Linear Algebra and its Applications 432 (2010) 499–504

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

The minimum Laplacian spread of unicyclic graphs ☆

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ARTICLE INFO

Article history:
Received 9 June 2009
Accepted 27 August 2009
Available online 23 September 2009

Submitted by R.A. Brualdi

AMS classification:
15A18
05C50

Keywords:
Laplacian spread
Unicyclic graphs
Algebraic connectivity

ABSTRACT

The Laplacian spread of a graph [1] is defined as the difference between the largest eigenvalue and the second-smallest eigenvalue of the associated Laplacian matrix. In this paper, the minimum Laplacian spread of unicyclic graphs with given order is determined.

1. Introduction

Let G be a simple graph with n vertices and m edges. A connected graph is called a unicyclic graph if $m = n$. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and the maximum degree, respectively. Let $D = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is defined as $L = D - A$, where A is the adjacency matrix of G. The Laplacian spectrum of G is the spectrum of its Laplacian matrix, and consists of the values $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$. Especially, $\mu_{n-1}(G) > 0$ if and only if G is connected. Fiedler [6] called $\mu_{n-1}(G)$ (or $\alpha(G)$) the algebraic connectivity of G.

The Laplacian spread of a graph G is defined as [1]

$$LS(G) = \mu_1 - \mu_{n-1}.$$

Fan et al. [1] showed that the star is the unique tree with maximum Laplacian spread, and the path is the unique one with minimum Laplacian spread among all trees of given order.

☆ This work was supported by the NNSF of China (No. 10771080) and by DF of the Ministry of Education (No. 20070574006).

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2009.08.027

Li et al. [8] determined the unicyclic graph with maximum Laplacian spread of given order not less than 10. In a recent work [9], using different way, Bao et al. have demonstrated that the unique unicyclic graph with maximum Laplacian spread among all unicyclic graphs of fixed order, which is obtained from a star by adding one edge between two pendant vertices.

As a consequence, in this paper we characterize the unique unicyclic graph with minimum Laplacian spread among all connected unicyclic graphs of given order.

2. Main results

The following properties of the Laplacian eigenvalues can be found in [2].

The Laplacian eigenvalues of graph G and its complement $\overline{G}$ (or G^c) have the relation $\mu_i(G) = n - \mu_{n-i}(\overline{G}), i = 1, 2, \ldots, n-1$.

Lemma 2.1 [2]. *Let G be an n-vertex graph with at least one edge and maximum vertex degree Δ. Then $\mu_1 \geqslant 1 + \Delta$ with equality for connected graph if and only if $\Delta = n - 1$.*

By the definition of Laplacian spread and the property of the complement, it is easy to check that [3] $LS(G) = \mu_1(G) + \mu_1(\overline{G}) - n$. By Lemma 2.1, Liu and You [3] obtained:

Lemma 2.2 [3]. *Let G be an n-vertex graph with minimum degree δ and maximum degree Δ. Then $LS(G) \geqslant \Delta - \delta + 1$.*

Lemma 2.3. *Let C_n be a cycle with n vertices. Then $LS(C_n) < 4$.*

Proof. Note that $\mu_1(C_n) \leqslant 4$ and the equality holds if and only if n is even. Thus we have $LS(C_n) \leqslant 4 - \mu_{n-1}(C_n) < 4$. □

Theorem 2.4. *Let G be an n-vertex unicyclic graph with $\Delta \geqslant 4$. Then $LS(G) \geqslant 4 > LS(C_n)$.*

Proof. Let G be a unicyclic graph with $\Delta \geqslant 4$. Then $\delta(G) = 1$. By Lemmas 2.2 and 2.3, we have $LS(G) \geqslant 4 - 1 + 1 = 4 > LS(C_n)$. □

Following, we assume that G is a unicyclic graph with $\Delta = 3$.

Lemma 2.5 [4]. *For $e \notin E(G)$, the Laplacian eigenvalues of G and $G' = G + e$ interlace, i.e., $\mu_1(G') \geqslant \mu_1(G) \geqslant \mu_2(G') \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G') = \mu_n(G) = 0$.*

Remark. In Lemma 2.5, if v is a pendant vertex of G and e the pendant edge incident with v, then $\mu_1(G - v) = \mu_1(G - e) \leqslant \mu_1(G)$.

From Lemma 2.5, Liu et al. [5] obtained the following result.

