Partial Orders and Equivalence Relations

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Relation

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If R is a partial order on X, we call (X, R)(or simply call X) a partially ordered set or poset (偏序集). In this case we usually write $x \leq y$ for xRy.

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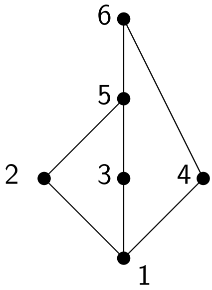

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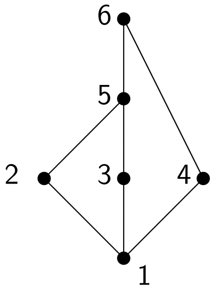

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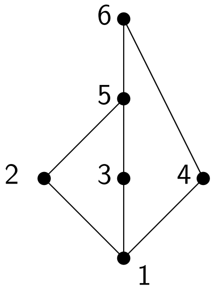

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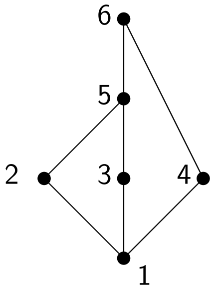

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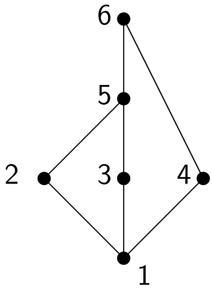

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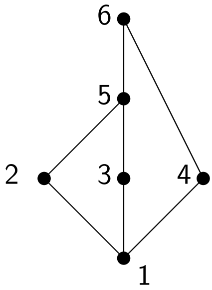

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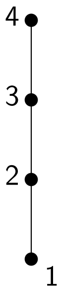

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(等價關係與分割是同一回事)

Theorem 2

Theorem

(i) *Let R be an equivalence relation on X. Then $\{R(x) \mid x \in X\}$ is a partition of X. (Here $R(x)$ is called an equivalence class 等價類).*

(ii) *Let $A_1, A_2, \ldots, A_t$ be a partition of X. Define a relation R on X by*

$$R = \{(x, y) \mid \text{there exists } A_i \text{ such that } x, y \in A_i\}.$$

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Proof.

Routine.

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Remark

The equivalence relation and partition are two important concepts in the algebra course.

Homework

4.6: 45, 46, 48, 49.