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Constructably Laplacian integral graphs ☆

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Abstract

A graph is Laplacian integral if the spectrum of its Laplacian matrix consists entirely of integers. We consider the class of constructably Laplacian integral graphs – those graphs that be constructed from an empty graph by adding a sequence of edges in such a way that each time a new edge is added, the resulting graph is Laplacian integral. We characterize the constructably Laplacian integral graphs in terms of certain forbidden vertex-induced subgraphs, and consider the number of nonisomorphic Laplacian integral graphs that can be constructed by adding a suitable edge to a constructably Laplacian integral graph. We also discuss the eigenvalues of constructably Laplacian integral graphs, and identify families of isospectral nonisomorphic graphs within the class.
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1. Introduction

Given a graph G on n vertices, its *Laplacian matrix* is the $n \times n$ matrix L given by $L = D - A$, where A is the (0, 1) adjacency matrix, and D is the diagonal matrix of vertex degrees. Motivated in part by a parallel question for the spectrum of the adjacency matrix (see [6]), a number of papers on Laplacian matrices investigate the class of *Laplacian integral* graphs – i.e. those graphs

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with the property that the spectrum of the Laplacian matrix consists entirely of integers (see for example [3,8–10,13]).

In particular, a paper of So [13] suggests a strategy for constructing Laplacian integral graphs. So observes that if an edge is added into a graph G in such a way that the Laplacian eigenvalues of G change only by integer quantities, then only one of two situations can occur:

(a) one eigenvalue of G increases by 2 upon addition of the edge; or
(b) two eigenvalues of G increase by 1 upon addition of the edge.

These two cases are known as *spectral integral variation in one place*, and *spectral integral variation in two places*, respectively. The following two results characterize situations (a) and (b).

Note that throughout this paper, for each $i \in \mathbb{N}$, we use e_i to denote the vector with a 1 in the ith position and zeros elsewhere; the order of the vector will always be clear from the context.

Theorem 1.1 [13]. *Let G be a graph such that vertices* 1 *and* 2 *are not adjacent. Form $\widehat{G}$ from G from by adding the edge e between vertices* 1 *and* 2. *Then spectral integral variation in one place occurs under the addition of e if and only if vertices* 1 *and* 2 *have the same neighbours in G. In the case that spectral integral variation in one place occurs by adding e, the eigenvalue of G that increases is equal to the degree of vertex* 1, *say d; further, $e_1 - e_2$ is an eigenvector for G corresponding to d, and for $\widehat{G}$ corresponding to $d + 2$.*

Henceforth, we use $\mathbf{1}_k$ to denote an all-ones vector of order k; the subscript will be suppressed only when the order is clear from the context.

Theorem 1.2 [7]. *Let G be a graph on n vertices with Laplacian matrix L given by*

$$L = \left[\begin{array}{cc|c|c|c|c} d_1 & 0 & -\mathbf{1}^{\mathrm{T}} & 0^{\mathrm{T}} & -\mathbf{1}^{\mathrm{T}} & 0^{\mathrm{T}} \\ 0 & d_2 & 0^{\mathrm{T}} & -\mathbf{1}^{\mathrm{T}} & -\mathbf{1}^{\mathrm{T}} & 0^{\mathrm{T}} \\ \hline -\mathbf{1} & 0 & L_{11} & L_{12} & L_{13} & L_{14} \\ \hline 0 & -\mathbf{1} & L_{21} & L_{22} & L_{23} & L_{24} \\ \hline -\mathbf{1} & -\mathbf{1} & L_{31} & L_{32} & L_{33} & L_{34} \\ \hline 0 & 0 & L_{41} & L_{42} & L_{43} & L_{44} \end{array}\right], \tag{1.1}$$

where the blocks $L_{11}, \dots, L_{44}$ are of sizes $d_1 - t$, $d_2 - t$, t and $n - 2 - d_1 - d_2 + t$, respectively. Suppose that $d_1 \geqslant d_2$. Form $\widehat{G}$ from G by adding the edge e between vertices 1 *and* 2. *Then spectral integral variation occurs in two places under the addition of e if and only if the following conditions hold*:

$$L_{11}\mathbf{1} - L_{12}\mathbf{1} = (d_2 + 1)\mathbf{1}, \tag{1.2}$$
$$L_{21}\mathbf{1} - L_{22}\mathbf{1} = -(d_1 + 1)\mathbf{1}, \tag{1.3}$$
$$L_{31}\mathbf{1} - L_{32}\mathbf{1} = -(d_1 - d_2)\mathbf{1}, \tag{1.4}$$
$$L_{41}\mathbf{1} - L_{42}\mathbf{1} = 0. \tag{1.5}$$

In the case that conditions (1.2)–(1.5) *hold, the two eigenvalues of L that are changed under the addition of e are*

$$\lambda_{i_1} = \frac{d_1 + d_2 + 1 - \sqrt{(d_1 + d_2 + 1)^2 - 4(d_1 d_2 + t)}}{2} \tag{1.6}$$

and

$$\lambda_{i_2} = \frac{d_1 + d_2 + 1 + \sqrt{(d_1 + d_2 + 1)^2 - 4(d_1 d_2 + t)}}{2}, \tag{1.7}$$

and the vectors $u_1 = \left[\begin{array}{c} d_2 + 1 - \lambda_{i_1} \\ \lambda_{i_1} - d_1 - 1 \\ \hline \mathbf{1} \\ \hline -\mathbf{1} \\ \hline 0 \\ \hline 0 \end{array}\right]$ *and* $u_2 = \left[\begin{array}{c} d_2 + 1 - \lambda_{i_2} \\ \lambda_{i_2} - d_1 - 1 \\ \hline \mathbf{1} \\ \hline -\mathbf{1} \\ \hline 0 \\ \hline 0 \end{array}\right]$ *are eigenvectors of L corresponding to* λ_{i_1} *and* λ_{i_2}, *respectively.*

Remark 1.3. In Theorem 1.2, one or more of the last four sets in the partitioning of L may be empty. In that case, the result carries through with the corresponding members of (1.2)–(1.5) omitted.

As mentioned above, So's notion of spectral integral variation suggests a strategy for constructing Laplacian integral graphs: starting with a known Laplacian integral graph G, add an edge into G so that spectral integral variation occurs (if that is possible); the resulting graph will then also be Laplacian integral. That strategy is employed in [8], which deals with the class of *integrally completable graphs*, i.e., those Laplacian integral graphs having the property that a sequence of edges can be added, with spectral integral variation occurring with each addition, and that such edge additions can continue until a complete graph is obtained.

In this paper, we continue in a similar vein. Specifically, we consider the class of graphs defined as follows.

Let G be a graph on n vertices with at least one edge. Denote the (empty) graph on n vertices with no edges by O_n. We say that G is *constructably Laplacian integral* if there is a sequence of graphs $O_n \equiv G_0, G_1, \dots, G_k \equiv G$ such that

(i) G_i is Laplacian integral for $i = 0, \dots, k$, and
(ii) for each $i = 0, \dots, k-1$, G_{i+1} is constructed from G_i by the addition of some edge.

We also take the convention that O_n is constructably Laplacian integral. We use $\mathscr{C}_n$ to denote the set of constructably Laplacian integral graphs on n vertices.

It is not difficult to see that the constructably Laplacian integral graphs are just the complements of the integrally completable graphs studied in [8], and consequently the present paper can be seen as a companion piece to [8]. In this paper, we characterize the constructably Laplacian integral graphs, discuss their eigenvalues, consider the number of nonisomorphic graphs in $\mathscr{C}_n$ that differ from a given graph in $\mathscr{C}_n$ by a single edge, construct families of isospectral nonisomorphic graphs in $\mathscr{C}_n$, and discuss the subclass of threshold graphs.

Throughout, we adopt the following notation and terminology. For a vertex v of a graph G, the *neighbourhood of* v is the set of vertices of G that are adjacent to v. Given a collection of vertices in a graph G, the corresponding *vertex-induced subgraph*, say S, is the graph on that collection of vertices with two vertices of S adjacent in S if and only if they are adjacent in G; we use ι_S to denote the $(0, 1)$ vector with entries equal to 1 in positions corresponding to vertices in S, and entries equal to 0 otherwise. We use P_4 and C_4 to denote the path on four vertices and the cycle on four vertices, respectively. Given graphs G and H, their union is denoted $G \cup H$, while their *join*, $G \vee H$, is the graph formed from $G \cup H$ by adding all possible edges between vertices of G and vertices of H. Finally, we use J to denote an all-ones matrix; the order will be made clear from the context.

2. Basic results

Recall that a graph G is a *complement reducible graph*, or *co-graph* for short, if it has the property that for each collection of four vertices, the corresponding vertex-induced subgraph of G is not P_4. Complement reducible graphs, also known as *decomposable graphs*, are well-studied (see [2], for an introduction) and in particular, it is straightforward to determine that any co-graph is Laplacian integral. The following result discusses spectral integral variation for co-graphs.

