A Combinatorial Proof of the Cyclic Sieving Phenomenon for Faces of Coxeterhedra

Tung-Shan Fu

Pingtung Institute of Commerce

Based on joint work with S.-P. Eu and Y.-J. Pan

Cyclic sieving phenomenon

- X: a finite set
- $X(q)$: a polynomial in $\mathbb{Z}[q]$ $(X(1) = |X|)$
- C: a finite cyclic group acting on X

Cyclic sieving phenomenon

- X: a finite set
- $X(q)$: a polynomial in $\mathbb{Z}[q]$ $(X(1) = |X|)$
- C: a finite cyclic group acting on X

If $c \in C$, we let

$X^c = \{x \in X : c(x) = x\}$ and $o(c) =$ order of c in C.

Cyclic sieving phenomenon

- X: a finite set
- $X(q)$: a polynomial in $\mathbb{Z}[q]$ $(X(1) = |X|)$
- C: a finite cyclic group acting on X

If $c \in C$, we let

$$X^c = \{x \in X : c(x) = x\} \text{ and } o(c) = \text{ order of } c \text{ in } C.$$

We also let ω_d be the primitive dth root of unity.

Cyclic sieving phenomenon

- X: a finite set
- $X(q)$: a polynomial in $\mathbb{Z}[q]$ $(X(1) = |X|)$
- C: a finite cyclic group acting on X

If $c \in C$, we let

$$X^c = \{x \in X : c(x) = x\} \text{ and } o(c) = \text{ order of } c \text{ in } C.$$

We also let ω_d be the primitive dth root of unity.

Definition (Reiner-Stanton-White 2004)

The triple $(X, X(q), C)$ exhibits the cyclic sieving phenomenon (CSP) if, for every $c \in C$, we have

$$|X^c| = X(\omega_{o(c)}).$$

Cyclic sieving phenomenon

- X: a finite set
- $X(q)$: a polynomial in $\mathbb{Z}[q]$ $(X(1) = |X|)$
- C: a finite cyclic group acting on X

If $c \in C$, we let

$$X^c = \{x \in X : c(x) = x\} \text{ and } o(c) = \text{ order of } c \text{ in } C.$$

We also let ω_d be the primitive dth root of unity.

Definition (Reiner-Stanton-White 2004)

The triple $(X, X(q), C)$ exhibits the cyclic sieving phenomenon *(CSP) if, for every $c \in C$, we have*

$$|X^c| = X(\omega_{o(c)}).$$

Note. The case $|C| = 2$ was first studied by Stembridge and called the "$q = -1$ phenomenon".

Example

Let $[n] = \{1, \ldots, n\}$ and

$$X = \binom{[n]}{k} = \{T \subseteq [n] : |T| = k\}.$$

Example

Let $[n] = \{1, \ldots, n\}$ and

$$X = \binom{[n]}{k} = \{T \subseteq [n] : |T| = k\}.$$

Let $C = \langle (1, \ldots, n) \rangle$. Now $c \in C$ acts on $T = \{t_1, \ldots, t_k\}$ by

$$c(T) = \{c(t_1), \ldots, c(t_k)\}.$$

Example

Let $[n] = \{1, \ldots, n\}$ and

$$X = \binom{[n]}{k} = \{T \subseteq [n] : |T| = k\}.$$

Let $C = \langle (1, \ldots, n) \rangle$. Now $c \in C$ acts on $T = \{t_1, \ldots, t_k\}$ by

$$c(T) = \{c(t_1), \ldots, c(t_k)\}.$$

For example, consider $n = 4$ and $k = 2$. We have

$$\begin{aligned} X &= \{12, 13, 14, 23, 24, 34\} \\ C &= \{e, (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}. \end{aligned}$$

Example

Let $[n] = \{1, \dots, n\}$ and

$$X = \binom{[n]}{k} = \{T \subseteq [n] : |T| = k\}.$$

Let $C = \langle (1, \dots, n) \rangle$. Now $c \in C$ acts on $T = \{t_1, \dots, t_k\}$ by

$$c(T) = \{c(t_1), \dots, c(t_k)\}.$$

For example, consider $n = 4$ and $k = 2$. We have

$$\begin{aligned} X &= \{12, 13, 14, 23, 24, 34\} \\ C &= \{e, (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}. \end{aligned}$$

For $c = (1,3)(2,4)$, we have

$$\begin{array}{lll} c(12) = 34, & c(13) = 13, & c(14) = 23 \\ c(34) = 12, & c(24) = 24, & c(23) = 14 \end{array}$$

A q-polynomial for $X(q)$

Let $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [1]_q[2]_q \cdots [n]_q$.