Lemma 2.6 [5]. *Let G be a connected graph on n vertices. If v is a pendant vertex of G, then $\mu_i(G) \leqslant \mu_{i-1}(G - v), 2 \leqslant i \leqslant n$. Particularly, $\alpha(G) \leqslant \alpha(G - v)$.*

Lemma 2.7. *Let G be an n-vertex unicyclic graph with $\Delta = 3$. Then $LS(G) \geqslant LS(G - v)$, where v is a pendant vertex.*

Proof. Let G be a unicyclic graph with $\Delta = 3$. Then $\delta(G) = 1$. Let v be a pendant vertex. By Lemmas 2.5 and 2.6, $\mu_1(G) \geqslant \mu_1(G - v)$ and $\alpha(G) \leqslant \alpha(G - v)$. Hence $LS(G) = \mu_1(G) - \alpha(G) \geqslant \mu_1(G - v) - \alpha(G - v) = LS(G - v)$. □

Table 1
The largest Laplacian eigenvalue and algebraic connectivity of $C_k + v$ with $k = 9, \ldots, 16$.

k	9	10	11	12	13	14	15	16
$\mu_1(C_k + v)$	4.37720	4.38595	4.38131	4.38387	4.38249	4.38324	4.38283	4.38305
$\alpha(C_k + v)$	0.34891	0.29680	0.25454	0.22012	0.19189	0.16856	0.14910	0.13275

Table 2
The largest Laplacian eigenvalue and algebraic connectivity of $C_k + v$ with $k = 17, \ldots, 23$.

k	17	18	19	20	21	22	23
$\mu_1(C_k + v)$	4.38293	4.38300	4.38296	4.38298	4.38297	4.38298	4.38297
$\alpha(C_k + v)$	0.11889	0.10705	0.09687	0.08806	0.08039	0.07367	0.06774

Throughout this paper, let $C_k + v$ be the cycle C_k added a pendant vertex v.
By Lemma 2.7 and finite steps deleting pendant vertices, we arrive at:

Theorem 2.8. *Let G be an n-vertex unicyclic graph with $\Delta = 3$ and the length of the cycle be k. Then $LS(G) \geqslant LS(C_k + v)$.*

Lemma 2.9 [7]. *Let G be a simple graph with at least one edge. If μ_1 be the largest Laplacian eigenvalue of G, then*

$$\mu_1 \geqslant max\left\{\sqrt{\frac{d_i^2 + 2d_i - 2d_j - 2 + \sqrt{\left(d_i^2 + 2d_i + 2d_j + 4\right)^2 + 4(d_i - c_{ij} - 1)(d_j - c_{ij} - 1)}}{2}} : v_iv_j \in E\right\}, \tag{1}$$

where v_i and v_j are the vertices of degree d_i and d_j respectively, and c_{ij} is the cardinality of the set of common neighbors between v_i and v_j.

It is well known that $\mu_{n-1}(C_n) = 2\left(1 - cos\left(\frac{2\pi}{n}\right)\right)$. Then we have

Lemma 2.10. *The algebraic connectivity of C_n is a decreasing function on n.*

Lemma 2.11. *If $k \geqslant 61$, then $LS(C_k + v) \geqslant 4 > LS(C_n)$, where n $(n \geqslant 3)$ is an arbitrary positive integer.*

Proof. Let u be the neighbor of v and let w be another neighbor of u. Then $d_i = \deg(u) = 3$ and $d_j = \deg(w) = 2$. Note that $c_{ij} = 0$. Then the right hand side of (1) is equal to $\sqrt{\frac{9+\sqrt{537}}{2}}$ which is greater than 4.0408. By Lemmas 2.6, 2.10 and direct calculation we have $\alpha(C_k + v) \leqslant \alpha(C_k) \leqslant \alpha(C_{61}) \doteq 0.0106$ for $k \geqslant 61$. Thus we have $LS(C_k + v) > 4.01081 - 0.0106 > 4 > LS(C_n)$. □

With the computer direct calculations, we straightforwardly have:

Lemma 2.12. *If $9 \leqslant k \leqslant 60$, then $LS(C_k + v) > 4 > LS(C_n)$, where n $(n \geqslant 3)$ is an arbitrary positive integer.*

Proof. If $24 \leqslant k \leqslant 60$, then $\mu_1(C_{24} + v) \doteq 4.38298$ and $\alpha(C_k + v) \leqslant \alpha(C_{24} + v) \doteq 0.06251$. Thus $LS(C_k + v) \geqslant LS(C_{24} + v) \doteq 4.38298 - 0.06251 = 4.32047 > 4 > LS(C_n)$.
If $9 \leqslant k \leqslant 23$, by Tables 1 and 2, then

$$LS(C_k + v) \geqslant LS(C_9 + v) \doteq 4.37720 - 0.34891 = 4.02829 > 4 > LS(C_n).$$