Theorem 2.1. *Let G be a co-graph on n vertices, and suppose that vertices 1 and 2 of G are not adjacent. Let $\widehat{G}$ be the graph constructed from G by adding the edge e between vertices 1 and 2. Then spectral integral variation occurs upon the addition of e if and only if $\widehat{G}$ is also a co-graph.*

Proof. First, suppose that $\widehat{G}$ is a co-graph; then $\widehat{G}$ is necessarily Laplacian integral. As G is also Laplacian integral, we conclude that spectral integral variation must take place upon adding the edge e to G.

Now suppose that spectral integral variation occurs when the e is added to G. Let N_1 and N_2 denote the neighbourhoods of vertices 1 and 2 in G, respectively. If spectral integral variation occurs in one place, then by Theorem 1.1, necessarily $N_1 = N_2$. Thus, each vertex of G is adjacent to either both of 1 and 2 or neither 1 nor 2. Consider a vertex-induced subgraph H of $\widehat{G}$ on four vertices. If H does not contain both vertices 1 and 2, then it is also a vertex-induced subgraph of G, and so is not equal to P_4. If H contains both 1 and 2, then since each vertex of $\widehat{G}$ that is distinct from 1 and 2 is adjacent to either both of 1 and 2 or neither 1 nor 2, it follows readily that H cannot equal P_4. Thus, if spectral integral variation occurs in one place, then $\widehat{G}$ is also a co-graph.

Finally, suppose that spectral integral variation occurs in two places upon adding the edge e to G. Then $N_1 \neq N_2$, and we consider the following subsets of vertices: $S_1 = N_1 \backslash N_2,\ S_2 = N_2 \backslash N_1,\ S_3 = N_1 \cap N_2$ and S_4, the set of vertices distinct from 1 and 2 that are adjacent to neither 1 nor 2. Observe that these subsets correspond to the last four subsets that generate the partitioning of the Laplacian matrix L in (1.1). Since $N_1 \neq N_2$, we may assume without loss of generality that $S_1 \neq \emptyset$. Note that since G contains no vertex-induced P_4 subgraphs there are no edges between any vertex in S_1 and any vertex in S_2. Hence either $S_2 = \emptyset$, or $L_{12} = 0$. Suppose that in G there is a vertex $v \in S_4$ that is adjacent to a vertex $u \in S_1$. Since $L_{41}\mathbf{1} = L_{42}\mathbf{1}$, there is necessarily a vertex $w \in S_2$ such that v is adjacent to w. But then the vertices $1, u, v, w$ induce a P_4 in G (observe that u and w are not adjacent) a contradiction. We conclude that either $S_4 = \emptyset$ or $L_{14} = 0$. It now follows that $L_{11}\mathbf{1} \leqslant (t+1)\mathbf{1}$, and since we must have $L_{11}\mathbf{1} = (d_2+1)\mathbf{1}$, we conclude that in fact $t = d_2$, $L_{13} = -J$, and $S_2 = \emptyset$. Thus we see that in G, each vertex of S_1 is adjacent to each vertex of S_3, and that no vertex of S_1 is adjacent to any vertex of S_4. It now follows readily that no vertex-induced subgraph of $\widehat{G}$ including both vertices 1 and 2 is equal to P_4. We conclude then that $\widehat{G}$ is a co-graph. □

Our next result characterizes constructably Laplacian integral graphs.

Theorem 2.2. *Let G be a graph on n vertices. Then G is constructably Laplacian integral if and only if it has no vertex-induced P_4 subgraphs and no vertex-induced C_4 subgraphs.*

Proof. Suppose that G is constructably Laplacian integral, and let $O_n \equiv G_0, G_1, \ldots, G_m \equiv G$ be a sequence of graphs such that each G_i is Laplacian integral, and for each $i = 0, \ldots, m-1$, G_{i+1} is formed from G_i by the addition of an edge. Evidently O_n contains no vertex-induced P_4 subgraphs; further, for each $i = 0, \ldots, m-1$, spectral integral variation occurs when constructing G_{i+1} from G_i, so we conclude that G_{i+1} is also a co-graph. Hence it follows that G contains no vertex-induced P_4 subgraphs. Further, since G_i contains no vertex-induced P_4 subgraphs, G_{i+1} cannot contain any vertex-induced C_4 subgraphs. We deduce then that G contains no vertex-induced C_4 subgraphs.

Next, suppose that G is a graph on n vertices that contains no vertex-induced subgraphs equal to either P_4 or C_4. We claim by induction on n that G is constructably Laplacian integral, and note that the claim certainly holds for $n \leqslant 4$. Suppose that the claim holds for some $n-1 \geqslant 4$ and that G is on n vertices. Evidently G is constructably Laplacian integral if and only if each of its connected components is, so without loss of generality, we take G to be connected. Since G has no vertex-induced P_4 subgraphs, it follows that G can be written as $H_1 \vee H_2$ for some pair of graphs H_1 and H_2 (see [2]). If neither H_1 nor H_2 is complete, then G has a vertex-induced C_4, contrary to hypothesis. Hence G must have a vertex of degree $n-1$, so that G can be written as $K_1 \vee H_3$ for some graph H_3 having no vertex-induced P_4 subgraphs or C_4 subgraphs. By the induction hypothesis, H_3 is constructably Laplacian integral, say with the sequence of Laplacian integral graphs $O_{n-1} \equiv A_0, A_1, \ldots, A_p \equiv H_3$ having the property that for each $i = 0, \ldots, p-1$, A_{i+1} is formed from A_i by adding an edge. By considering the sequence of Laplacian integral graphs O_n $(K_{i,1} \cup O_{n-i-1})$, $i = 1, \ldots, n-1$, followed by $A_j \vee K_1$, $j = 1, \ldots, p$, we find readily that G is a constructably Laplacian integral graph. □

Remark 2.3. Observe that G is constructably Laplacian integral if and only if its complement, $\overline{G}$, is integrally completable. According to a result in [8], $\overline{G}$ is integrally completable if and only if $\overline{G}$ has no vertex-induced P_4 subgraphs and no vertex-induced $K_2 \cup K_2$ subgraphs. Thus we have another proof that G is constructably Laplacian integral if and only if G has no vertex-induced P_4 subgraphs and no vertex-induced C_4 subgraphs.

The following is immediate from Theorem 2.2.

Corollary 2.4. *Suppose that G is constructably Laplacian integral and that vertices* 1 *and* 2 *of G are not adjacent. Denote the neighbourhoods of* 1 *and* 2 *by N_1 and N_2, respectively, and let S denote the set of vertices distinct from* 1 *and* 2 *that are adjacent to neither of vertices* 1 *and* 2. *Let $\widehat{G}$ denote the graph constructed from G by adding the edge between vertices* 1 *and* 2. *Then $\widehat{G}$ is Laplacian integral* (*and hence constructably Laplacian integral*) *if and only if one of the following holds*:

(a) $N_1 = N_2$;
(b) $N_2 \subset N_1$ *and no vertex in $N_1 \backslash N_2$ is adjacent to any vertex in S*;
(c) $N_1 \subset N_2$ *and no vertex in $N_2 \backslash N_1$ is adjacent to any vertex in S*.

Remark 2.5. Consider Corollary 2.4, and suppose that d_1 and d_2 are the degrees of vertices 1 and 2, respectively. If condition (a) of Corollary 2.4 holds, then the vector $e_1 - e_2$ is an eigenvector for the Laplacian matrix of G corresponding to the eigenvalue d_1, and $e_1 - e_2$ is an eigenvector for the Laplacian matrix of $\widehat{G}$ corresponding to the eigenvalue $d_1 + 2$.

If condition (b) of Corollary 2.4 holds, then partitioning the vectors below conformally with (1.1), we find that the vectors

$$\begin{bmatrix} d_2 - d_1 \\ 0 \\ \hline \mathbf{1} \\ \hline 0 \\ \hline 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ d_2 - d_1 - 1 \\ \hline \mathbf{1} \\ \hline 0 \\ \hline 0 \end{bmatrix}$$

are eigenvectors for the Laplacian matrix of G corresponding to eigenvalues $d_1 + 1$ and d_2, respectively, while the vectors

$$\begin{bmatrix} d_2 - d_1 - 1 \\ 1 \\ \hline \mathbf{1} \\ \hline 0 \\ \hline 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ d_2 - d_1 \\ \hline \mathbf{1} \\ \hline 0 \\ \hline 0 \end{bmatrix}$$

are eigenvectors for the Laplacian matrix of $\widehat{G}$ corresponding to eigenvalues $d_1 + 2$ and $d_2 + 1$, respectively.

Remark 2.6. It arises from the proof of Theorem 2.2 that if $n \geqslant 2$ and G is a connected graph in $\mathscr{C}_n$, then $G = K_1 \vee \widetilde{G}$ for some $\widetilde{G} \in \mathscr{C}_{n-1}$. In particular, G necessarily has one or more vertices of degree $n - 1$. If $G \neq K_n$, and has say $p < n$ vertices of degree $n - 1$, we find that $G = K_p \vee H$, where $H \in \mathscr{C}_{n-p}$, and H has no vertices of degree $n - p - 1$. It follows then that G can be written as $G = K_p \vee (H_1 \cup \cdots \cup H_q)$, where $q \geqslant 2$ and each H_i is a connected constructably Laplacian integral graph of lower order.