A q-polynomial for $X(q)$

Let $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [1]_q[2]_q \cdots [n]_q$.
Define the *Gaussian coefficients* by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q[n-k]_q}.$$

A q-polynomial for $X(q)$

Let $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [1]_q[2]_q \cdots [n]_q$. Define the *Gaussian coefficients* by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q[n-k]_q}.$$

For example, take $n = 4$ and $k = 2$. We have

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4.$$

A q-polynomial for $X(q)$

Let $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [1]_q[2]_q \cdots [n]_q$. Define the *Gaussian coefficients* by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q[n-k]_q}.$$

For example, take $n = 4$ and $k = 2$. We have

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4.$$

Then
$$\begin{aligned} \omega = 1 \quad &\Rightarrow \quad \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q=1} = 1 + 1 + 2 + 1 + 1 = 6 \\ \omega = -1 \quad &\Rightarrow \quad \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q=-1} = 1 - 1 + 2 - 1 + 1 = 2 \\ \omega = -i \quad &\Rightarrow \quad \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q=-i} = 1 - i - 2 + i + 1 = 0 \end{aligned}$$

An instance of CSP

Theorem (Reiner-Stanton-White)

The following triple exhibits the CSP

$$\left(\binom{[n]}{k}, \begin{bmatrix} n \\ k \end{bmatrix}_q, C \right),$$

where $C = \langle (1, \ldots, n) \rangle$.

An equivalent condition for CSP

If $X(q)$ is expanded as

$$X(q) \equiv a_0 + a_1 q + \cdots + a_{n-1} q^{n-1} \pmod{q^n - 1},$$

where $n = |C|$, then a_k counts the number of orbits whose stabilizer-order divides k.

An equivalent condition for CSP

If $X(q)$ is expanded as

$$X(q) \equiv a_0 + a_1 q + \cdots + a_{n-1} q^{n-1} \pmod{q^n - 1},$$

where $n = |C|$, then a_k counts the number of orbits whose stabilizer-order divides k.

In particular,

- a_0 is the total number of orbits.
- a_1 the number of free orbits (i.e., of size n).
- $a_2 - a_1$ is the number of orbits of size $\frac{n}{2}$.

Permutation polytopes

The *permutohedron* PA_{n-1} of dimension $n-1$ is the the convex hull of all permutations of the vector $(1,\ldots,n) \in \mathbb{R}^n$.

Permutation polytopes

The *permutohedron* PA_{n-1} of dimension $n-1$ is the the convex hull of all permutations of the vector $(1,\ldots,n) \in \mathbb{R}^n$.

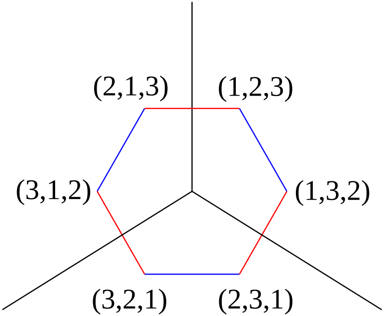


Figure: The permutohedron PA_2

An instance of CSP

- X: vertex set of PA_2
- $X(q) = [3]_q! \equiv 2q^2 + 2q + 2 \pmod{q^3 - 1}$
- $C = \mathbb{Z}/3\mathbb{Z}$ acts on X by rotating the coordinates

Then $(X, X(q), C)$ exhibits the CSP.

An instance of CSP

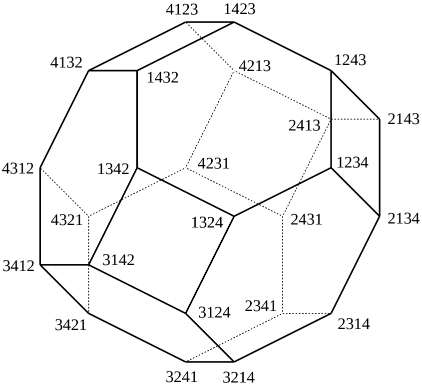
- X: vertex set of PA_2
- $X(q) = [3]_q! \equiv 2q^2 + 2q + 2 \pmod{q^3 - 1}$
- $C = \mathbb{Z}/3\mathbb{Z}$ acts on X by rotating the coordinates

Then $(X, X(q), C)$ exhibits the CSP.

