Lecture 6 Monday February 2

More intersection numbers.

Given a DRG $\Gamma = (X, R)$ with diameter D, since the distance matrices $\{A_i\}_{i=0}^D$ form a basis for the Bose-Mesner algebra $\exists$ scalars

$$p^h_{ij} \in \mathbb{F} \qquad (0 \le h, i, j \le D)$$

such that

$$A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h \qquad (*)$$

for $0 \le i, j \le D$. Since A_i, A_j commute

$$p^h_{ij} = p^h_{ji} \qquad (0 \le h, i, j \le D)$$

For $0 \le h \le D$ and $x, y \in X$ at $\partial(x, y) = h$ compute the (x,y)-entry in $(*)$ to find

$$p^h_{ij} = \left| \Gamma_i(x) \cap \Gamma_j(y) \right|$$

So p^h_{ij} is a nonnegative integer. For notational convenience define

$$p^h_{ij} = 0 \quad \text{unless } 0 \le i, j \le D \qquad (i, j \in \mathbb{Z})$$

Note that

$$c_i = p^i_{1\,i-1}, \qquad a_i = p^i_{1i}, \qquad b_i = p^i_{1\,i+1}$$

for $0 \le i \le D$. We often call any p^h_{ij} an <u>intersection number</u> of Γ.

By simple counting arguments we find

$$p^h_{0j} = \delta_{hj} \qquad (0 \le h, j \le D)$$

$$p^h_{i0} = \delta_{hi} \qquad (0 \le h, i \le D)$$

$$p^0_{ij} = \delta_{ij} k_i \qquad (0 \le i, j \le D)$$

$$\sum_{i=0}^{D} p^h_{ij} = k_j \qquad (0 \le h, j \le D)$$

LEM 44 With above notation

$$k_h p^h_{ij} = k_i p^i_{hj} = k_j p^j_{ih} \qquad (0 \le h, i, j \le D)$$

pf Each of the three products equals $|X|^{-1}$ times the number of triples

$$\left| \left\{ xyz \;\middle|\; x, y, z \in X, \quad \partial(x,y) = h, \quad \partial(y,z) = i, \quad \partial(x,z) = j \right\} \right| \qquad \square$$

LEM 45 For a DRG Γ with diameter D

and for $0 \le h, i, j \le D$

(i) $p^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two

(ii) $p^h_{ij} \neq 0$ if one of h, i, j is equal to the sum of the other two.

pf. The distance function ∂ satisfies the triangle inequality $\square$

Next goal: show each p^h_{ij} is determined by $\{c_i\}_{i=1}^{D}$, $\{b_i\}_{i=0}^{D-1}$.

LEM 46 For a DRG $\Gamma = (X, R)$ with diameter D

and for $0 \le h, i, j, r \le D$

$$\sum_{\alpha=0}^{D} p^r_{h\alpha} \, p^\alpha_{ij} = \sum_{\beta=0}^{D} p^r_{j\beta} \, p^\beta_{hi}$$

pf Expand each side of

$$A_h (A_i A_j) = (A_h A_i) A_j$$

as a linear combination of the distance matrices,

and compare coefficients. $\square$

LEM 47 For a DRG Γ with diameter D and for

$$0 \le r, i, j \le D$$

$$c_r p^{r-1}_{ij} + a_r p^{r}_{ij} + b_r p^{r+1}_{ij} = b_{i-1} p^{r}_{i-1,j} + a_i p^{r}_{ij} + c_{i+1} p^{r}_{i+1,j}$$

Pf Take $h=1$ in LEM 46 $\square$

Note 48 For a DRG Γ with diameter D we can compute

$$p^h_{ij} \qquad 0 \le h, i, j \le D$$

from $\{c_i\}_{i=1}^{D}$, $\{b_i\}_{i=0}^{D-1}$ using the recursion in LEM 47.

Example 49 For a DRG Γ with diameter D

$$p^i_{2\,i-1} = c_i \frac{a_i + a_{i-1} - a_1}{c_2} \qquad 1 \le i \le D$$

$$p^i_{2\,i+1} = b_i \frac{a_i + a_{i+1} - a_1}{c_2} \qquad 0 \le i \le D-1$$

$$p^i_{2\,i} = \frac{c_i(b_{i-1}-1) + a_i(a_i - a_1 - 1) + b_i(c_{i+1}-1)}{c_2} \qquad 0 \le i \le D$$

For a DRG $\Gamma = (X, R)$ with diameter D

recall the polynomials $\{u_i\}_{i=0}^{D}$ a $\{v_i\}_{i=0}^{D}$ from Defs 23, 30

We wish to say these are <u>orthogonal</u> <u>polynomials</u>. What does this mean? what is orthogonal to what?