From a standard result on the Laplacian spectrum of a join of graphs (see Corollary 9.25 of [11]), we find then that G has p as an eigenvalue of multiplicity $q - 1$, while the remaining nonzero eigenvalues of G are of the form $\lambda + p$, where λ is a nonzero eigenvalue of some H_i. Further, each λ-eigenvector for H_i lifts to a $(\lambda + p)$-eigenvector of G by appending zeros in the positions corresponding to the vertices of $G \backslash H_i$. In particular, it follows that the algebraic connectivity α of G, i.e. the smallest positive eigenvalue for G, is the number of vertices of G having degree $n - 1$, and the multiplicity of α is one less than the number of connected components in the graph formed from G by deleting all vertices of degree $n - 1$.

3. Eigenvalues of a graph in $\mathscr{C}_n$

In this section, we provide some graph-theoretic interpretations of eigenvalues for constructably Laplacian integral graphs. Remark 2.5 suggests a connection between vertex degrees and eigenvalues for graphs in $\mathscr{C}_n$, and the next result reinforces that connection.

Theorem 3.1. *Let $G \in \mathscr{C}_n$, and let v be a vertex of G of degree d. Then one of the following holds.*

(a) $d+1$ *is an eigenvalue of* G. *In that case, either there is a vertex* u *adjacent to* v *such that* $N_v \backslash \{u\} = N_u \backslash \{v\}$ *and* $e_u - e_v$ *is a* $(d+1)$*-eigenvector, or the set* A *of vertices in* N_v *of degree less than* d *is not empty, and* $-\iota_A + |A|e_v$ *is a* $(d+1)$*-eigenvector.*

(b) $d+1$ *is not an eigenvalue of* G. *In that case,* d *is an eigenvalue of* G. *Further, there is a vertex* u *not adjacent to* v *such that* $N_u = N_v$, *and* $e_u - e_v$ *is a* d*-eigenvector.*

Proof. We proceed by induction on n, and note that for $n = 2$ the conclusion is readily verified. Suppose now that $n \geqslant 3$, and without loss of generality, we take G to be connected. If $G = K_n$, then any vertex v has degree $n-1$, and certainly n is an eigenvalue. Further, observe that for a vertex $u \neq v$, $N_v \backslash \{u\} = N_u \backslash \{v\}$ and $e_u - e_v$ is an n-eigenvector, so that (a) holds.

Now suppose that $G \neq K_n$, in which case we have $G = K_p \vee (H_1 \cup \cdots \cup H_q)$ for some $p \geqslant 1$ and $q \geqslant 2$, where $H_1, \ldots, H_q$ are connected constructably Laplacian integral graphs of lower order. If the degree of v is $n-1$, then note that certainly n is an eigenvalue for G. Letting A denote the set of vertices of degree less than n, we find that $-\iota_A + |A|e_v$ is an n-eigenvector. Further, if $p \geqslant 2$, then for a vertex $u \neq v$ of degree $n-1$ we have $N_v \backslash \{u\} = N_u \backslash \{v\}$ and $e_u - e_v$ is an n-eigenvector.

If the degree of v is $d < n$, then without loss of generality, we can take v to be in H_1. Recall from Remark 2.6 that each nonzero eigenvalue λ of H_1 generates the eigenvalue $\lambda + p$ of G, and that the corresponding λ-eigenvectors lift to $(\lambda + p)$-eigenvectors of G. Since the d is the sum of p with the degree of v as a vertex of H_1, conclusions (a) and (b) now follow readily from the induction hypothesis. □

Let G be a connected graph that is constructively Laplacian integral. We inductively construct a rooted, directed tree $\vec{T}(G)$ having a weight m_v associated with each vertex v of the tree as follows:

1. If $G = K_m$ for some $m \geqslant 1$, then $\vec{T}(G)$ is a single vertex, the root, with weight m.
2. Suppose that G is not a complete graph, say $G = K_p \vee (H_1 \cup \cdots \cup H_q)$ for some $p, q \in \mathbb{N}$ with $q \geqslant 2$, where each H_i is a connected, constructably Laplacian integral graph. For each $i = 1, \ldots, q$, let v_i be the root vertex of $\vec{T}(H_i)$, and form $\vec{T}(G)$ from $\vec{T}(H_1) \cup \cdots \cup \vec{T}(H_q)$ by adding a new root vertex v_0, with weight p, and the arcs $v_i \to v_0$, $i = 1, \ldots, q$. Observe that each arc in $\vec{T}(G)$ is oriented towards the root vertex v_0, that for each vertex of $\vec{T}(G)$, there is a unique directed path to the root vertex, and that each vertex of $\vec{T}(G)$ either has indegree zero, or has indegree at least two. We note in passing that the directed tree $\vec{T}(G)$ is similar in approach to the so-called *composition tree* for a co-graph described in [5].

Let A denote the set of vertices of $\vec{T}(G)$ of indegree at least two, and let B denote the set of vertices of $\vec{T}(G)$ of indegree zero. For any vertex v of $\vec{T}(G)$, let s_v denote the sum of the weights of the vertices on the unique path from v to v_0 (here we admit the empty path if $v = v_0$, with $s_{v_0} = m_{v_0}$). Finally, for each $v \in A$, let r_v denote the sum of the weights of the vertices distinct from v whose path to v_0 goes through v, and let d_v denote the indegree of v. We now construct the following multisets of integers. For each $v \in A$, let $L_1(v) = \{s_v^{(d_v-1)}\}$ and $L_2(v) = \{(r_v + s_v)^{(m_v)}\}$, and for each $v \in B$, let $L_3(v) = \{s_v^{(m_v-1)}\}$; here we adopt the convention

$a^{(b)}$ to indicate that the number a is repeated b times. Our next result shows how $\overrightarrow{T}(G)$ can be used to find the spectrum of a graph $G \in \mathscr{C}_n$.

Theorem 3.2. *Suppose that G is a connected constructably Laplacian integral graph. Let $\Lambda(G)$ denote the nonzero part of the spectrum of G. Then $\Lambda(G)$ is given by the multiset $\bigcup_{v \in A}(L_1(v) \cup L_2(v)) \cup \bigcup_{v \in B} L_3(v)$.*

Proof. We proceed by induction on the number of vertices of G. Note that if G happens to be a complete graph, say on m vertices, then $\overrightarrow{T}(G)$ is a single vertex v_0 of weight m, $A = \emptyset$, and $L_3(v_0) = \{m^{(m-1)}\}$, which coincides $\Lambda(G)$ (observe that both sets are empty in the case that $m = 1$). In particular, note that if G has two vertices (and so is necessarily equal to K_2), we have the desired conclusion.

Now suppose that G has more than two vertices, and is not a complete graph. Then G can be written as $G = K_p \vee (H_1 \cup \cdots \cup H_q)$ for some $p \in \mathbb{N}$ and $q \geqslant 2$, where each H_i is a connected, constructably Laplacian integral graph. Suppose that for each i, H_i has n_i vertices. Let v_0 denote the root vertex of $\overrightarrow{T}(G)$, which has weight p. Then $v_0 \in A$, $L_1(v_0) = \{p^{(d_{v_0}-1)}\}$ while $L_2(v_0) = \{(p + \sum_{i=1}^{q} n_i)^{(p)}\}$. Further, for each vertex $v \neq v_0$, we have $v \in H_i$ and note that v is in the set A or the set B for $\overrightarrow{T}(G)$ according as v is in the corresponding set for $\overrightarrow{T}(H_i)$. Further, in order to compute s_v for the vertex v of $\overrightarrow{T}(G)$, we simply add p to the corresponding value of s_v considered as a vertex of $\overrightarrow{T}(H_i)$. It now follows readily that $\bigcup_{v \in A}(L_1(v) \cup L_2(v)) \cup \bigcup_{v \in B} L_3(v) = \{p^{(d_{v_0}-1)}\} \cup \{(p + \sum_{i=1}^{q} n_i)^{(p)}\} \bigcup_{i=1}^{q} \{\Lambda(H_i) + p\}$, the latter union from the induction hypothesis. This last is easily seen to coincide with $\Lambda(G)$. □

Example 3.3. In this example we illustrate the technique of Theorem 3.2. Consider the following three graphs, each on 12 vertices: $H_1 = K_1 \vee ((K_1 \vee (K_2 \cup K_2)) \cup (K_1 \vee O_5))$, $H_2 = K_1 \vee ((K_1 \vee (K_2 \cup O_2)) \cup (K_2 \vee O_3))$, $H_3 = K_1 \vee ((K_1 \vee O_4) \cup (K_1 \vee (K_2 \cup K_2 \cup K_1)))$. The weighted digraphs $\overrightarrow{T}(H_1)$, $\overrightarrow{T}(H_2)$, $\overrightarrow{T}(H_3)$ are given in Figs. 1–3. From those digraphs, it is straightforward to determine that each of these graphs has the following Laplacian spectrum: 0, 1, $2^{(5)}$, $4^{(2)}$, 6, 7, 12. It is not difficult to see that in each digraph, the two eigenvalues equal to 4 arise from the vertices in B, while the remaining nonzero eigenvalues correspond to vertices in A.