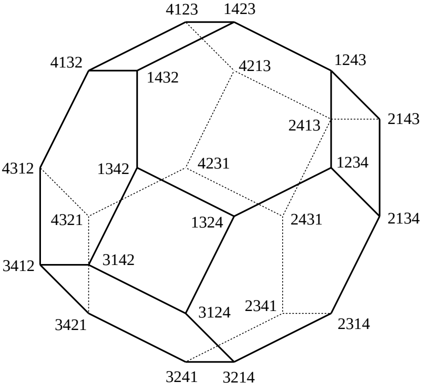
- X: edge set of PA_2
- $X(q) = \begin{bmatrix}3\\1\end{bmatrix}_q + \begin{bmatrix}3\\2\end{bmatrix}_q \equiv 2q^2 + 2q + 2 \pmod{q^3 - 1}$
- $C = \mathbb{Z}/3\mathbb{Z}$ acts on X by rotating the coordinates

Then $(X, X(q), C)$ exhibits the CSP.

The permutohedron PA_3



The permutohedron PA_3



- Vertex $(\sigma^{-1}(1), \ldots, \sigma^{-1}(n)) \in \mathbb{R}^n$ is labeled by $\sigma \in \mathfrak{S}_n$.
- Two vertices are adjacent iff the corresponding permutations differ by an adjacent transposition.

Description for faces of PA_{n-1}

Theorem (Billera-Sarangarajan 1996)

The face lattice of the permutohedron PA_{n-1} is isomorphic to the lattice of all ordered partitions of the set $\{1, \ldots, n\}$, ordered by refinement.

Description for faces of PA_{n-1}

Theorem (Billera-Sarangarajan 1996)

The face lattice of the permutohedron PA_{n-1} is isomorphic to the lattice of all ordered partitions of the set $\{1, \ldots, n\}$, ordered by refinement.

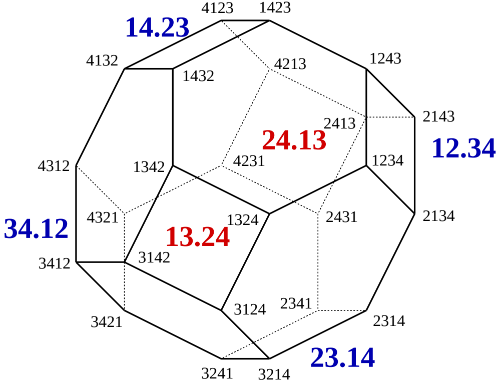
Face numbers

For $2 \leq k \leq n$, the number of $(n-k)$-faces in PA_{n-1} is given by

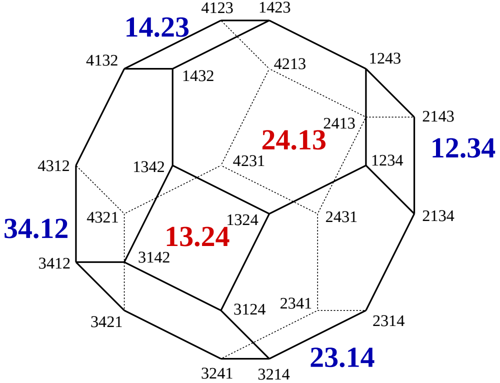
$$k! \cdot S_{n,k},$$

where $S_{n,k}$ is the Stirling number of the second kind.

The facets of PA_{n-1}



The facets of PA_{n-1}



facet-orbits:

1.234	12.34	13.24	123.4
2.134	23.14	24.13	234.1
3.124	34.12		134.2
4.123	14.23		124.3

Face numbers of PA_{n-1}

Let $x_{n,k} = k! S_{n,k}$. Then $x_{n,k}$ satisfies the following recurrence relation

$$x_{n,k} = \begin{cases} 1 & \text{if } k = 1 \\ \sum_{i=1}^{n-k+1} \binom{n}{i} x_{(n-i,k-1)} & \text{if } 2 \le k \le n. \end{cases}$$

Face numbers of PA_{n-1}

Let $x_{n,k} = k!S_{n,k}$. Then $x_{n,k}$ satisfies the following recurrence relation

$$x_{n,k} = \begin{cases} 1 & \text{if } k = 1 \\ \sum_{i=1}^{n-k+1} \binom{n}{i} x_{(n-i,k-1)} & \text{if } 2 \leq k \leq n. \end{cases}$$

For example,

$$\begin{aligned} x_{n,2} &= \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1}, \\ x_{n,3} &= \binom{n}{1} x_{n-1,2} + \binom{n}{2} x_{n-2,2} + \cdots + \binom{n}{n-2} x_{2,2}. \end{aligned}$$

Face numbers of PA_{n-1}

Let $x_{n,k} = k!S_{n,k}$. Then $x_{n,k}$ satisfies the following recurrence relation

$$x_{n,k} = \begin{cases} 1 & \text{if } k = 1 \\ \sum_{i=1}^{n-k+1} \binom{n}{i} x_{(n-i,k-1)} & \text{if } 2 \leq k \leq n. \end{cases}$$

For example,

$$\begin{aligned} x_{n,2} &= \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1}, \\ x_{n,3} &= \binom{n}{1} x_{n-1,2} + \binom{n}{2} x_{n-2,2} + \cdots + \binom{n}{n-2} x_{2,2}. \end{aligned}$$

Note that $x_{n,2}$ is number of facets of PA_{n-1}.

