



Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Bounds on the index of the signless Laplacian of a graph

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ARTICLE INFO

Article history:

Received 23 October 2007

Received in revised form 13 February 2009

Accepted 12 June 2009

Available online xxxx

Keywords:

Index

Signless Laplacian

Bounds

Degrees

Average degree of neighbors

ABSTRACT

Let $G = (V, E)$ be a simple, undirected graph of order n and size m with vertex set V , edge set E , adjacency matrix A and vertex degrees $\Delta = d_1 \geq d_2 \geq \dots \geq d_n = \delta$. The average degree of the neighbor of vertex v_i is $m_i = \frac{1}{d_i} \sum_{j=1}^n a_{ij} d_j$. Let D be the diagonal matrix of degrees of G . Then $L(G) = D(G) - A(G)$ is the Laplacian matrix of G and $Q(G) = D(G) + A(G)$ the signless Laplacian matrix of G . Let $\mu_1(G)$ denote the index of $L(G)$ and $q_1(G)$ the index of $Q(G)$. We survey upper bounds on $\mu_1(G)$ and $q_1(G)$ given in terms of the d_i and m_i , as well as the numbers of common neighbors of pairs of vertices. It is well known that $\mu_1(G) \leq q_1(G)$. We show that many but not all upper bounds on $\mu_1(G)$ are still valid for $q_1(G)$.

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1. Introduction

Let $G = (V, E)$ be a simple undirected graph with n vertices $v_i \in V$ and m edges $\{v_i, v_j\} \in E$, for $i, j = 1, 2, \dots, n$ and $i \neq j$. When v_i is adjacent to v_j , we denote this fact by $v_i \sim v_j$. The vertex degree of v_i is d_i , and the degree sequence of G is $d(G) = (d_1, d_2, \dots, d_n)$, where, possibly after relabeling, $\Delta = d_1 \geq d_2 \geq \dots \geq d_n = \delta$. The average degree of G is $\bar{d} = \frac{\sum_{i=1}^n d_i}{n}$ and the average degree of the neighbors of v_i is $m_i = \frac{1}{d_i} \sum_{v_j \sim v_i} d_j$. The eigenvalues of G are the eigenvalues of the adjacency matrix $A(G)$, given as $\lambda_1 \geq \dots \geq \lambda_{n-1} \geq \lambda_n$, where, λ_1 is called the index of G . Consider $D(G)$ as the diagonal matrix of vertex degrees of G . The Laplacian matrix of G is $L(G) = D(G) - A(G)$, its eigenvalues are displayed as $\mu_1 \geq \dots \geq \mu_{n-1} \geq \mu_n$ and μ_1 is the index of Laplacian matrix. Since $L(G)$ and $A(G)$ are well known, there are many results on their spectra, see e.g. [2, 10, 14].

The matrix $Q(G)$ was introduced in the classical book of Cvetković, Doob and Sachs on "Spectra of Graphs" [5], but without a name being given to it at that time. Later it was called "quasi-Laplacian matrix" (essentially by Chinese researchers) and more recently "signless Laplacian" [4, 6]. As a consequence, papers in which $Q(G)$ plays a role belong to several categories:

- (i) papers explicitly devoted to $Q(G)$, called by either of its two names;
- (ii) papers devoted wholly or in part to $Q(G)$, without giving it a name;
- (iii) papers in which $Q(G)$ plays a role (which may be an essential one in some proofs) but is not studied *per se*.

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Several researchers have observed that

$$\mu_1(G) \leq q_1(G). \tag{1}$$

Moreover, Yan [18] derived from the Courant–Weyl inequalities the relation

$$2\lambda_1(G) \leq q_1(G) \tag{2}$$

where λ_1 is the index of the adjacency matrix of G . These relations immediately imply that any lower bound on $\mu_1(G)$ is a valid lower bound on $q_1(G)$ and that doubling any lower bound on $\lambda_1(G)$ also yields a valid lower bound on $q_1(G)$. For instance, the relation $\lambda_1(G) \geq \bar{d}$ [5] implies $q_1(G) \geq 2\bar{d} \geq 2\delta$. Similarly, upper bounds on $q_1(G)$ yield valid upper bounds on $\mu_1(G)$ and $\lambda_1(G)$. But when are upper bounds on $\mu_1(G)$ or twice upper bounds on λ_1 valid bounds on $q_1(G)$?

The purpose of the present paper is to answer that question for a series of bounds on $\mu_1(G)$ expressed in terms of d_i and m_i as well as the number of common neighbors for pairs of vertices. Bounds which remain valid are considered in Section 2 and bounds which do not in Section 3.