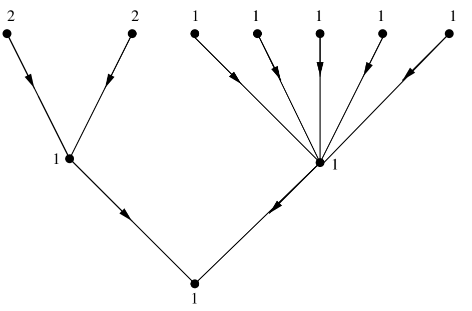

Fig. 1. $\overrightarrow{T}(H_1)$.

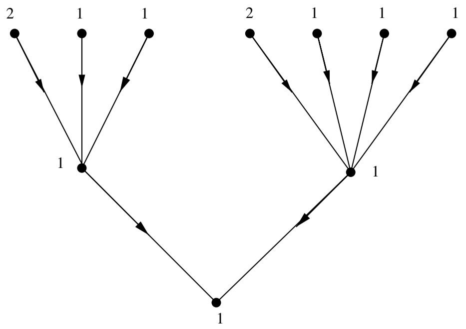

Fig. 2. $\vec{T}(H_2)$.

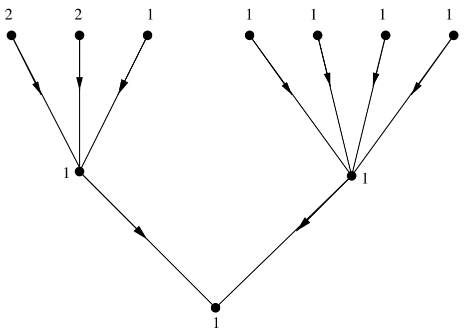

Fig. 3. $\vec{T}(H_3)$.

Remark 3.4. Let G be a connected constructably Laplacian integral graph, and suppose, adopting the notation of Theorem 3.2, that v is a vertex of $\vec{T}(G)$ with $v \in A$. Evidently there are m_v vertices of G, each of common degree $\delta = r_v + s_v - 1$. Considering the set $L_2(V) = \{(r_v + s_v)^{(m_v)}\}$, we see that each of those m_v vertices of degree δ gives rise to an eigenvalue of the Laplacian matrix equal to $\delta + 1$, illustrating Corollary 3.1(a). A similar conclusion holds if $v \in B$ and $m_v \geqslant 2$, as we generate (from $L_3(v)$) m_v vertices of G having degree $\delta = s_v - 1$, and a corresponding Laplacian eigenvalue $\delta + 1$ of multiplicity $m_v - 1$, again illustrating Corollary 3.1(a). Now suppose that $v \in B$ and that $m_v = 1$, say with $v \to u$ as the arc in $\vec{T}(G)$. If there is a $w \in B$ such that $w \to u$ and $m_w = 1$, then note that the eigenvalue s_u is the common degree of the vertices in G corresponding to u and w, both of which have the same neighbourhood in G. That illustrates Corollary 3.1(b).

Finally, we provide an interpretation of the eigenvalues of G arising from $L_1(v)$ for some $v \in A$. We say that a subset S of vertices of G is a *splitting clique* if the vertices of G induce a clique, and $G\backslash S$ is disconnected. It is straightforward to see that if $v \in A$, then there is a splitting clique S of cardinality s_v such that $G\backslash S$ has exactly d_v connected components. Thus we see that a Laplacian eigenvalue arising from some $L_1(v)$ corresponds to the cardinality of a certain splitting clique, and that the corresponding multiplicity arises from the number of connected components formed by deleting that splitting clique.

Remark 3.5. In Remark 2.5, we saw that the eigenvectors

$$\begin{bmatrix} d_2 - d_1 \\ 0 \\ \hline \mathbf{1} \\ \hline 0 \\ \hline 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} d_2 - d_1 - 1 \\ 1 \\ \hline \mathbf{1} \\ \hline 0 \\ \hline 0 \end{bmatrix}$$

correspond, respectively, to the eigenvalues $d_1 + 1$ for G and $d_1 + 2$ for $G \cup \{e\}$, where e denotes the edge between vertices 1 and 2. Note that each of these vectors is of the form $\iota_A - |A|e_1$, where A denotes the set of vertices in N_1 of degree less than d_1.

Consider the eigenvalues d_2 and $d_2 + 1$ of G and $G \cup \{e\}$ respectively, and let S denote the set of vertices that are adjacent to both vertices 1 and 2. Observe that the vertices of S must induce a complete subgraph of G, for if not, there are nonadjacent vertices u, v of S, so that the vertices $\{1, 2, u, v\}$ induce a C_4 in G, a contradiction. Since S induces a complete subgraph of G, it follows that S is in fact a splitting clique, so that $d_2 = |S|$ is the cardinality of a splitting clique in G. A similar argument shows that in $G \cup \{e\}$, the set of vertices $S \cup \{1\}$ is a splitting clique, which evidently has cardinality $d_2 + 1$.

4. Graphs in $\mathscr{C}_n$ differing by one edge

Given a graph in $\mathscr{C}_n$, there might be several different edges that can be added in order to yield other constructably Laplacian integral graphs; it is natural to wonder how many nonisomorphic graphs in $\mathscr{C}_n$ can be constructed from a given graph in $\mathscr{C}_n$ by adding an edge. Similarly we might ask how many different graphs in $\mathscr{C}_n$ will yield a given graph in $\mathscr{C}_n$ via the addition of a suitable edge. In this section, we address that topic.

Suppose that $G \in \mathscr{C}_n$, and let $\bar{\delta}(G)$ be the number of nonisomorphic graphs in $\mathscr{C}_n$ that can be constructed from G by the addition of a single edge. Let $\sigma_n = \max\{\bar{\delta}(G) | G \in \mathscr{C}_n\}$. Our next result yields the value of σ_n.

Theorem 4.1. *For each* $n \in \mathbb{N}$, $\sigma_n = \lfloor \frac{2n-1}{3} \rfloor$.

Proof. It is straightforward to verify the formula for σ_n for $1 \leqslant n \leqslant 4$. We first claim that if $G \in \mathscr{C}_n$ then $\bar{\delta}(G) \leqslant \lfloor \frac{2n-1}{3} \rfloor$, and we proceed by induction on n. Suppose that $n \geqslant 5$ and that $G \in \mathscr{C}_n$. If G is connected, then $G = K_1 \vee H$ where $H \in \mathscr{C}_{n-1}$, and so from the induction hypothesis, we find that $\bar{\delta}(G) = \bar{\delta}(H) \leqslant \lfloor \frac{2(n-1)-1}{3} \rfloor \leqslant \lfloor \frac{2n-1}{3} \rfloor$, as desired.

Next, suppose that G is not connected, say with $G = O_p \cup G_1 \cup \cdots \cup G_k$, where for each $i = 1, \ldots, k$, G_i is a connected graph in $\mathscr{C}_{n_i}$, with $n_i \geqslant 2$ and where $p + \sum_{i=1}^k n_i = n$. Note that if we add an edge to G that joins G_1 and G_2, say, then that creates a P_4, and so spectral integral variation cannot occur. Similarly, suppose that G has an isolated vertex u, and consider the graph formed by adding an edge of the form $\{u, v\}$. It is straightforward to see that spectral integral variation occurs if and only if either v is also an isolated vertex, or v is a vertex of some G_i that is adjacent to every other vertex of G_i. Also, observe that as above, since each G_i is connected, $\bar{\delta}(G_i) \leqslant \frac{2(n_i-1)-1}{3}$. If $p = 0$, then $k \geqslant 2$ and $\bar{\delta}(G) \leqslant \sum_{i=1}^k \bar{\delta}(G_i) \leqslant \sum_{i=1}^k \frac{2(n_i-1)-1}{3} = \frac{2n}{3} - k \leqslant \frac{2n-1}{3}$, as desired. If $p = 1$ then $k \geqslant 1$ and $\bar{\delta}(G) \leqslant \sum_{i=1}^k \bar{\delta}(G_i) + k \leqslant \sum_{i=1}^k \frac{2(n_i-1)-1}{3} + k = \frac{2(n-1)}{3} \leqslant \frac{2n-1}{3}$, again, as desired. Finally, if $p \geqslant 2$, we find that $\bar{\delta}(G) \leqslant \sum_{i=1}^k \bar{\delta}(G_i) + k + 1 \leqslant \sum_{i=1}^k \frac{2(n_i-1)-1}{3} + k + 1 = \frac{2(n-p)}{3} + 1 \leqslant \frac{2n-1}{3}$. This completes the induction proof of the claim.