A feasible q-polynomial for face numbers

Let $X(n,k;q) \in \mathbb{Z}[q]$ be the polynomial recursively defined by

$$X(n,k;q) = \begin{cases} 1 & \text{if } k = 1 \\ \displaystyle\sum_{i=1}^{n-k+1} \begin{bmatrix} n \\ i \end{bmatrix}_q X(n-i,k-1;q) & \text{if } 2 \leq k \leq n. \end{cases}$$

A feasible q-polynomial for face numbers

Let $X(n,k;q) \in \mathbb{Z}[q]$ be the polynomial recursively defined by

$$X(n,k;q) = \begin{cases} 1 & \text{if } k = 1 \\ \displaystyle\sum_{i=1}^{n-k+1} \begin{bmatrix} n \\ i \end{bmatrix}_q X(n-i,k-1;q) & \text{if } 2 \le k \le n. \end{cases}$$

For example, take $n = 4$ and $k = 2$,

$$\begin{aligned} X(4,2;q) &= \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q + \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q \\ &\equiv 4 + 3q + 4q^2 + 3q^3 \pmod{q^4 - 1}. \end{aligned}$$

q-Lucas Theorem

Theorem (q-Lucas Theorem)

Let ω be a primitive dth root of unity. If $n = ad + b$ and $k = rd + s$, where $0 \leq b, s \leq q - 1$, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=\omega} = \binom{a}{r} \begin{bmatrix} b \\ s \end{bmatrix}_{q=\omega}.$$

q-Lucas Theorem

Theorem (q-Lucas Theorem)

Let ω be a primitive dth root of unity. If $n = ad + b$ and $k = rd + s$, where $0 \leq b, s \leq q - 1$, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=\omega} = \binom{a}{r} \begin{bmatrix} b \\ s \end{bmatrix}_{q=\omega}.$$

If $d \geq 2$ is a divisor of n, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=\omega} = \begin{cases} \binom{\frac{n}{d}}{\frac{k}{d}} & d|k \\ 0 & \text{otherwise,} \end{cases}$$

q-Lucas Theorem

Theorem (q-Lucas Theorem)

Let ω be a primitive dth root of unity. If $n = ad + b$ and $k = rd + s$, where $0 \le b, s \le q - 1$, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=\omega} = \binom{a}{r} \begin{bmatrix} b \\ s \end{bmatrix}_{q=\omega}.$$

If $d \ge 2$ is a divisor of n, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=\omega} = \begin{cases} \binom{\frac{n}{d}}{\frac{k}{d}} & d | k \\ 0 & \text{otherwise,} \end{cases}$$

e.g., for $n = 4$ and $d = 2$, then $\omega = -1$ and $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q=-1} = \binom{2}{1}$.

The CSP for faces of PA_{n-1}

Proposition

For $d \geq 2$ a divisor of n, let ω be a primitive dth root of unity. Then

$$[X(n,k;q)]_{q=\omega} = \begin{cases} x_{(\frac{n}{d},k)} & \text{if } n \geq kd \\ 0 & \text{otherwise.} \end{cases}$$

The CSP for faces of PA_{n-1}

Proposition

For $d \geq 2$ a divisor of n, let C_d be the subgroup of order d of C, and let $X_{n,k,d}$ be the set of $(n-k)$-faces of PA_{n-1} that are invariant under C_d. Then

$$|X_{n,k,d}| = \begin{cases} x_{(\frac{n}{d},k)} & \text{if } n \geq kd \\ 0 & \text{otherwise.} \end{cases}$$

The CSP for faces of PA_{n-1}

Proposition

For $d \geq 2$ a divisor of n, let C_d be the subgroup of order d of C, and let $X_{n,k,d}$ be the set of $(n-k)$-faces of PA_{n-1} that are invariant under C_d. Then

$$|X_{n,k,d}| = \begin{cases} x_{(\frac{n}{d},k)} & \text{if } n \geq kd \\ 0 & \text{otherwise.} \end{cases}$$