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

LEM 50 Let X be a finite nonempty set. For $B, C \in \mathrm{Mat}_X(\mathbb{F})$ put

$$\langle B, C \rangle = \mathrm{tr}(B \bar{C}^t)$$

$$\|B\|^2 = \langle B, B \rangle$$

then for all $B, B', C \in \mathrm{Mat}_X(\mathbb{F})$ and $\alpha \in \mathbb{F}$

(i) $\langle \alpha B, C \rangle = \alpha \langle B, C \rangle$

(ii) $\langle B, C \rangle = \overline{\langle C, B \rangle}$

(iii) $\langle B + B', C \rangle = \langle B, C \rangle + \langle B', C \rangle$

(iv) $\|B\|^2$ is a nonneg real number

(v) $\|B\|^2 = 0$ iff $B = 0$

In other words $\langle\ ,\ \rangle$ is a positive definite Hermitean form on $\mathrm{Mat}_X(\mathbb{F})$

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LEM 51 Referring to LEM 50

$$\langle AB, C \rangle = \langle B, \bar{A}^t C \rangle$$

$$= \langle A, C \bar{B}^t \rangle$$

for all $A, B, C \in Mat_{\times}(\mathbb{F})$

Pf Routine using $tr(uv) = tr(vu)$ $\square$

LEM 52 For a DRG $\Gamma = (X, R)$ with diameter D

(i) $\langle A_i, A_j \rangle = \delta_{ij} k_i |X|$ $(0 \le i, j \le D)$

(ii) $\langle E_i, E_j \rangle = \delta_{ij} m_i$ $(0 \le i, j \le D)$

pf (i)

$$\begin{aligned}
\langle A_i, A_j \rangle &= \langle A_i, I A_j \rangle \\
&= \langle A_i \overline{A_j}^t, I \rangle \\
&= \langle A_i A_j, I \rangle \\
&= \sum_{h=0}^{D} p_{ij}^h \underbrace{\langle A_h, I \rangle}_{\operatorname{tr} A_h = \delta_{h,0} |X|} \\
&= p_{ij}^0 |X| \\
&= \delta_{ij} k_i |X|
\end{aligned}$$

(ii)

$$\begin{aligned}
\langle E_i, E_j \rangle &= \langle E_i, I E_j \rangle \\
&= \langle E_i \overline{E_j}^t, I \rangle \\
&= \langle E_i E_j, I \rangle \\
&= \delta_{ij} \langle E_i, I \rangle \\
&= \delta_{ij} \operatorname{tr}(E_i) = \delta_{ij} m_i
\end{aligned}$$

$\square$

thm 53 For a DRG $\Gamma = (X, R)$ with diameter D

(i) For $0 \le i, j \le D$

$$\sum_{r=0}^{D} v_i(\theta_r)\, v_j(\theta_r)\, m_r = \delta_{ij}\, k_i\, |X|$$ "row orthog"

(ii) For $0 \le r, s \le D$

$$\sum_{i=0}^{D} v_i(\theta_r)\, v_i(\theta_s)\, k_i^{-1} = \delta_{rs}\, m_r^{-1}\, |X|$$ "column orthog"

pf (i) In the equation

$$\langle A_i, A_j \rangle = \delta_{ij}\, k_i\, |X|$$

write each of A_i, A_j as a linear combination of the primitive idempotents using LEM 33 (i) and simplify using LEM 52 (ii).

(ii) In the equation

$$\langle E_r, E_s \rangle = \delta_{rs}\, m_r$$

write each of E_r, E_s as a linear combination of the distance matrices and simplify using LEM 52 (i). $\square$

thm 54 For a DRG $\Gamma = (X, R)$ with diameter D

(i) For $0 \le i, j \le D$

$$\sum_{r=0}^{D} u_i(\theta_r)\, u_j(\theta_r)\, m_r = \delta_{i,j}\, k_i^{-1} / |X|$$