Lastly, to show that $\sigma_n = \lfloor \frac{2n-1}{3} \rfloor$, we exhibit, for each $n \geqslant 5$, a graph $G \in \mathscr{C}_n$ such that $\bar{\delta}(G) = \lfloor \frac{2n-1}{3} \rfloor$. To construct these graphs, let $A(1) = O_2$, $B(1) = K_{1,2}$, and $C(1) = K_2 \cup O_2$. For each $i \in \mathbb{N}$, we define $A(i+1) = (K_1 \vee A(i)) \cup O_2$, $B(i+1) = (K_1 \vee B(i)) \cup O_2$, and $C(i+1) = (K_1 \vee C(i)) \cup O_2$. Note that for each $i \in \mathbb{N}$, $A(i)$ has $3i-1$ vertices, $B(i)$ has $3i$ vertices, and that $C(i)$ has $3i+1$ vertices. We claim that for each $i \in \mathbb{N}$, $\bar{\delta}(A(i)) = 2i-1$, $\bar{\delta}(B(i)) = 2i-1$ and $\bar{\delta}(C(i)) = 2i$, which will yield the desired conclusion. To show that $\bar{\delta}(A(i)) = 2i-1$, we proceed by induction on i, and note that the case for $i = 1$ is evident. Suppose that $i \geqslant 2$. We find that $\bar{\delta}(A(i)) = \bar{\delta}(A(i-1)) + 2 = 2(i-1) - 1 + 2 = 2i - 1$, as desired. The proofs that $\bar{\delta}(B(i)) = 2i-1$ and $\bar{\delta}(C(i)) = 2i$, are analogous, and are omitted. □

Corollary 4.2. *Suppose that $n \in \mathbb{N}$ with $n \geqslant 2$. Then for each $k \in \mathbb{N}$ with $1 \leqslant k \leqslant \lfloor \frac{2n-1}{3} \rfloor$, there is a graph $G \in \mathscr{C}_n$ such that $\bar{\delta}(G) = k$.*

Proof. We proceed by induction on n, and note that the result certainly holds for $n = 2, 3, 4$. Suppose that $n \geqslant 5$. From Theorem 4.1, there is certainly a $G \in \mathscr{C}_n$ such that $\bar{\delta}(G) = \lfloor \frac{2n-1}{3} \rfloor$, so suppose that $1 \leqslant k \leqslant \lfloor \frac{2n-1}{3} \rfloor - 1$. Then $k \leqslant \lfloor \frac{2(n-1)-1}{3} \rfloor$, so from the induction hypothesis, there is a graph $H \in \mathscr{C}_{n-1}$ such that $\bar{\delta}(H) = k$. It now follows that the graph $G = K_1 \vee H$ is in $\mathscr{C}_n$ and that $\bar{\delta}(G) = \bar{\delta}(H) = k$. □

For each $G \in \mathscr{C}_n$, let $\underline{\delta}(G)$ be the number of nonisomorphic graphs in $\mathscr{C}_n$ to which an edge can be added that will yield G. Evidently $\underline{\delta}(G)$ is the number of nonisomorphic graphs in $\mathscr{C}_n$ that can be formed by deleting a suitable edge in G. Let $\tau_n = \max\{\underline{\delta}(G) | G \in \mathscr{C}_n\}$. We have the following observations.

Observation 1. If $G = G_1 \cup \cdots \cup G_k$ where each G_i is connected and is a constructably Laplacian integral graph, then $\underline{\delta}(G) \leqslant \sum_{i=1}^{k} \underline{\delta}(G_i)$.

Observation 2. If H is a connected constructably Laplacian integral graph, then $\underline{\delta}(K_1 \vee H) = \underline{\delta}(H)$. The equality is obvious if H is a complete graph, so suppose that H is not a complete graph, so that H has the form $H = K_1 \vee \widehat{H}$ for some noncomplete constructably Laplacian integral graph $\widehat{H}$. Let $G = K_1 \vee H$, let u be the vertex of G not in H, and let v be a vertex of H that is adjacent to every other vertex of H. Let $\underline{\delta}(H) = m$, and suppose that $H_1, \ldots, H_m$ are the nonisomorphic constructably Laplacian integral graphs that can be formed by deleting an edge from H. It now follows that if any edge other than $\{u, v\}$ is deleted from G, the resulting graph, if it is constructably Laplacian integral, is isomorphic to one of $K_1 \vee H_i$, $i = 1, \ldots, m$. Finally, let x and y be nonadjacent vertices of H. Observe that in the graph $G \backslash \{u, v\}$, the vertices u, v, x and y induce a C_4, so that $G \backslash \{u, v\}$ is not constructably Laplacian integral. It now follows that $\underline{\delta}(G) = m$, as desired.

Observation 3. Suppose that for each $i = 1, \ldots, m$, G_i is a connected constructably Laplacian integral graph on at least two vertices. Suppose that $m + p \geqslant 2$ and let $G = K_1 \vee (G_1 \cup \cdots \cup G_m \cup O_p)$. Then

$$\underline{\delta}(G) \leqslant \sum_{i=1}^{m} \underline{\delta}(G_i) + \begin{cases} 1, & p \geqslant 1, \\ 0, & p = 0. \end{cases}$$

To see this, note first that if $p \geqslant 1$, then all graphs formed from deleting a pendant edge of G are isomorphic. Next, let u be a vertex of G of maximum degree and suppose that v is a vertex of G_1. Observe that there is a vertex w of G_1 that is adjacent to v, and that since $m + p \geqslant 2$, there is a vertex x that is adjacent to neither v nor w. If we delete the edge $\{u, v\}$ from G, it follows that in the resulting graph, the subgraph induced by vertices u, v, w and x is isomorphic to P_4, so that $G\backslash\{u, v\}$ is not constructably Laplacian integral. It now follows that if a nonpendant edge is deleted from G that yields a constructably Laplacian integral graph then that edge must be an edge e in some G_i such that $G_i - e$ is a constructably Laplacian integral graph. We find that

$$\underline{\delta}(G) \leqslant \sum_{i=1}^{m} \underline{\delta}(G_i) + \begin{cases} 1, & p \geqslant 1, \\ 0, & p = 0, \end{cases}$$

as desired.

Our next result gives the value of τ_n.

Theorem 4.3. *For each $n \in \mathbb{N}$, we have $\tau_n = \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. We begin by proving by induction on n that if $G \in \mathscr{C}_n$, then $\underline{\delta}(G) \leqslant \lfloor \frac{n}{2} \rfloor$, and note that the result is evident for $n = 1, 2, 3$. Suppose that $n \geqslant 4$, and that $G \in \mathscr{C}_n$. First suppose that G is not connected, and has the form $G = G_1 \cup \cdots \cup G_k$ where each G_i is connected, on n_i vertices. By Observation 1 and the induction hypothesis, we have $\underline{\delta}(G) \leqslant \sum_{i=1}^{k} \underline{\delta}(G_i) \leqslant \sum_{i=1}^{k} \lfloor \frac{n_i}{2} \rfloor \leqslant \lfloor \frac{n}{2} \rfloor$. If G is connected, and is of the form $G = K_1 \vee H$, then by Observation 2, we have $\underline{\delta}(G) = \underline{\delta}(H) \leqslant \lfloor \frac{n-1}{2} \rfloor \leqslant \lfloor \frac{n}{2} \rfloor$, the first inequality following from the induction hypothesis. Finally, if G is connected and is of the form $G = K_1 \vee (G_1 \cup \cdots \cup G_m \cup O_p)$ where each G_i is connected on $n_i \geqslant 2$ vertices, and where $m + p \geqslant 2$, then by Observation 3 we have $\underline{\delta}(G) \leqslant \sum_{i=1}^{m} \underline{\delta}(G_i) + \begin{cases} 1, & p \geqslant 1 \\ 0, & p = 0 \end{cases}$ If $p = 0$, then applying the induction hypothesis, to each $\underline{\delta}(G_i)$ it follows that $\underline{\delta}(G) \leqslant \lfloor \frac{n-1}{2} \rfloor$, while if $p \geqslant 1$, a similar argument yields $\underline{\delta}(G) \leqslant \frac{n-p-1}{2} + 1 \leqslant \frac{n}{2}$, and the desired inequality follows. Thus $\underline{\delta}(G) \leqslant \lfloor \frac{n}{2} \rfloor$.