Count the number of k-block ordered partitions of $[n]$ that are invariant under

$$\begin{aligned} C_d &= \langle (1, \tfrac{n}{d}+1, \ldots, \tfrac{n}{d}(d-1)+1) \\ &\qquad (2, \tfrac{n}{d}+2, \ldots, \tfrac{n}{d}(d-1)+2) \cdots (\tfrac{n}{d}, \tfrac{2n}{d}, \ldots, n) \rangle. \end{aligned}$$

Algebraic Background: Coxeter system (W, S)

$W = A_{n-1}$, the Coxeter group of type A

- Group $A_{n-1} = \mathfrak{S}_n$, the *symmetric group* on the set $[n]$

Algebraic Background: Coxeter system (W, S)

$W = A_{n-1}$, the Coxeter group of type A

- Group $A_{n-1} = \mathfrak{S}_n$, the *symmetric group* on the set $[n]$
- The *Coxeter generators* $S = \{s_1, \ldots, s_{n-1}\}$ of A_{n-1} consists of adjacent transpositions

$$s_i = (i, i+1).$$

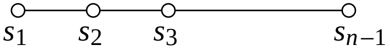
Algebraic Background: Coxeter system (W, S)

$W = A_{n-1}$, the Coxeter group of type A

- Group $A_{n-1} = \mathfrak{S}_n$, the *symmetric group* on the set $[n]$
- The *Coxeter generators* $S = \{s_1, \ldots, s_{n-1}\}$ of A_{n-1} consists of adjacent transpositions

$$s_i = (i, i+1).$$

- The diagram



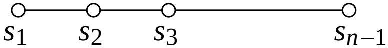
Algebraic Background: Coxeter system (W, S)

$W = A_{n-1}$, the Coxeter group of type A

- Group $A_{n-1} = \mathfrak{S}_n$, the *symmetric group* on the set $[n]$
- The *Coxeter generators* $S = \{s_1, \ldots, s_{n-1}\}$ of A_{n-1} consists of adjacent transpositions

$$s_i = (i, i+1).$$

- The diagram



- The *Coxeter element* $c = s_1 s_2 \cdots s_{n-1} = (1, 2, \ldots, n) \in \mathfrak{S}_n$ generates a cyclic group of order n.

Example: permutohedron A_3

- $W = \mathfrak{S}_4$.
- $S = \{s_1, s_2, s_3\}$, i.e., $s_1 = (1,2)$, $s_2 = (2,3)$, $s_3 = (3,4)$.

Example: permutohedron A_3

- $W = \mathfrak{S}_4$.
- $S = \{s_1, s_2, s_3\}$, i.e., $s_1 = (1,2)$, $s_2 = (2,3)$, $s_3 = (3,4)$.

$J \subseteq S$	$\{s_2, s_3\}$	$\{s_1, s_3\}$	$\{s_1, s_2\}$

Example: permutohedron A_3

- $W = \mathfrak{S}_4$.
- $S = \{s_1, s_2, s_3\}$, i.e., $s_1 = (1,2)$, $s_2 = (2,3)$, $s_3 = (3,4)$.

$J \subseteq S$	$\{s_2, s_3\}$	$\{s_1, s_3\}$	$\{s_1, s_2\}$
W_J	$1234, 1342$ $1243, 1423$ $1324, 1432$	1234 2134 2143 1243	$1234, 2314$ $1324, 3124$ $2134, 3214$

Example: permutohedron A_3

- $W = \mathfrak{S}_4$.
- $S = \{s_1, s_2, s_3\}$, i.e., $s_1 = (1,2)$, $s_2 = (2,3)$, $s_3 = (3,4)$.

$J \subseteq S$	$\{s_2, s_3\}$	$\{s_1, s_3\}$	$\{s_1, s_2\}$
W_J	$1234, 1342$ $1243, 1423$ $1324, 1432$	1234 2134 2143 1243	$1234, 2314$ $1324, 3124$ $2134, 3214$
	1.234	**12.34**	**123.4**

Example: permutohedron A_3

- $W = \mathfrak{S}_4$.
- $S = \{s_1, s_2, s_3\}$, i.e., $s_1 = (1,2)$, $s_2 = (2,3)$, $s_3 = (3,4)$.

$J \subseteq S$	$\{s_2, s_3\}$	$\{s_1, s_3\}$	$\{s_1, s_2\}$
W_J	$1234, 1342$ $1243, 1423$ $1324, 1432$	1234 2134 2143 1243	$1234, 2314$ $1324, 3124$ $2134, 3214$
wW_J (cosets)	**1.234** **2.134** **3.124** **4.134**	**12.34** **13.24** **14.23** **23.14** **24.13** **34.12**	**123.4** **234.1** **134.2** **124.3**

Coxeterhedron

For a Coxeter system (W, S), the subgroups W_J generated by subsets $J \subseteq S$ are called *parabolic subgroups* of W.

