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# The Laplacian spectral radius of some graphs☆

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ABSTRACT

The Laplacian spectral radius of a graph is the largest eigenvalue of the associated Laplacian matrix. In this paper, we determine those graphs which maximize the Laplacian spectral radius among all bipartite graphs with (edge-)connectivity at most $k$. We also characterize graphs of order $n$ with $k$ cut-edges, having Laplacian spectral radius equal to $n$.



## 1. Introduction

Let $G$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For $v \in V(G)$, let $N_G(v)$ (or $N(v)$ for short) be the set of vertices which are adjacent to $v$ in $G$ and let $d(v) = |N(v)|$ be the degree of $v$. The maximum degree of $G$ is denoted by $\triangle(G)$. For any $e \in E(\overline{G})$, where $\overline{G}$ is the complement of the graph $G$, we use $G + e$ to denote the graph obtained by adding $e$ to $G$. Similarly, for any set $W$ of vertices (edges), $G - W$ and $G + W$ are the graphs obtained by deleting the vertices (edges) in $W$ from $G$ and by adding the vertices (edges) in $W$ to $G$, respectively. If $G$ is connected and $G - W$ is disconnected, then we say that $W$ is a *w-vertex (-edge) cut* of $G$, where $w = |W|$. For any nonempty subset $V_1$ of $V(G)$, the subgraph of $G$ induced by $V_1$ is denoted by $G[V_1]$. Readers are referred to [1] for undefined terms.

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Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of $G$, respectively. The *Laplacian matrix* of $G$ is defined as $L(G) = D(G) - A(G)$. It is easy to see that $L(G)$ is a symmetric positive semidefinite matrix having 0 as an eigenvalue. Thus, the eigenvalues $\mu_i(G)$'s of $L(G)$ (or the Laplacian eigenvalues of $G$) satisfy

$$\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) = 0,$$

repeated according to their multiplicities. We also call $\mu_1(G)$ and $\mu_{n-1}(G)$ the *Laplacian spectral radius* (denoted by $\mu(G)$) and the *algebraic connectivity* (denoted by $\alpha(G)$) of the graph $G$, respectively. It is known that a graph is connected if and only if its algebraic connectivity is different from zero. Since $L(G) + L(\overline{G}) = nI - J$, where $I$ and $J$ denote respectively the identity matrix and the matrix all of whose entries being equal to 1, $\mu_i(G) = n - \mu_{n-i}(\overline{G})$ for $1 \leqslant i \leqslant n - 1$. In particular, $\mu(G) = n - \alpha(\overline{G})$ and the following corollary is immediate.

**Corollary 1.1.** *Let $G$ be a graph of order $n$. Then $\mu(G) \leqslant n$ with the equality if and only if $\overline{G}$ is disconnected.*

The Laplacian spectral radius of graphs is an important topic in the theory of graph spectra, not only because of its relations to numerous graph invariants (e.g., connectivity, expanding property, isoperimetric number, maximum cut, independence number, genus, diameter, mean distance, and bandwidth-type parameters of a graph), but also because of its applications in diverse disciplines (e.g., theoretical chemistry, combinatorial optimization and communication networks). See, for example, [4,9] and the references therein for a sample of results. Recently, the problem of determining graphs having maximum or minimum Laplacian spectral radius among given classes of graphs has received a good deal of attention. Hong and Zhang [8] determined the tree with maximum Laplacian spectral radius among all trees with given number of pendant edges. Guo [5,7] determined the trees with maximum Laplacian spectral radius among all trees with given diameter and independence number, respectively. Zhang and Zhang [12] determined the graphs with maximum Laplacian spectral radius among the bipartite graphs with $k$ cut-edges and the bipartite bicyclic graphs, respectively. Yu and Lu [10] determined the tree with maximum Laplacian spectral radius among all trees with given maximum degree. Zhang [11] characterized all extremal trees with the maximum Laplacian spectral radius in the set of all trees with a given degree sequence.

Motivated by the above mentioned recent work, we consider in the present paper the problem of determining those graphs which maximize the Laplacian spectral radius over some special classes of graphs.

This paper is organized as follows: in Section 2, we present some useful lemmas. In Section 3, we determine the graph which maximizes the Laplacian spectral radius among all bipartite graphs with (edge-)connectivity at most $k$. In Section 4, we characterize all graphs of order $n$ with $k$ cut-edges and whose Laplacian spectral radius equal to $n$.