Finally, we claim that for each $n \in \mathbb{N}$, there is a graph $G \in \mathscr{C}_n$ such that $\underline{\delta}(G) = \lfloor \frac{n}{2} \rfloor$. We proceed by induction on n, and note that this is straightforward to see for $n = 1, 2, 3$, so suppose that $n \geqslant 4$. Select a graph $H \in \mathscr{C}_{n-2}$ such that $\underline{\delta}(H) = \lfloor \frac{n-2}{2} \rfloor$. Letting $G = K_1 \vee (H \cup K_1)$, it is straightforward to see that $\underline{\delta}(G) \geqslant \underline{\delta}(H) + 1 = \lfloor \frac{n-2}{2} \rfloor + 1 = \lfloor \frac{n}{2} \rfloor$. The fact that $\underline{\delta}(G) = \lfloor \frac{n}{2} \rfloor$ now follows immediately. □

Corollary 4.4. *Suppose that $n \in \mathbb{N}$ with $n \geqslant 2$. Then for each $k \in \mathbb{N}$ with $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$, there is a connected graph $G \in \mathscr{C}_n$ such that $\underline{\delta}(G) = k$.*

Proof. We use induction on n, and note that the cases $n = 2, 3$ are straightforward. Suppose that $1 \leqslant k \leqslant \lfloor \frac{n-2}{2} \rfloor$, and, applying the induction hypothesis, let H be a connected graph in $\mathscr{C}_{n-2}$ such that $\underline{\delta}(H) = k$. Now let $G = K_1 \vee (H \cup K_1)$; it is not difficult to see then that $\underline{\delta}(G) = k + 1$. Consequently, for each $2 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$, there is a connected graph $G \in \mathscr{C}_n$ such that $\underline{\delta}(G) = k$, and observing that $\underline{\delta}(K_{1,n-1}) = 1$, the statement follows. □

A graph $G \in \mathscr{C}_n$ is called a *terminal graph* if adding any edge into G fails to yield a Laplacian integral graph. Our next result characterizes those graphs.

Theorem 4.5. *A graph G is a terminal graph if and only if both of the following are satisfied*:

(i) *G has no vertex-induced P_4 or C_4 subgraphs*;
(ii) *each pair of nonadjacent vertices in G sits on a vertex-induced $K_2 \cup K_2$.*

Proof. First, note that if both (i) and (ii) hold, then from (i), G is constructably Laplacian integral. Further, from (ii), if any edge e is added into G, then $G \cup \{e\}$ contains a vertex-induced P_4. Hence $G \cup \{e\}$ is not constructably Laplacian integral, from which it follows that $G \cup \{e\}$ is not Laplacian integral.

Now suppose that G is a terminal graph on n vertices. Since G is constructably Laplacian integral, certainly (i) holds. We claim that (ii) also holds, and we proceed to establish the claim by induction on n. Note that if $n = 2$, then $G = K_2$, and the claim holds vacuously. Suppose now that $n \geqslant 3$. If G is connected, then $G = K_1 \vee H$ for some terminal graph H on $n - 1$ vertices. Applying the induction hypothesis to H now yields that (ii) holds for G. Now suppose that G is not connected, say $G = G_1 \cup \cdots \cup G_k$, where each G_i is connected and constructably Laplacian integral. Evidently each G_i must be a terminal graph, and note that no G_i can consist of a single vertex, otherwise we can an add at least one edge incident with that isolated vertex to yield another Laplacian integral graph. Thus each G_i is a connected constructably Laplacian integral on at least two vertices. Note then that any pair of (necessarily nonadjacent) vertices belonging to distinct G_i's sits on a vertex-induced $K_2 \cup K_2$. Further, from the induction hypothesis, any pair of nonadjacent vertices of G belonging to the same G_i sits on a vertex-induced $K_2 \cup K_2$. Thus (ii) holds, as desired. □

Next, we consider the connected terminal graphs on n vertices having a minimum number of edges.

Theorem 4.6. *Let G be a connected terminal graph on $n \geqslant 5$ vertices. Then G has at least $3\lfloor \frac{n}{2} \rfloor$ edges. If n is odd, then equality holds in that lower bound if and only if $G = K_1 \vee (K_2 \cup \cdots \cup K_2)$. If n is even, then equality holds in that lower bound if and only if $G = K_1 \vee (K_3 \cup K_2 \cup \cdots \cup K_2)$.*

Proof. Suppose that G is a connected terminal graph. Then $G = K_1 \vee H$ for some terminal graph H on $n - 1$ vertices. Let $\epsilon(H)$ be the number of edges in H, and let the degree sequence for H be $d_i, i = 1, \ldots, n - 1$. Observe that if at least one d_i is zero, then H is not a terminal graph (since we could add an edge incident with the isolated vertex of H and preserve Laplacian integrality). Hence we have $2\epsilon(H) = \sum_{i=1}^{n-1} d_i \geqslant n - 1$, so that $\epsilon(H) \geqslant \frac{n-1}{2}$.

In the case that n is odd, we have $\epsilon(H) \geqslant \frac{n-1}{2}$, with equality if any only if H is a union of $\frac{n-1}{2}$ independent edges. The lower bound on the number of edges in G, along with the characterization of the equality case, now follows readily when n is odd.

Now suppose that n is even. Since $\epsilon(H) \geqslant \frac{n-1}{2}$, in fact it must be the case that $\epsilon(H) \geqslant \frac{n}{2}$. If it were the case that $\epsilon(H) = \frac{n}{2}$, then since each $d_i \geqslant 1$, we would necessarily have that H has one vertex of degree 2, and the remaining vertices of degree 1. It follows then that $H = P_3 \cup K_2 \cup \cdots \cup K_2$, which is not a terminal graph, a contradiction.

Hence, we must have that $\epsilon(H) \geqslant \frac{n+2}{2}$, which readily yields the lower bound on the number of edges in G. Suppose next that G has $\frac{3n}{2}$ edges, so that necessarily $\epsilon(H) = \frac{n+2}{2}$. Let d_1 be the maximum degree for H, and note that since $d_1 = n + 2 - \sum_{i=2}^{n-1} d_i \leqslant 4$, it follows that the possible degree sequences for H are: $4, 1^{(n-2)}$ if $d_1 = 4$; $3, 2, 1^{(n-3)}$ if $d_1 = 3$; and $2^{(3)}, 1^{(n-4)}$ if $d_1 = 2$.

If H has the degree sequence $4, 1^{(n-2)}$, then since H is constructably Laplacian integral, the vertex of maximum degree 4 is necessarily in a connected component of H on 5 vertices, and it follows that in that case, $H = K_{1,4} \cup K_2 \cup \cdots \cup K_2$, which is not a terminal graph. We then conclude that G is not a terminal graph, a contradiction.

If H has the degree sequence $3, 2, 1^{(n-3)}$, then one connected component of H has 4 vertices and a spanning star (corresponding to the vertex of degree 3), but that component has at most one vertex of degree 2. Hence that component must be $K_{1,3}$, which is not a terminal graph. It follows that H, and hence G, is not a terminal graph, a contradiction.

Finally, suppose that H has degree sequence $2^{(3)}, 1^{(n-4)}$. It follows that either $H = P_3 \cup P_3 \cup K_2 \cup \cdots \cup K_2$, or $H = K_3 \cup K_2 \cup \cdots \cup K_2$. The former is not a terminal graph, while the latter is certainly a terminal graph. We thus conclude that if G is a terminal graph with $\frac{3n}{2}$ edges, then $G = K_1 \vee (K_3 \cup K_2 \cup \cdots \cup K_2)$, as desired. □

5. Classes of isospectral graphs in $\mathscr{C}_n$

In this section we focus on families of nonisomorphic isospectral constructably Laplacian integral graphs. The next result exhibits one such family.

Theorem 5.1. *For each $n \geqslant 12$, there is a collection of $3^{\lfloor \frac{n}{12} \rfloor}$ connected graphs in $\mathscr{C}_n$ that are isospectral but pairwise nonisomorphic.*

Proof. Consider the following three graphs, each of which has 11 vertices: $G_1 = (K_1 \vee (K_2 \cup K_2)) \cup (K_1 \vee O_5)$, $G_2 = (K_1 \vee (K_2 \cup O_2)) \cup (K_2 \vee O_3)$, $G_3 = (K_1 \vee O_4) \cup (K_1 \vee (K_2 \cup K_2 \cup K_1))$. It is straightforward to see that the graphs G_1, G_2 and G_3 are pairwise nonisomorphic, and that each has Laplacian spectrum given by $0^{(2)}, 1^{(5)}, 3^{(2)}, 5, 6$.

Consider the following sets of graphs: $S = \{G_1, G_2, G_3\}$ and $C_{12} = \{H_1, H_2, H_3\}$, where $H_i = K_1 \vee G_i$, $i = 1, 2, 3$. Evidently the graphs in C_{12} are connected, constructably Laplacian integral, isospectral, and nonisomorphic. For each $k \geqslant 2$, let $C_{12k} = \{K_1 \vee (A \cup B) | A \in C_{12(k-1)}, B \in S\}$. A straightforward induction proof on k shows that C_{12k} is a set of cardinality 3^k, that each graph in C_{12k} is a connected graph in $\mathscr{C}_{12k}$, and that the graphs in C_{12k} are isospectral and pairwise nonisomorphic. In particular, if n is divisible by 12, then there is a collection of $3^{\frac{n}{12}}$ connected graphs in $\mathscr{C}_n$ that are isospectral but pairwise nonisomorphic.