Coxeterhedron

For a Coxeter system (W, S), the subgroups W_J generated by subsets $J \subseteq S$ are called *parabolic subgroups* of W.

The *Coxeterhedron* PW associated to (W, S) is the finite poset of all cosets $\{wW_J\}_{w \in W, J \subseteq S}$ of all parabolic subgroups of W, ordered by inclusion.

$W = B_n$, the Coxeter group of type B

- The group B_n is the group of all *signed permutations* w on the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ such that $w(-i) = -w(i)$ for $1 \leq i \leq n$.

$W = B_n$, the Coxeter group of type B

- The group B_n is the group of all *signed permutations* w on the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ such that $w(-i) = -w(i)$ for $1 \leq i \leq n$.
- The *Coxeter generators* $\{s_1, \ldots, s_n\}$ of B_n are defined by

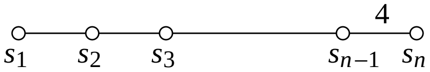
$$\begin{cases} s_i = (i, i+1)(-i, -i-1), & 1 \leq i \leq n-1 \\ s_n = (n, -n). \end{cases}$$

$W = B_n$, the Coxeter group of type B

- The group B_n is the group of all *signed permutations* w on the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ such that $w(-i) = -w(i)$ for $1 \le i \le n$.
- The *Coxeter generators* $\{s_1, \ldots, s_n\}$ of B_n are defined by

$$\begin{cases} s_i = (i, i+1)(-i, -i-1), & 1 \le i \le n-1 \\ s_n = (n, -n). \end{cases}$$

- The diagram

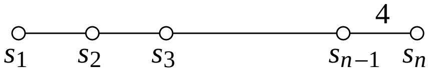


$W = B_n$, the Coxeter group of type B

- The group B_n is the group of all *signed permutations* w on the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ such that $w(-i) = -w(i)$ for $1 \leq i \leq n$.
- The *Coxeter generators* $\{s_1, \ldots, s_n\}$ of B_n are defined by

$$\begin{cases} s_i = (i, i+1)(-i, -i-1), & 1 \leq i \leq n-1 \\ s_n = (n, -n). \end{cases}$$

- The diagram



- The *Coxeter element* $c = s_1 \cdots s_n = (1, 2, \ldots, n, -1, -2, \ldots, -n)$ generates a cyclic group of order $2n$.

Notation for signed permutations

Given $w \in B_n$, let $w = w_1 w_2 \cdots w_n$, where

$$w_i = \begin{cases} j & \text{if } w_i = +j \\ \overline{j} & \text{if } w_i = -j. \end{cases}$$

Notation for signed permutations

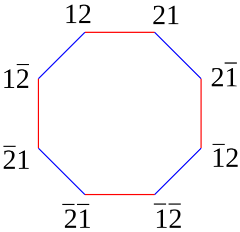
Given $w \in B_n$, let $w = w_1 w_2 \cdots w_n$, where

$$w_i = \begin{cases} j & \text{if } w_i = +j \\ \overline{j} & \text{if } w_i = -j. \end{cases}$$

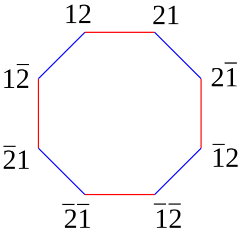
For example,

$$B_2 \text{ consists of } \begin{array}{cccc} 12, & \overline{1}2, & 1\overline{2}, & \overline{1}\overline{2} \\ 21, & \overline{2}1, & 2\overline{1}, & \overline{2}\overline{1} \end{array}$$

The coxeterhedron PB_2



The coxeterhedron PB_2



Under the cyclic group action generated by $c = (1, 2, -1, -2)$, there are 2 free vertex-orbits and 2 free edge-orbits.

$W = D_n$, the Coxeter group of type D

- The group D_n is the subgroup of B_n consisting of all signed permutations with an *even* number of sign changes.