## 2. Preliminaries

In this section, we present some lemmas, which will be useful in the subsequent sections.

**Lemma 2.1** ([3]). *Let $G$ be a connected bipartite graph and $H$ be a subgraph of $G$. Then $\mu(H) \leqslant \mu(G)$, with the equality holds if and only if $G = H$.*

**Lemma 2.2** ([6]). *Let $u$ and $v$ be two vertices of a connected bipartite graph $G = (V_1, V_2, E)$. Suppose that $v_1, v_2, \ldots, v_s$ $(1 \leqslant s \leqslant d(v))$ are some vertices of $N(v)\backslash N(u)$ different from $u$. Let $\mathbf{x}$ be a unit eigenvector of $L(G)$ corresponding to $\mu(G)$, and let $G^*$ be the graph obtained from $G$ by deleting the edges $vv_i$ and adding the edges $uv_i$ $(1 \leqslant i \leqslant s)$. If $|x_u| \geqslant |x_v|$ and $G^*$ is also a bipartite graph, then $\mu(G^*) > \mu(G)$.*

**Lemma 2.3** ([2]). *Let $G$ be a connected graph of order $n$. Then,*

$$\mu(G) \leqslant \max\{d(u) + d(v) - |N(u) \cap N(v)| : uv \in E(G)\}.$$

*Hence this upper bound for $\mu(G)$ does not exceed $n$.*

## 3. Graphs with connectivity at most $k$

Let $k \geqslant 1$. We say that a graph $G$ is *k-connected* if either $G$ is the complete graph $K_{k+1}$, or $G$ has at least $k+2$ vertices and contains no $(k-1)$-vertex cut. Similarly, $G$ is *k-edge-connected* if it has at least two vertices and does not contain $(k-1)$-edge cut. The maximum value of $k$ for which a connected graph $G$ being $k$-connected is the *connectivity* of $G$, denoted by $\kappa(G)$. If $G$ is disconnected, we define $\kappa(G)=0$. The *edge-connectivity* $\kappa'(G)$ is defined analogously. We remark that if $G$ is of order $n$, then

(1) $\kappa(G) \leqslant \kappa'(G) \leqslant n-1$, and
(2) the three statements $\kappa(G)=n-1, \kappa'(G)=n-1$ and $G \cong K_n$ are equivalent.

We denote by $\mathcal{V}_n^k$ the set of graphs of order $n$ with $\kappa(G) \leqslant k \leqslant n-1$, and by $\mathcal{E}_n^k$ the set of graphs of order $n$ with $\kappa'(G) \leqslant k \leqslant n-1$. Let $\mathcal{V}_n^{k+}$ and $\mathcal{E}_n^{k+}$ denote the set of bipartite graphs of order $n$ with $\kappa(G) \leqslant k$ and $\kappa'(G) \leqslant k$, respectively.

**Theorem 3.1.** *Among all graphs $G$ in $\mathcal{V}_n^{k+}$, the maximum Laplacian spectral radius is attained uniquely when $G \cong K_{n-k,k}$.*

**Proof.** We have to prove that for every $G \in \mathcal{V}_n^{k+}, \mu(G) \leqslant \mu(K_{n-k,k})$, and the equality holds if and only if $G \cong K_{n-k,k}$. Let $G^*$ with $V(G^*)=\{v_1,v_2,\ldots,v_n\}$ be the graph having the maximum Laplacian spectral radius among all graphs in $\mathcal{V}_n^{k+}$, i.e. $\mu(G) \leqslant \mu(G^*)$ for every $G \in \mathcal{V}_n^{k+}$. Denote the unit eigenvector corresponding to $\mu(G^*)$ by $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, where $x_i$ corresponds to the vertex $v_i$ $(i=1,2,\ldots,n)$. Since $G^* \in \mathcal{V}_n^{k+}, G^*$ has a $k$-vertex cut. Without loss of generality, we may let $V_1=\{v_1,v_2,\ldots,v_k\}$ be a $k$-vertex cut of $G^*$. We first prove the following two claims.

