(w_{jk})\right\}.$$

Using Corollary 3.1, we get

(i) G is a bipartite semiregular graph;
(ii) w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j.

From earlier calculations (15), we get

$$\gamma_i=\lambda_1(w_i), \quad \text{for all } i.$$

We can assume that $V=U\cup W$, where $U=\{i:\ \gamma_i=\gamma_1\}$ and $W=\{i:\ \gamma_i=\gamma_2\}$. Two cases arise (i) $1\sim 2$, (ii) $1\nsim 2$.

Case (i) $1\sim 2$.

Two subcases arise (a) $|U|\geqslant 2$, (b) $|U|=1$.

Subcase (a) $|U|\geqslant 2$.

In this subcase there exists vertex $k\ (\neq 1)\in U$ such that $\gamma_k=\gamma_1$. Since $\gamma_k\leqslant\gamma_2$, therefore $\gamma_1=\gamma_2$. Thus G is a bipartite regular graph.

Subcase (b) $|U|=1$.

In this subcase G is a star graph whose edge weights all have the same largest eigenvalue.

Case (ii) $1\nsim 2$.

In this case both vertices 1 and 2 are in the same partite set. So, $\gamma_1=\gamma_2$. Hence G is a bipartite regular graph.

Conversely, suppose that the two conditions hold for the graph G. We have to show that $\lambda_1=\gamma_1+\gamma_2$.

Since w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all $i,\ j$, then we have

$$\gamma_i = \lambda_1\left(\sum_{k:k\sim i} w_{ij}\right) = \lambda_1(w_i). \tag{19}$$

Using Lemma 3.4 we get that

$$\begin{aligned}\lambda_1 &= \lambda_1(w_i) + \lambda_1(w_j)\\ &= \gamma_i + \gamma_j,\ i \sim j, \quad \text{by (19)}.\end{aligned} \tag{20}$$

Also, we have $\gamma_i = \lambda_1(w_i)$ is constant for each partite set. For star graph whose edge weights all have the same largest eigenvalue, we get from (20),

$$\lambda_1 = \gamma_1 + \gamma_2.$$

For bipartite regular graph, $\gamma_i,\ i = 1, 2, \ldots, n$ are equal. From (20), we get

$$\lambda_1 = \gamma_1 + \gamma_2.$$

Hence the theorem is proved. □

Corollary 3.6. *Let G be a simple connected weighted graph where each vertex weight w_i is a positive number and suppose $w_1 \geqslant \cdots \geqslant w_n$. Then $\lambda_1 = w_1 + w_2$ if and only if G is a star graph with equal edge weights or G is a bipartite regular graph.*

Proof. When w_{ij}'s are positive numbers in place of matrices,

$$\lambda_1(w_{ij}) = w_{ij}.$$

Since $w_i = \sum_{j:j\sim i} w_{ij}$ and using the above result in Theorem 3.5, we get the required result. □

Corollary 3.7 [3]. *Let G be a simple connected unweighted graph and let d_i be the degree of vertex i and suppose $d_1 \geqslant \cdots \geqslant d_n$. Then $\lambda_1 = d_1 + d_2$ if and only if G is a star graph or G is a bipartite regular graph.*

Proof. The proof follows directly from Corollary 3.6. □

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