$W = D_n$, the Coxeter group of type D

- The group D_n is the subgroup of B_n consisting of all signed permutations with an *even* number of sign changes.
- The *Coxeter generators* $\{s_1, \dots, s_n\}$ of D_n are defined by

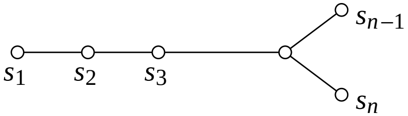
$$\begin{cases} s_i = (i, i+1)(-i, -i-1), & 1 \leq i \leq n-1 \\ s_n = (n, -n+1)(n-1, -n). \end{cases}$$

$W = D_n$, the Coxeter group of type D

- The group D_n is the subgroup of B_n consisting of all signed permutations with an *even* number of sign changes.
- The *Coxeter generators* $\{s_1, \ldots, s_n\}$ of D_n are defined by

$$\begin{cases} s_i = (i, i+1)(-i, -i-1), & 1 \leq i \leq n-1 \\ s_n = (n, -n+1)(n-1, -n). \end{cases}$$

- The diagram

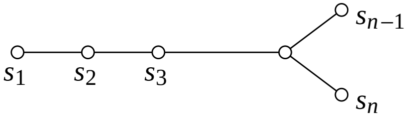


$W = D_n$, the Coxeter group of type D

- The group D_n is the subgroup of B_n consisting of all signed permutations with an *even* number of sign changes.
- The *Coxeter generators* $\{s_1, \ldots, s_n\}$ of D_n are defined by

$$\begin{cases} s_i = (i, i+1)(-i, -i-1), & 1 \leq i \leq n-1 \\ s_n = (n, -n+1)(n-1, -n). \end{cases}$$

- The diagram



- The *Coxeter element* $c = s_1 \cdots s_n = (1, 2, \ldots, n-1, -1, -2, \ldots, -n+1)(n, -n)$ generates a cyclic group of order $2n-2$.

Reiner-Ziegler's representation for faces of PW

Representing the faces wW_J of PW by boxed ordered partitions:

$$\begin{array}{llll}
13.4.256 & \longleftrightarrow & 314652W_{\{s_1,s_4,s_5\}} & \text{in } \mathsf{PA}_5 \\
1\overline{3}.4.\overline{2}5\overline{6} & \longleftrightarrow & \overline{3}14\overline{6}5\overline{2}W_{\{s_1,s_4,s_5\}} & \text{in } \mathsf{PB}_6 \\
1\overline{3}.4.\boxed{256} & \longleftrightarrow & \overline{3}14\overline{6}5\overline{2}W_{\{s_1,s_4,s_5,s_6\}} & \text{in } \mathsf{PB}_6 \\
1\overline{3}.4.\overline{256} & \longleftrightarrow & \overline{3}14\overline{652}W_{\{s_1,s_4,s_5\}} & \text{in } \mathsf{PD}_6 \\
1\overline{3}.4.\boxed{25\overline{6}} & \longleftrightarrow & \overline{3}14\overline{652}W_{\{s_1,s_4,s_5,s_6\}} & \text{in } \mathsf{PD}_6 \\
1\overline{3}.4.\boxed{2\overline{5}.6} & \longleftrightarrow & \overline{3}14\overline{652}W_{\{s_1,s_4,s_6\}} & \text{in } \mathsf{PD}_6
\end{array}$$

Face numbers of PW

For the groups $W = A_{n-1}$, B_n, D_n, the number $f_W(k)$ of $(n-k)$-faces of the Coxeterhedron PW is given by

$$
\begin{aligned}
f_{A_{n-1}}(k) &= x_{n,k}, \\
f_{B_n}(k) &= \sum_{j=0}^{n-k} \binom{n}{j} x_{(n-j,k)} \cdot 2^{n-j}, \\
f_{D_n}(k) &= \left(2x_{n,k} - n \cdot x_{(n-1,k-1)}\right) \cdot 2^{n-1} \\
&\qquad + \sum_{j=2}^{n-k} \binom{n}{j} x_{(n-j,k)} \cdot 2^{n-j},
\end{aligned}
$$

where $x_{n,k} = \begin{cases} 1 & \text{if } k = 1 \\ \sum_{i=1}^{n-k+1} \binom{n}{i} x_{(n-i,k-1)} & \text{if } 2 \leq k \leq n. \end{cases}$

q-polynomials for face numbers of PW

For the groups $W = A_{n-1}$, B_n, D_n, the number $f_W(k)$ of $(n-k)$-faces of the Coxeterhedron PW is given by

$$\begin{aligned}
f_{A_{n-1}}(k;q) &= X(n,k;q), \\
f_{B_n}(k;q) &= \sum_{j=0}^{n-k} \begin{bmatrix} n \\ j \end{bmatrix}_q X(n-j,k;q) \prod_{i=j+1}^{n} (1+q^i), \\
f_{D_n}(k;q) &= \left(2X(n,k;q) - \begin{bmatrix} n \\ 1 \end{bmatrix}_q X(n-1,k-1;q)\right) \prod_{i=1}^{n-1} (1+q^i) \\
&\quad + \sum_{j=2}^{n-k} \begin{bmatrix} n \\ j \end{bmatrix}_q X(n-j,k;q) \prod_{i=j}^{n-1} (1+q^i).
\end{aligned}$$

Poincaré polynomials

For a subset $W' \subseteq W$, let $W'(q)$ be the *Poincaré polynomial* of W', which is defined by

$$W'(q) := \sum_{w \in W'} q^{\ell(w)},$$

where $\ell(\cdot)$ is the length function of W.