To cover the case that n is not divisible by 12, we consider the following collections of graphs. For each $j = 1, \ldots, 11$, let $H_i(j) = K_1 \vee (G_i \cup O_j)$, $i = 1, 2, 3$, and let $C_{12+j} = \{H_1(j), H_2(j), H_3(j)\}$. For each $k \geqslant 2$ and $j = 1, \ldots, 11$, let $C_{12k+j} = \{K_1 \vee (A \cup B) | A \in C_{12(k-1)+j}, B \in S\}$. Again, a proof by induction on k shows that C_{12k+j} is a collection of 3^k connected, isospectral pairwise nonisomorphic graphs in $\mathscr{C}_{12k+j}$. The conclusion now follows. □

Remark 5.2. In [10], Merris constructs, for each $r \in \mathbb{N}$, a family of $\binom{2^{r-2}}{2^{r-3}}$ Laplacian integral graphs on $n = 2^{r-3}(2r + 1)$ vertices, where the graphs in the family are isospectral and pairwise nonisomorphic. Indeed, following the details of the construction in [10], it is not difficult to verify that each of the graphs in that family is constructably Laplacian integral.

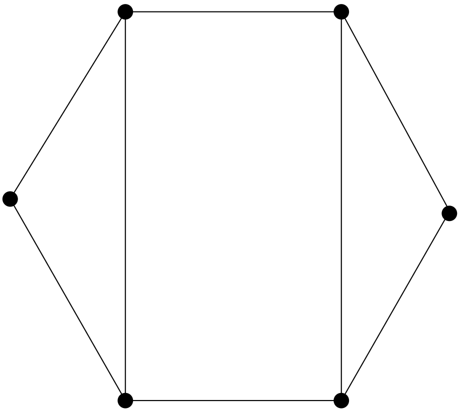

Fig. 4. H.

In this remark, we estimate, for large values of n, the number of graphs in the family constructed by Merris. In order to facilitate the estimation, we let $k = 2^{r-3}$. Recall Stirling's asymptotic formula for $m!$, namely that as $m \to \infty$, we have $m! \approx \sqrt{2\pi} m^{m+\frac{1}{2}} \mathrm{e}^{-m}$ (see [4]). Using Stirling's formula, it follows that as $r \to \infty$, we have $\binom{2^{r-2}}{2^{r-3}} = \binom{2k}{k} = \frac{(2k)!}{(k!)^2} \approx \frac{2^{2k}}{\sqrt{\pi k}}$. Since $n = 2^{r-3}(2r+1)$, we find that for all sufficiently large r, $\log_2(n) = r - 3 + \log_2(2r+1) \geqslant r + \frac{1}{2}$, and $2r + 1 \geqslant r - 3 + \log_2(2r+1) = \log_2(n)$. Since $n = k(2r+1)$, it follows from these inequalities that $\frac{n}{2\log_2(n)} \leqslant k = \frac{n}{2r+1} \leqslant \frac{n}{\log_2(n)}$. Applying these lower and upper bounds on k, it follows that

$$\frac{1}{\sqrt{\pi}} \left(\frac{\log_2(n)}{n} \right)^{\frac{1}{2}} 2^{\frac{n}{\log_2(n)}} \leqslant \frac{2^{2k}}{\sqrt{\pi k}} \leqslant \frac{\sqrt{2}}{\sqrt{\pi}} \left(\frac{\log_2(n)}{n} \right)^{\frac{1}{2}} 2^{\frac{2n}{\log_2(n)}}.$$

Referring to the inequalities above, we see that for all sufficiently large n, the family of Laplacian integral nonisomorphic isospectral graphs on n vertices constructed in Theorem 5.1, which has $3^{\lfloor \frac{n}{12} \rfloor}$ members, is larger than the family of graphs constructed in [10].

Example 5.3. In this example, we show that a constructably Laplacian integral graph can be isospectral with a graph that is not constructably Laplacian integral. We begin by noting that the Laplacian spectrum of $K_{1,4}$ is $0, 1^{(3)}, 5$, that the Laplacian spectrum of K_2 is $0, 2$, and that the obtained from K_4 by deleting an edge, $K_4 \backslash \{e\}$, say, has Laplacian spectrum $0, 2, 4^{(2)}$. Hence the graph $G_1 = K_{1,4} \cup K_2 \cup (K_4 \backslash \{e\})$ has Laplacian spectrum given by $0^{(3)}, 1^{(3)}, 2^{(2)}, 4^{(2)}, 5$. It is straightforward to see that $G_1 \in \mathscr{C}_{11}$.

Next, consider the graph H shown in Fig. 4. It turns out that the Laplacian spectrum of H is $0, 1, 2^{(2)}, 4, 5$. Note also that the Laplacian spectrum of K_1 is 0, while that of $K_{1,3}$ is $0, 1^{(2)}, 4$. Hence, the graph $G_2 = K_1 \cup K_{1,3} \cup H$ has Laplacian spectrum given by $0^{(3)}, 1^{(3)}, 2^{(2)}, 4^{(2)}, 5$, and so is isospectral with G_1. Evidently $G_2 \notin \mathscr{C}_{11}$, since G_2 contains both a vertex-induced P_4 subgraph and a vertex-induced C_4 subgraph. Observe that for any $p \in \mathbb{N}$, the connected graphs $K_p \vee G_1$ and $K_p \vee G_2$ are isospectral, with the former being constructably Laplacian integral and the latter failing to be a co-graph.

Given a square matrix whose entries consist of integers, one of the invariants associated with it is the Smith normal form. Recall that two square integer matrices M_1 and M_2 are *equivalent* if there are integer matrices U, V each of determinant 1 or -1, such that $UM_1V = M_2$. Evidently M_1 and M_2 are equivalent provided that one can be obtained from the other via a sequence of row or column operations of the following type: permutation of rows (columns); addition of a multiple of one row (column) to another row (column); multiplication of a row (column) by -1. A standard result asserts that if M_1 and M_2 are equivalent, then they have the same Smith normal form. See [12] for further details.

The Smith normal form for the Laplacian matrix of a graph has been investigated in several papers, see for example [1] and the references therein. Here we investigate a family of constructably Laplacian integral graphs sharing the same spectrum, degree sequence, and Smith normal form.

Suppose that $p, q \in \mathbb{N}$ with $q \geqslant 2$ and consider the graph

$$G(p,q) = K_1 \vee (\overbrace{K_2 \cup \cdots \cup K_2}^{p} \cup O_q).$$

Now for parameters $p_1, p_1, q_1, q_2 \in \mathbb{N}$ with $q_1, q_2 \geqslant 2$, let $H(p_1, q_1, p_2, q_2) = K_1 \vee (G(p_1, q_1) \cup G(p_2, q_2))$. Using the technique of Theorem 3.2, it follows that the Laplacian spectrum of $H(p_1, q_1, p_2, q_2)$ is given by

$$0,\ 1,\ 2^{(p_1+p_2+q_1+q_2-1)},\ 4^{(p_1+p_2)},\ 2p_1+q_1+2,\ 2p_2+q_2+2,$$
$$2p_1+2p_2+q_1+q_2+3.$$

In particular if $q_4 \geqslant 4$, $q_2 \geqslant 2$ and $p_2 \geqslant 2$, then the graphs $H(p_1+i, q_1-2i, p_2-i, q_2+2i)$, $0 \leqslant i \leqslant \min\{p_2-1, \frac{q_1-2}{2}\}$, are all isospectral and all have the same degree sequence. Our next result helps to discus the Smith normal form for this family of graphs.

Lemma 5.4. *Suppose that $p, q \in \mathbb{N}$ with $q \geqslant 2$, and let L be the Laplacian matrix of $G(p,q)$. Then $L + I$ is equivalent to a diagonal matrix whose entries consist of $1^{(p+2)}, 2^{(q-2)}, 8^{(p)}$ and $2(2p+q+2)$.*

Proof. Throughout this proof, I_k and $0_{k\times j}$ will denote the $k \times k$ identity matrix and the $k \times j$ zero matrix, respectively. Subscripts will be suppressed only when the order is clear from the context.

Let $U = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$, and note that $L + I$ can be written as

$$L+I = \left[\begin{array}{ccc|c|c} U & & & 0_{2p\times q} & -\mathbf{1}_{2p} \\ & \ddots & & & \\ & & U & & \\ \hline 0_{q\times 2p} & & & 2I_q & -\mathbf{1}_q \\ \hline -\mathbf{1}_{2p}^{\mathrm{T}} & & & -\mathbf{1}_q^{\mathrm{T}} & 2p+q+1 \end{array}\right].$$

Letting $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$, we have AUB $= \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} \equiv D$. It follows readily that $L + I$ is equivalent to the matrix

$$M = \left[\begin{array}{ccc|c|c} D & & & 0_{2p\times q} & -c \\ & \ddots & & & \\ & & D & & \\ \hline 0_{q\times 2p} & & & 2I_q & -\mathbf{1}_q \\ \hline -c^{\mathrm{T}} & & & -\mathbf{1}_q^{\mathrm{T}} & 2p+q+1 \end{array}\right],$$

where $c = \mathbf{1}_p \otimes \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. It is readily seen that M permutationally similar to

$$\left[\begin{array}{c|c|c|c} -I_p & 0 & 0 & -\mathbf{1}_p \\ \hline 0 & 8I_p & 0 & -4\mathbf{1}_p \\ \hline 0 & 0 & 2I_q & -\mathbf{1}_q \\ \hline -\mathbf{1}_p^{\mathrm{T}} & -4\mathbf{1}_p^{\mathrm{T}} & -\mathbf{1}_q^{\mathrm{T}} & 2p+q+1 \end{array}\right],$$

which is in turn equivalent to $I_p \oplus \widetilde{M}$, where $\widetilde{M}$ is given by

$$\widetilde{M} = \left[\begin{array}{c|c|c} 8I_p & 0 & -4\mathbf{1}_p \\ \hline 0 & 2I_p & -\mathbf{1}_q \\ \hline -4\mathbf{1}_p^{\mathrm{T}} & -\mathbf{1}_q^{\mathrm{T}} & 3p+q+1 \end{array}\right].$$