2. Bounds on $\mu_1(G)$ in terms of vertex degrees and average degrees of neighbors valid for $q_1(G)$

A number of upper bounds on μ_1 given as functions of the degree and of the average degree of the neighbors of a vertex have been proposed in the literature. Brankov et al. [3] gathered some of them and classified them into two classes:

(1) Upper bounds depending on the vertices v_i , the d_i and m_i :

$$\mu_1 \leq \max_{v_i} f(d_i, m_i).$$

These bounds are:

$$\mu_1(G) \leq \max\{2d_i | v_i \in V(G)\}, \tag{3}$$

$$\mu_1(G) \leq \max\{d_i + m_i | v_i \in V(G)\}, \quad [15] \tag{4}$$

$$\mu_1(G) \leq \max\{d_i + \sqrt{d_i m_i} | v_i \in V(G)\}, \quad [19] \tag{5}$$

$$\mu_1(G) \leq \max\{\sqrt{2d_i(d_i + m_i)} | v_i \in V(G)\}, \quad [12] \tag{6}$$

$$\mu_1(G) \leq \max\left\{\frac{d_i + \sqrt{(d_i)^2 + 8(d_i m_i)}}{2} | v_i \in V(G)\right\} \quad [11]. \tag{7}$$

(2) Upper bounds depending on the edges $\{v_i, v_j\}$, the d_i and d_j , as well as the m_i and m_j :

$$\mu_1 \leq \max_{v_i \sim v_j} f(d_i, m_i, d_j, m_j).$$

Examples of such bounds are:

$$\mu_1(G) \leq \max_{v_i \sim v_j} \{d_i + d_j\}, \quad [1] \tag{8}$$

$$\mu_1(G) \leq \max_{v_i \sim v_j} \left\{ \frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j} \right\}, \quad [13] \tag{9}$$

$$\mu_1(G) \leq \max_{v_i \sim v_j} \left\{ \sqrt{d_i(d_i + m_i) + d_j(d_j + m_j)} \right\}, \quad [19] \tag{10}$$

$$\mu_1(G) \leq \max_{v_i \sim v_j} \left\{ 2 + \sqrt{d_i(d_i + m_i - 4) + d_j(d_j + m_j - 4) + 4} \right\}, \quad [19] \tag{11}$$

$$\mu_1(G) \leq \max_{v_i \sim v_j} \left\{ \frac{(d_i + d_j) + \sqrt{(d_i - d_j)^2 + 4m_i m_j}}{2} \right\} \quad [7,20]. \tag{12}$$

Remark. It is easy to see that the bound given by the inequality (5) is better than that given by (6).

Let us consider first the series of bounds (3)–(7). The upper bound (3) was proved to be valid for $q_1(G)$ by several researchers e.g. [6], and the bound (4) also by Das [8] in a section devoted to the signless Laplacian without giving it a name. Based on a simple technique suggested in [3] we next prove that the bound (7) is also valid for $q_1(G)$.

Theorem 1. *Let G be a simple and connected graph. Then*

$$q_1(G) \leq \max_{v_i} 2d_i \tag{13}$$

and

$$q_1(G) \leq \max_{v_i} \frac{d_i + \sqrt{(d_i)^2 + 8(d_i m_i)}}{2}. \tag{14}$$

Proof. Let $x = (x_1, x_2, \dots, x_n)$ be a non-negative eigenvector corresponding to the eigenvalue q_1 . So,

$$q_1 x = Q(G)x = Dx + Ax.$$

Let $x_i = \max_{1 \leq j \leq n} \{x_j\}$. Then,

$$q_1 x_i = d_i x_i + \sum_{v_j \sim v_i} x_j \leq d_i x_i + \sum_{v_j \sim v_i} x_i = 2d_i x_i$$

from where the first bound follows.

Similarly, we have

$$q_1^2 x = Q^2 x = (D + A)^2 x = D^2 x + DAx + ADx + A^2 x,$$

from where it follows that

$$q_1^2 x_i = (d_i)^2 x_i + d_i \sum_{v_j \sim v_i} x_j + \sum_{v_j \sim v_i} d_j x_j + \sum_{v_j \sim v_i} \sum_{v_k \sim v_j} x_k.$$

In order to prove the inequality (14), we consider a simple quadratic function of q_1 :

$$(q_1^2 + b q_1) x = (D^2 x + DAx + ADx + A^2 x) + b(Dx + Ax).$$

Since $x_i = \max_{1 \leq j \leq n} \{x_j\}$, it follows that

$$(q_1^2 + b q_1) x_i = (d_i)^2 x_i + d_i \sum_{v_j \sim v_i} x_j + \sum_{v_j \sim v_i} d_j x_j + \sum_{v_j \sim v_i} \sum_{v_k \sim v_j} x_k + b \left(d_i x_i + \sum_{v_j \sim v_i} x_j \right).$$