Poincaré polynomials

For a subset $W' \subseteq W$, let $W'(q)$ be the *Poincaré polynomial* of W', which is defined by

$$W'(q) := \sum_{w \in W'} q^{\ell(w)},$$

where $\ell(\cdot)$ is the length function of W.
The cardinality and Poincaré polynomial of W are given by

$$|W| = \prod_{i=1}^{|S|} (e_i + 1), \quad W(q) = \prod_{i=1}^{|S|} [e_i + 1]_q,$$

where e_i are the *exponents* of W.

Poincaré polynomials

For a subset $W' \subseteq W$, let $W'(q)$ be the *Poincaré polynomial* of W', which is defined by

$$W'(q) := \sum_{w \in W'} q^{\ell(w)},$$

where $\ell(\cdot)$ is the length function of W.
The cardinality and Poincaré polynomial of W are given by

$$|W| = \prod_{i=1}^{|S|} (e_i + 1), \quad W(q) = \prod_{i=1}^{|S|} [e_i + 1]_q,$$

where e_i are the *exponents* of W.

Φ	$e_1, \ldots, e_n$
A_n	$1, 2, 3, \ldots, n$
B_n	$1, 3, 5, \ldots, 2n-1$
D_n	$1, 3, 5, \ldots, 2n-3, n-1$

The number of cosets for parabolic subgroups

For any parabolic subgroup W_J and $J \subseteq S$,

- the diagram for (W_J, J) is obtained from the diagram for (W, S) by removing all nodes in $S \backslash J$,

The number of cosets for parabolic subgroups

For any parabolic subgroup W_J and $J \subseteq S$,

- the diagram for (W_J, J) is obtained from the diagram for (W, S) by removing all nodes in $S \backslash J$,
- $|W_J|$ and $W_J(q)$ can be expressed in terms exponents as a product derived from the connected components of the diagram for W_J,

The number of cosets for parabolic subgroups

For any parabolic subgroup W_J and $J \subseteq S$,

- the diagram for (W_J, J) is obtained from the diagram for (W, S) by removing all nodes in $S \backslash J$,
- $|W_J|$ and $W_J(q)$ can be expressed in terms exponents as a product derived from the connected components of the diagram for W_J,
- $|W^J| = |W|/|W_J|$ and $W^J(q) = W(q)/W_J(q)$.

The CSP for faces of Coxeterhedron

Theorem (Reiner-Stanton-White 2004)

For a Coxeter system (W, S) *and* $J \subseteq S$*, let* C *be a cyclic group generated by a regular element. Let* X *be the set of cosets* W/W_J*, and* $X(q) := W^J(q)$*. Then the triple* $(X, X(q), C)$ *exhibits the cyclic sieving phenomenon.*

Remarks

We prove a special case of Theorem [RSW] with the following restrictions.

- The cyclic group we considered is generated by a Coxeter element, while Theorem [RSW] holds for the cyclic group generated by a regular element.

Remarks

We prove a special case of Theorem [RSW] with the following restrictions.

- The cyclic group we considered is generated by a Coxeter element, while Theorem [RSW] holds for the cyclic group generated by a regular element.

- The CSP that we show is collectively on the set of all cosets $\cup_{J \subseteq S, |J| = n-k} W/W_J$, while Theorem [RSW] shows a refinement of such phenomenon that holds individually for each W_J on the cosets W/W_J.

Remarks

We prove a special case of Theorem [RSW] with the following restrictions.

- The cyclic group we considered is generated by a Coxeter element, while Theorem [RSW] holds for the cyclic group generated by a regular element.

- The CSP that we show is collectively on the set of all cosets $\cup_{J \subseteq S, |J|=n-k} W/W_J$, while Theorem [RSW] shows a refinement of such phenomenon that holds individually for each W_J on the cosets W/W_J.

- The polynomial $f_W(k;q)$ that we use is exactly the sum of the Poincaré polynomials $W^J(q)$ for all $J \subseteq S$ and $|J| = n - k$, while in Theorem [RSW] a single polynomial $W^J(q)$ is used.