(ii) For $0 \le r, s \le D$

$$\sum_{i=0}^{D} u_i(\theta_r)\, u_i(\theta_s)\, k_i = \delta_{r,s}\, m_r^{-1} / |X|$$

pf Evaluate thm 53 using $v_i = u_i k_i$ $(0 \le i \le D)$. $\square$

Another formula for the p^h_{ij}

LEM 55 For a DRG $\Gamma = (X, R)$ with diameter D

and for $0 \le h, i, j \le D$

$$p^h_{ij} = |X|^{-1} k_h^{-1} \langle A_i A_j, A_h \rangle$$

$$= |X|^{-1} k_h^{-1} \langle A_h, A_i A_j \rangle$$

pf Expand $A_i A_j$ using

$$A_i A_j = \sum_{\ell=0}^{D} p^\ell_{ij} A_\ell$$

and use LEM 52 (i) $\square$

thm 5.6 For a DRG $\Gamma = (X, R)$ with diameter D

and for $0 \le h, i, j \le D$

$$p^h_{ij} = |X|^{-1} \sum_{r=0}^{D} v_i(\theta_r)\, v_j(\theta_r)\, u_h(\theta_r)\, m_r$$

Pf Obs

$$\langle A_i A_j, A_h \rangle = \left\langle \sum_{r=0}^{D} v_i(\theta_r)\, v_j(\theta_r)\, E_r, \quad \sum_{s=0}^{D} v_h(\theta_s)\, E_s \right\rangle$$

$$= \sum_{r=0}^{D} v_i(\theta_r)\, v_j(\theta_r)\, v_h(\theta_r)\, m_r$$

Now evaluate the above equation using LEM 5.5 $\square$

thm 57 For a DRG $\Gamma = (X, R)$ with diameter D

the intersection numbers c_i, a_i, b_i are determined by

the spectrum
$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_D \\ m_0 & m_1 & \cdots & m_D \end{pmatrix}$$

as follows

$$|X| = \sum_{i=0}^{D} m_i$$

$$b_0 = k = \theta_0$$

$$v_1 = \lambda \qquad u_1 = \frac{\lambda}{k}$$

$$c_1 = 1$$

$$a_1 = p_{11}^1 = |X|^{-1} \sum_{r=0}^{D} v_1(\theta_r)\, v_1(\theta_r)\, u_1(\theta_r)\, m_r$$

$$b_1 = k - a_1 - 1$$

$$u_2 = \frac{\lambda u_1 - a_1 u_1 - c_1 u_0}{b_1}$$

$$c_2 = p_{11}^2 = |X|^{-1} \sum_{r=0}^{D} v_1(\theta_r)\, v_1(\theta_r)\, u_2(\theta_r)\, m_r$$

$$k_2 = \frac{k b_1}{c_2}$$

$$v_2 = u_2 k_2$$

$$a_2 = \rho_{21}^2 = |X|^{-1} \sum_{r=0}^{p} v_2(\theta_r)\, v_1(\theta_r)\, u_2(\theta_r)\, m_r$$

$$b_2 = k - a_2 - c_2$$

$$u_3 = \ldots$$

etc.

$\square$

Lecture 7 Wednesday February 4

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Given DRG $\Gamma = (X, R)$ with diameter D and standard module $V = \mathbb{F}^X$. Recall for $0 \leq j \leq D$ the subspace $E_j V$ is the eigenspace of adjacency matrix A for the eigenvalue θ_j. We now examine the geometry of the vectors $\{ E_j \hat{x} \mid x \in X \}$

LEM 58. Given DRG $\Gamma = (X, R)$ with diameter D.

For $0 \leq i, j \leq D$ and for $x, y \in X$ at $\partial(x,y) = i$

(i) $\langle E_j \hat{x}, E_j \hat{y} \rangle = |X|^{-1} m_j u_i(\theta_j)$

(ii) $\| E_j \hat{x} \|^2 = |X|^{-1} m_j$

(iii) $$u_i(\theta_j) = \frac{\langle E_j \hat{x}, E_j \hat{y} \rangle}{\| E_j \hat{x} \| \, \| E_j \hat{y} \|}$$

"cosine of angle between $E_j \hat{x}$, $E_j \hat{y}$"

Pf (i) Recall by Lem 33

$$E_j = |X|^{-1} m_j \sum_{h=0}^{D} u_h(\theta_j) A_h$$

Obs

$$\langle E_j \hat{x}, E_j \hat{y} \rangle = \langle \overline{E_j}^t E_j \hat{x}, \hat{y} \rangle$$

$$= \langle E_j^2 \hat{x}, \hat{y} \rangle$$

$$= \langle E_j \hat{x}, \hat{y} \rangle$$

$$= (E_j)_{xy}$$

$$= |X|^{-1} m_j u_i(\theta_j)$$

(ii) Set $i=0$ $x=y$ in (i)