**Claim 1.** All the components of $G^*-V_1$ are complete bipartite graphs.

Since $G^* \in \mathcal{V}_n^{k+}$, all the components of $G^*-V_1$ are bipartite graphs. Suppose on the contrary that $G_1=(X,Y)$ is a non-complete bipartite component of $G^*-V_1$. Then we can add one edge from $u \in X$ to $v \in Y$ to form a new graph $G^*+uv$. It is obvious that $V_1$ is also a $k$-vertex cut of $G^*+uv$, i.e. $G^*+uv \in \mathcal{V}_n^{k+}$. By Lemma 2.1, we have $\mu(G^*)<\mu(G^*+uv)$, which contradicts the maximality of $G^*$.

Thus, we may assume that all components of $G^*-V_1$ are complete bipartite graphs, say, $K_{m_1,n_1}$, $K_{m_2,n_2},\ldots,K_{m_t,n_t}$, and let $a_i=m_i+n_i$ for $1 \leqslant i \leqslant t$. Then $k+\sum_{i=1}^t a_i=n$.

**Claim 2.** At most one of $a_1,a_2,\ldots,a_t$ is greater than 1.

Otherwise, for some $1 \leqslant i<j \leqslant t$, we have $a_i>1$ and $a_j>1$. Let $v_i \in K_{m_i,n_i}, v_j \in K_{m_j,n_j}$, and $v_i$, $v_j \in N(V_1)$. If $N_{G^*}(v_i) \backslash V_1=\{v_{i1},v_{i2},\ldots,v_{in_i}\}$ and $N_{G^*}(v_j) \setminus V_1=\{v_{j1},v_{j2},\ldots,v_{jn_j}\}$, we let

$$G=\begin{cases} G^*-\{v_jv_{j1},v_jv_{j2},\ldots,v_jv_{jn_j}\}+\{v_iv_{j1},v_iv_{j2},\ldots,v_iv_{jn_j}\} & \text{if } |x_i| \geqslant |x_j|;\\ G^*-\{v_iv_{i1},v_iv_{i2},\ldots,v_iv_{in_i}\}+\{v_jv_{i1},v_jv_{i2},\ldots,v_jv_{in_i}\} & \text{otherwise.}\end{cases}$$

It is clear that $G \in \mathcal{V}_n^{k+}$ so that by Lemma 2.2, we have $\mu(G^*)<\mu(G)$, which contradicts the maximality of $G^*$.

Since $G^* \in \mathcal{V}_n^{k+}$, combining Claims 1 and 2, we have that $G[V_1]=kK_1$. If $a_i=1, 1 \leqslant i \leqslant t$, then $G^* \cong K_{n-k,k}$; if exactly one of $a_1,a_2,\ldots,a_t$ is greater than 1, say $a_1>1$, then $G^*$ must be the graph $K_n^*$ shown in Fig. 1. By Corollary 1.1, we then have $\mu(K_n^*)<\mu(K_{n-k,k})=n$. □

Since $K_{n-k,k} \in \mathcal{E}_n^{k+} \subseteq \mathcal{V}_n^{k+}$, the following corollary is immediate.

**Corollary 3.2.** *Among all graphs in $\mathcal{E}_n^{k+}$, the maximum Laplacian spectral radius is attained uniquely at $K_{n-k,k}$.*

We refigure the graph $K_{n-k,k}$ as shown in Fig. 2, where $|V_1|=k, |V_2|=r>0$ and $|V_3|=n-k-r>0$. It is clear that when edges are added to join vertices within $V_1$, vertices within $V_2$, and those within

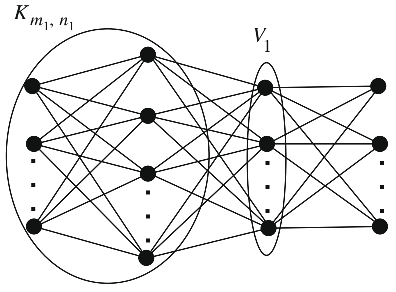


Fig. 1. Bipartite graph: $K_n^*$.

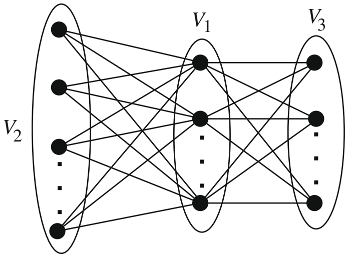


Fig. 2. Complete bipartite graph: $K_{n-k,\,k}$.

$V_3$, the Laplacian spectral radius of the resulting graph will not be affected. Hence we may have the following lemma.