It is easy to see that the inequalities below are true

$$d_i \sum_{v_j \sim v_i} x_j \leq d_i^2 x_i; \quad \sum_{v_j \sim v_i} x_j \leq d_i x_i; \quad \sum_{v_j \sim v_i} d_j x_j \leq d_i m_i x_i; \quad \sum_{v_j \sim v_i} \sum_{v_k \sim v_j} x_k \leq d_i m_i x_i,$$

and we get

$$(q_1^2 + b q_1) x_i \leq 2d_i(d_i + m_i) x_i + 2bd_i x_i,$$

provided that $d_i + b \geq 0$.

Hence,

$$q_1^2 + b q_1 - 2d_i(d_i + m_i + b) \leq 0.$$

As $q_1 \geq 0$, we have

$$q_1 \leq \frac{-b + \sqrt{b^2 + 8d_i(d_i + m_i + b)}}{2}.$$

From the inequality above, for every different values to b , we can get several distinct upper bounds. In particular, if $b = -d_i$, we get the bound (14):

$$q_1 \leq \max_{v_i} \left\{ \frac{d_i + \sqrt{(d_i)^2 + 8d_i m_i}}{2} \right\}. \quad \square$$

Finally, using a technique to prove bounds on $\mu_1(G)$ due to Zhang [19], we show that (5) is also satisfied by $q_1(G)$.

Theorem 2. Let G be a simple and connected graph. Then

$$q_1(G) \leq \max \left\{ d_i + \sqrt{d_i m_i} \mid v_i \in V(G) \right\}. \tag{15}$$

Moreover, if G is a k -regular, bipartite regular or semi-regular graph the equality holds.

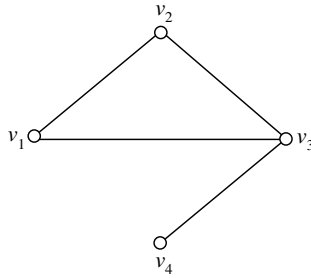


Fig. 1. Graph G.

Proposition 3. Let G be a simple and connected graph. Then,

$$\mu_1(G) \leq \max\{d_i + d_j - |N_i \cap N_j| : 1 \leq v_i < v_j \leq n\},$$

where $|N_i \cap N_j|$ is the number of common neighbors of v_i and v_j .

Proposition 4. Let G be a simple and connected graph. Then,

$$\mu_1(G) \leq \max\{d_i + d_j - |N_i \cap N_j| : 1 \leq v_i < v_j \leq n, v_i v_j \in E\},$$

where $|N_i \cap N_j|$ is the number of common neighbors of v_i and v_j .

Proposition 5. Let G be a simple and connected graph. Then,

$$\mu_1(G) \leq \max \left\{ \sqrt{2d_i(d_i + m'_i)} : v_i \in V \right\},$$

where $m'_i = \frac{\sum_{v_j v_i \in E} (d_j - |N_i \cap N_j|)}{d_i}$ and $|N_i \cap N_j|$ is the number of common neighbors of v_i and v_j .

Proposition 6. Let G be a simple and connected graph. Then,

$$\mu_1(G) \leq \max \left\{ \frac{d_i + \sqrt{(d_i)^2 + 8d_i m'_i}}{2} : v_i \in V \right\},$$

where $m'_i = \frac{\sum_{v_j v_i \in E} (d_j - |N_i \cap N_j|)}{d_i}$. Moreover, the equality holds if and only if G is a bipartite regular graph.

The graph G displayed in Fig. 1 suffices to show that all bounds on the index of the Laplacian matrix given in Propositions 3–6 do not hold for the index of the signless Laplacian. Indeed,

- $q_1(G) = 4, 5616$;
- $\max\{d_i + d_j - |N_i \cap N_j| : 1 \leq v_i < v_j \leq n\} = 4$;
- $\max\{d_i + d_j - |N_i \cap N_j| : 1 \leq v_i < v_j \leq n, \{v_i, v_j\} \in E\} = 4$;
- $\max \left\{ \sqrt{2d_i(d_i + m'_i)} : v_i \in V \right\} = 4, 4721$, and finally,
- $\max \left\{ \frac{d_i + \sqrt{(d_i)^2 + 8d_i m'_i}}{2} : v_i \in V \right\} = 4, 3723$.

Acknowledgements

The Brazilian authors are indebted to CNPq (Brazilian Council for the Scientific and Technological Development) for all the support received for this research. Work of the fourth author was done in part during a visit to COPPE, Rio de Janeiro in December 2006, whose kind hospitality is gratefully acknowledged.

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