The lemma follows. □

By Theorem 2.8, Lemmas 2.11 and 2.12, we arrive at:

Corollary 2.13. *Let G be an n-vertex unicyclic graph with $\Delta = 3$ and the length of the cycle be k. If $k \geqslant 9$, then $LS(G) > LS(C_n)$.*

Lemma 2.14. *Let G be an n-vertex unicyclic graph with $\Delta = 3$ and the length of the cycle be $k = 6, 7, 8$. Then $LS(G) > LS(C_n)$.*

Proof. When $k = 8$. We consider two cases according to the order of G.

Case 1. $k = 8$ and $n = 9$.

Then $G \cong C_8 + v$ and $LS(G) \doteq 4.39276 - 0.41309 > 3.87939 - 0.46791 \doteq LS(C_9)$.

Case 2. $k = 8$ and $n \geqslant 10$.

Let $C_8 = v_1v_2 \cdots v_8v_1$ and let G have a subgraph $C_8 + v_1v$. Since $n \geqslant 10$ and $\Delta = 3$, G has a subgraph obtained by adding a vertex u to $C_8 + v_1v$. There exist five such non-isomorphic graphs. The graph which attains the minimum Laplacian spread among these 5 graphs is $C_8 + v_1v + v_5u$. By gradually deleting pendant vertices from G, G can be transformed into $C_8 + v_1v + v_5u$.

By Lemma 2.7, we have $LS(G) \geqslant LS(C_8 + v_1v + v_5u) \doteq 4.15632 > LS(C_n)$.

When $k = 6, 7$, similar to the case $k = 8$, the results follow.

The proof of Lemma 2.14 is completed. □

Lemma 2.15. *Let G be an n-vertex unicyclic graph with $\Delta = 3$ and the length of the cycle be $k = 5$. Then $LS(G) > LS(C_n)$.*

Proof. If $n = 6$, then $G \cong C_5 + v$ for some v. Thus we have

$$LS(G) \doteq 4.30278 - 0.69722 > 3 = LS(C_6).$$

When $n \geqslant 7$. Let $C_5 = v_1v_2 \cdots v_5v_1$ and v be a vertex adjacent to v_1. Then $C_5 + v_1v$ is a subgraph of G. Note that $n \geqslant 7$ and $\Delta = 3$. A new vertex u may be adjacent to v, v_2, v_3, v_4 or v_5. We consider the following two cases.

Case 1. u is no adjacent to v_3.

By direct computation, the minimum LS value of $C_5 + v_1v + xu$ for $x \in \{v, v_2, v_4, v_5\}$ is attained when $x = v_2$ or v_5.

By Lemma 2.7, we have $LS(G) \geqslant LS(C_5 + v_1v + v_2u) \doteq 4.65109 - 0.62280 > 4 > LS(C_n)$.

Case 2. u is adjacent to v_3.

Subcase 2.1. $n = 7$.

Then $G \cong C_5 + v_1v + v_3u$ and $LS(G) \doteq 4.41421 - 0.51881 = 3.8954 > 3.04892 \doteq LS(C_7)$.

Subcase 2.2. $n \geqslant 8$.

By direct computation, the minimum LS value of $C_5 + v_1v + v_3u + xy$ for $y \in \{v, v_2, u, v_4, v_5\}$ is attained when $y = v$ or u.

By Lemma 2.7, we have $LS(G) \geqslant LS(C_5 + v_1v + v_3u + xv) \doteq 4.48119 - 0.32487 > 4 > LS(C_n)$.

Thus the proof of Lemma 2.15 is completed. □

Lemma 2.16. *Let G be an n-vertex unicyclic graph with $\Delta = 3$ and the length of the cycle be $k = 4$. Then $LS(G) > LS(C_n)$.*

Proof. There are two cases:

Case 1. $n = 5$.

Then $G \cong C_4 + v$ for some v and $LS(G) \doteq 4.48119 - 0.82991 = 3.65128 > 2.23606 \doteq LS(C_5)$.

Case 2. $n \geqslant 6$.