**Lemma 3.3.** *Let $K_{n-k,k}^{t_i,l_i,m_i}$ be the graph obtained from $K_{n-k,k}$ depicted in Fig. 2 by adding $t_i$ edges joining vertices within $V_1, l_i$ edges joining those within $V_2$, and $m_i$ edges joining those within $V_3$, where $0 \leqslant t_i \leqslant \frac{k(k-1)}{2}, 0 \leqslant l_i \leqslant \frac{r(r-1)}{2}$ and $0 \leqslant m_i \leqslant \frac{(n-r-k)(n-r-k-1)}{2}$. Then $K_{n-k,k}^{t_i,l_i\,m_i} \in \mathcal{V}_n^k$ and $\mu(K_{n-k,k}^{t_i,l_i\,m_i}) = n$.*

**Remark 3.1.** Lemma 3.3 exemplifies some graphs in $\mathcal{V}_n^k$ having Laplacian spectral radius equal to $n$. The problem of characterizing all graphs in $\mathcal{V}_n^k$ having Laplacian spectral radius equal to $n$ seems to be difficult.

## 4. Graphs with $k$ cut edges

Zhang and Zhang [12] had determined the graph with maximum Laplacian spectral radius among all bipartite graphs of order $n$ with $k$ cut-edges. In this section, we study the corresponding maximality problem among all simple graphs of order $n$ with $k$ cut-edges.

Let $\mathcal{G}_n^k$ be the set of all simple graphs with $n$ vertices and $k$ cut-edges. It is noted that $k$ must be less than or equal to $n-1$. Let $\mathfrak{B}_n^k$ be the set of graphs of order $n$ having maximum degree $n-1$ and exactly $k$ pendant edges. It is clearly that $\mathfrak{B}_n^k \subset \mathcal{G}_n^k$. Thus, we have the following lemma.

**Lemma 4.1.** *Let $G$ be a graph containing at least one cut-edge $e$, i.e., $G - e = G_1 \cup G_2$, where $G_1, G_2$ are two connected components of $G - e$. If $\overline{G}$ is disconnected, then $e$ is a pendant edge of $G$, i.e., $|G_1| = 1$ or $|G_2| = 1$.*

**Theorem 4.2.** *Let $G \in \mathcal{G}_n^k$ with $1 \leqslant k \leqslant n-1$. Then $\mu(G) = n$ if and only if $G \in \mathfrak{B}_n^k$.*

**Proof.** By Corollary 1.1, we only need to prove that if $G \notin \mathfrak{B}_n^k$, then $\overline{G}$ is connected, and hence $\mu(G) = n$. Let $E_1 = \{e_1, e_2, \ldots, e_k\}$ be the set of cut-edges of $G$. By Lemma 4.1, we may assume that every $e_i \in E_1$ is a pendant edge.

We next prove that all $k$ cut-edges are attached to a common vertex $u$. Otherwise, there exist two pendant edges $u'v' = e_i$, $x'y' = e_j \in E_1$. By Lemma 4.1, we may assume that $d(v') = d(y') = 1$.

If $u'x' \notin E(G)$, then it is easy to check that for each $e = uv \in E(G), d(u) + d(v) - |N(u) \cap N(v)| = |N(u) \cup N(v)| \leqslant n - 1$.

If $u'x' \in E(G)$, then one readily checks that for each $uv \in E(G) \setminus \{u'x'\}, |N(u) \cup N(v)| \leqslant n-1$. For $|N(u') \cup N(x')| \leqslant n$, the equality holds if and only if for every $w \in V(G) \setminus \{u', x'\}, w \in N(u') \cup N(x')$. Therefore it is obvious that when $|N(u') \cup N(x')| = n, \overline{G}$ is connected.

But then, Corollary 1.1 and Lemma 2.3 together imply that $\mu(G) < n$, which is a contradiction.

Thus, all $k$ cut-edges are attached to a common vertex $u$ so that $d(u) = n-1$. Since $G \in \mathcal{G}_n^k$, edges in $E(G) \setminus E_1$ must be those in a triangle (a cycle of length 3). Hence $G \in \mathfrak{B}_n^k$. finishing the proof. □

**Remark 4.1.** We have found in Theorem 4.2 all graphs of order $n$ with $k$ cut-edges with Laplacian spectral radius equal to $n$.

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