Adding twice the bottom row plus each of the first p rows of $\widetilde{M}$ to the $p+1$st row yields

$$\left[\begin{array}{c|c|c|c} 8I_p & 0 & 0 & -4\mathbf{1}_p \\ \hline 0^{\mathrm{T}} & 0 & -2\mathbf{1}_{q-1}^{\mathrm{T}} & 2p+2q+1 \\ \hline 0 & 0 & 2I_{q-1} & -\mathbf{1}_{q-1} \\ \hline -4\mathbf{1}_p^{\mathrm{T}} & -1 & -\mathbf{1}_{q-1}^{\mathrm{T}} & 3p+q+1 \end{array}\right],$$

which is in turn equivalent to $[1] \oplus \widehat{M}$, where

$$\widehat{M} = \left[\begin{array}{c|c|c} 8I_p & 0 & -4\mathbf{1}_p \\ \hline 0^{\mathrm{T}} & -2\mathbf{1}_{q-1}^{\mathrm{T}} & 2p+2q+1 \\ \hline 0 & 2I_{q-1} & -\mathbf{1}_{q-1} \end{array}\right].$$

Now $\widehat{M}$ is equivalent to

$$\left[\begin{array}{c|c|c} 8I_p & 0 & 0 \\ \hline 0^{\mathrm{T}} & -2\mathbf{1}_{q-1}^{\mathrm{T}} & 2p+2q+1 \\ \hline 0 & 2I_{q-1} & -\mathbf{1}_{q-1} \end{array}\right],$$

so the conclusion will follow once we discuss the matrix $\overline{M} = \left[\begin{array}{c|c} -2\mathbf{1}_{q-1}^{\mathrm{T}} & 2p+2q+1 \\ \hline 2I_{q-1} & -\mathbf{1}_{q-1} \end{array}\right]$. Using the second row of $\overline{M}$ to eliminate in the last column, we see that $\overline{M}$ is equivalent to

$$\left[\begin{array}{c|c|c} 2(2p+q+2) & 0^{\mathrm{T}} & 0 \\ \hline 2 & 0^{\mathrm{T}} & -1 \\ \hline -2\mathbf{1}_{q-2} & 2I_{q-2} & 0 \end{array}\right].$$

Using the last $q-1$ columns in this last matrix to eliminate in the first column, we arrive at

$$\left[\begin{array}{c|c|c} 2(2p+q+2) & 0^{\mathrm{T}} & 0 \\ \hline 0 & 0^{\mathrm{T}} & -1 \\ \hline 0 & 2I_{q-2} & 0 \end{array}\right],$$

which is evidently equivalent to

$$\left[\begin{array}{c|c|c} 2(2p+q+2) & 0^{\mathrm{T}} & 0 \\ \hline 0 & 2I_{q-2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

The conclusion now follows. □

Corollary 5.5. *Suppose that* $p_1, p_2, q_1, q_2 \in \mathbb{N}$ *with* $q_1, q_2 \geqslant 2$, *and* $2p_1 + q_1 \neq 2p_2 + q_2$. *Consider the family of graphs* $H(p_1 + i, q_1 - 2i, p_2 - i, q_2 + 2i),\ 0 \leqslant i \leqslant \min\{p_2 - 1, \frac{q_1-2}{2}\}$. *The members of this family are pairwise nonisomorphic, isospectral, have the same degree sequence, and the same Smith normal form.*

Proof. The fact that the graphs in this family are isospectral with the same degree sequence has already been observed; the fact that the members of this family are pairwise nonisomorphic follows readily from the hypothesis that $2p_1 + q_1 \neq 2p_2 + q_2$. It remains only to discuss the Smith normal form.

Fix $0 \leqslant i \leqslant \min\{p_2 - 1, \frac{q_1-2}{2}\}$, let L_1 and L_2 denote the Laplacian matrices for $G(p_1 + i, q_1 - 2i)$ and $G(p_2 - i, q_2 + 2i)$, respectively. A straightforward computation shows that the Laplacian matrix for $H(p_1 + i, q_1 - 2i, p_2 - i, q_2 + 2i)$ is equivalent to $M \equiv (L_1 + I) \oplus (L_2 + I) \oplus [0]$. Applying Lemma 5.4, it now follows that M is equivalent to a diagonal matrix whose diagonal entries consist of $1^{(p_1+p_2+4)},\ 2^{(q_1+q_2-4)},\ 8^{(p_1+p_2)},\ 2(2p_1 + q_1 + 2),\ 2(2p_2 + q_2 + 2)$, and 0. Thus, the Smith normal form for M is independent of i, and the result follows. $\square$

6. Threshold graphs

The class of *threshold graphs* can be characterized as the graphs having no vertex-induced subgraphs isomorphic to either P_4, C_4, or $K_2 \cup K_2$ (see [11]), and so by Theorem 2.2, any threshold is constructably Laplacian integral. In this section we make a few remarks on this class of graphs.

The following result of Hammer and Kelmans describes the spectrum of a threshold graph in terms of its degree sequence. We note that an alternate description of the spectrum of a threshold graph can be found in [9].

Proposition 6.1 [5]. *Let* G *be a connected graph on* $n \geqslant 2$ *vertices having degree sequence* $d_1^{(k_1)} > d_2^{(k_2)} > \cdots > d_m^{(k_m)}$. *Then* G *is a threshold graph if and only if one of the following holds*:

(i) m *is even, and the Laplacian spectrum of* G *is given by*: $(d_i + 1)^{(k_i)},\ i = 1, \ldots, \frac{m}{2}$; $d_{\frac{m+2}{2}}^{(k_{\frac{m+2}{2}}-1)}$; $d_i^{(k_i)},\ i = \frac{m+4}{2}, \ldots, m$; *and* 0.

(ii) m *is odd, and the Laplacian spectrum of* G *is given by*: $(d_i + 1)^{(k_i)},\ i = 1, \ldots, \frac{m-1}{2}$; $(d_{\frac{m+1}{2}} + 1)^{(k_{\frac{m+1}{2}}-1)}$; $d_i^{(k_i)},\ i = \frac{m+3}{2}, \ldots, m$; *and* 0.

Remark 6.2. Let G be a constructably Laplacian integral graph. From Theorem 3.1, we see that for each vertex of G, say of degree d, either d or $d + 1$ is an eigenvalue for the corresponding Laplacian matrix. In the special case that G is a threshold graph, a partial converse holds: we find from Proposition 6.1 that for each nonzero Laplacian eigenvalue λ of G, there is a vertex, say of degree d, such that λ is either d or $d + 1$.

Theorem 6.3. *Let* G *be a threshold graph on* n *vertices, and let* $G_0, G_1, \ldots, G_k$ *be a sequence of graphs such that* $G_0 = O_n$, $G_k = G$, *each* G_i *is Laplacian integral, and for each* $i = 0, \ldots,$

$k-1$, G_{i+1} is formed from G_i by the addition of an edge. Then for each $i=0,\dots,k$, the graph G_i is a threshold graph.

Proof. Consider the sequence of graphs $G_0, G_1, \dots, G_k$. We claim that each G_i is a threshold graph. To see the claim, observe that G_k is constructed from G_{k-1} by the addition of a single edge. Suppose that G_{k-1} is not a threshold graph; since G_{k-1} is itself constructably Laplacian integral, we deduce from Theorem 2.2 that G_{k-1} must contain a vertex-induced $K_2 \cup K_2$, say on vertices $1, 2, 3$ and 4. If the edge e added into G_{k-1} to construct G_k is incident with at most one of vertices $1, 2, 3, 4$, then G_k also has a vertex-induced $K_2 \cup K_2$ and so is not a threshold graph, contrary to our hypothesis. Hence both end points of e are in $\{1, 2, 3, 4\}$, and it follows that G contains a vertex-induced P_4, also contrary to our hypothesis. We conclude that necessarily G_{k-1} is also a threshold graph. Iterating the claim above, we find that each of $G_0, G_1, \dots, G_k$ is a threshold graph. □

Remark 6.4. As noted above, in [8] it is shown that a graph is Laplacian integrally completable if and only if it has no vertex-induced P_4 subgraphs, and no vertex-induced $K_2 \cup K_2$ subgraphs. Thus we see that a graph G is both constructably Laplacian integral and Laplacian integrally completable if and only if it has no vertex-induced subgraphs equal to either P_4, C_4 or $K_2 \cup K_2$ – i.e. if and only if G is a threshold graph.

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