Let $C_4 = v_1v_2 \cdots v_4v_1$ and v be a vertex adjacent to v_1. Then G contains the subgraph $C_4 + v_1v$. Since $n \geqslant 6$ and $\Delta = 3$, a new vertex u may be adjacent to v, v_2, v_3 or v_4. By direct computation, the subgraph which attains the minimum Laplacian spread is $C_4 + v_1v + vu$, where the value is $LS(C_4 + v_1v + vu) \doteq 4.56155 - 0.43845 = 4.12310 > 4 > LS(C_n)$. By gradually deleting pendant vertices from G, G can be transformed into $C_4 + v_1v + vu$. Then by Lemma 2.7, $LS(G) \geqslant LS(C_4 + v_1v + vu) > 4 > LS(C_n)$.

Thus the lemma follows. □

Lemma 2.17. *Let G be an n-vertex unicyclic graph with $\Delta = 3$ and the length of the cycle be $k = 3$. Then $LS(G) > LS(C_n)$.*

Proof. If $n = 4$, then $G \cong C_3 + v$ for some v. Hence $LS(G) = 4 - 1 = 3 > 2 = LS(C_4)$.

Let $C_3 = v_1v_2v_3v_1$ and v be a vertex adjacent to v_1. Then $C_3 + v_1v$ is a subgraph of G.

When $n \geqslant 5$. A new vertex u may be adjacent to v_2, v_3 or v. According to the vertex-induced subgraphs of G, we have:

Case 1. $C_3 + v_1v + v_2u$ is a subgraph of G.

Let $G_1 \cong C_3 + v_1v + v_2u$, $G_2 \cong C_3 + v_1v + v_2u + vx \cong C_3 + v_1v + v_2u + ux$, $G_3 \cong C_3 + v_1v + v_2u + v_3x$ for some x.

If $n = 5$, then $G \cong G_1$ and $LS(G) \doteq 4.30278 - 0.69722 = 3.60556 > LS(C_5) \doteq 2.23606$.

If $n = 6$, then $G \cong G_i$ $(i = 2, 3)$. By direct calculation, $LS(G_i) > LS(C_6) = 3$.

Suppose $n \geqslant 7$. By direct computation, the minimum LS value among all 7-vertex unicyclic graphs containing $C_3 + v_1v + v_2u$ is $LS(C_3 + v_1v + v_2u + vx + v_3y) \doteq 4.03224$ for some x and y. By Lemma 2.7, we have $LS(G) \geqslant LS(C_3 + v_1v + v_2u + vx + v_3y) \doteq 4.03224 > 4 > LS(C_n)$.

Case 2. $C_3 + v_1v + v_3u$ is a subgraph of G.

Since v_2 and v_3 are symmetric in $C_3 + v_1v$, this case is similar to Case 1.

Case 3. $C_3 + v_1v + vu$ is a subgraph of G.

Let $G_4 \cong C_3 + v_1v + vu$, $G_5 \cong C_3 + v_1v + vu + ux$, $G_6 \cong C_3 + v_1v + vu + vx$ and $G_7 \cong C_3 + v_1v + vu + v_2x \cong C_3 + v_1v + vu + v_3x$.

If $n = 5$, then $G \cong G_4$ and $LS(G) \doteq 4.17009 - 0.51881 = 3.65128 > LS(C_5) \doteq 2.23606$.

If $n = 6$, then $G \cong G_j$ $(j = 5, 6, 7)$. By direct calculation, we have $LS(G_j) > LS(C_n)$.

When $n \geqslant 7$.

Similar to Case 1 the graph $C_3 + v_1v + vu + ux + xy$ for some x and y attains the minimum LS value among all 7-vertex unicyclic graphs containing $C_3 + v_1v + vu$. By Lemma 2.7, we have $LS(G) \geqslant LS(C_3 + v_1v + vu + ux + xy) \doteq 4.22833 - 0.22538 > 4 > LS(C_n)$.

All possible cases are exhausted, and the proof of Lemma 2.17 is completed. □

By Corollary 2.13 and Lemmas 2.14–2.17, we have the following result:

Theorem 2.18. *Let G be a unicyclic graph with $\Delta = 3$. Then $LS(G) > LS(C_n)$.*

Combining Theorems 2.4 and 2.18, we arrive at the main result:

Theorem 2.19. *Let G be a unicyclic graph with n vertices. Then $LS(G) \geqslant LS(C_n)$ and the equality holds if and only if $G \cong C_n$.*

Acknowledgements

The authors are very grateful to the anonymous referees for their valuable comments and suggestions, which have improved the presentation of the paper.

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