(iii) Combine (i), (ii) $\square$

Another view

LEM 59 With ref to LEM 58

$$\det \begin{pmatrix} \| E_j \hat{x} \|^2 & \langle E_j \hat{x}, E_j \hat{y} \rangle \\ \langle E_j \hat{y}, E_j \hat{x} \rangle & \| E_j \hat{y} \|^2 \end{pmatrix} = |X|^{-2} m_j^2 \left(1 - |u_i(\theta_j)|^2 \right)$$

$$= |X|^{-1} m_j \| E_j \hat{x} - u_i(\theta_j) E_j \hat{y} \|^2$$

Pf Use LEM 58 $\square$

COR 60 Given DRG Γ with diameter D. Then

$$|u_i(\theta_j)| \leq 1 \qquad (0 \leq i, j \leq D)$$

Pf. Clear from LEM 59 $\square$

COR 61. Given a DRG $\Gamma = (X, R)$ with diameter D.

For $0 \le i, j \le D$ the following are equiv.

(i) $u_i(\theta_j) = 1$

(ii) $E_j \hat{x} = E_j \hat{y}$ for all $x, y \in X$ at $\partial(x, y) = i$

(iii) $\exists\ x, y \in X$ with $\partial(x, y) = i$ and $E_j \hat{x} = E_j \hat{y}$

pf Use LEM 59 $\square$

COR 62 Given DRG $\Gamma = (X, R)$ with diameter D

For $0 \le i, j \le D$ the following are equivalent

(i) $u_i(\theta_j) = -1$

(ii) $E_j \hat{x} = -E_j \hat{y}$ for all $x, y \in X$ at $\partial(x, y) = i$

(iii) $\exists\ x, y \in X$ with $\partial(x, y) = i$ and $E_j \hat{x} = -E_j \hat{y}$

pf Use LEM 59. $\square$

EXAMPLE 63 3-cube $\Gamma = H(3,2)$ $D=3$, $c_i = i$, $b_i = D-i$

Spectrum $\begin{pmatrix} 3 & 1 & -1 & -3 \\ 1 & 3 & 3 & 1 \end{pmatrix}$

i	0	1	2	3
u_i	1	$\frac{\lambda}{3}$	$\frac{\lambda^2-3}{6}$	$\frac{\lambda(\lambda^2-7)}{6}$

table of cosines is

$$\begin{array}{c} \\ A_0 \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} E_0 & E_1 & E_2 & E_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{3} & \frac{-1}{3} & -1 \\ 1 & \frac{-1}{3} & \frac{1}{3} & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \end{array}$$

(i,j)-entry is $u_i(\theta_j)$

Ex 63, cont Take $\mathbb{F} = \mathbb{R}$

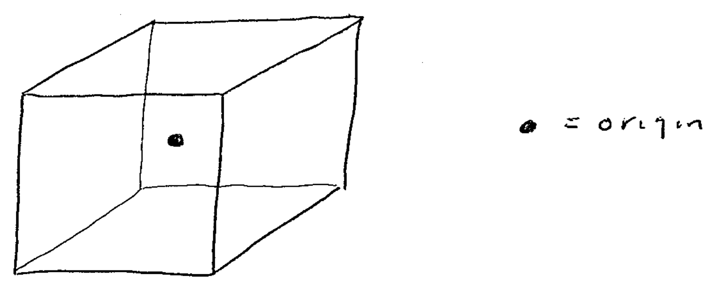
Consider cosines for $E = E_1$

$\dim EV = 3$

So view EV as Euclidean 3-space

the vectors $\{ E\hat{x} \mid x \in X \}$ form the vertices of a

geometric cube centered at the origin



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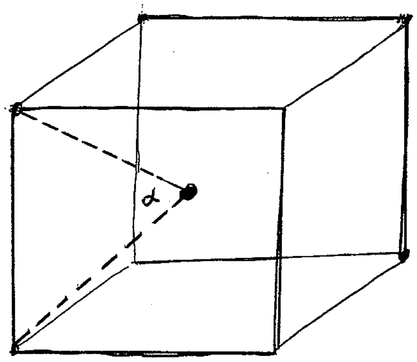
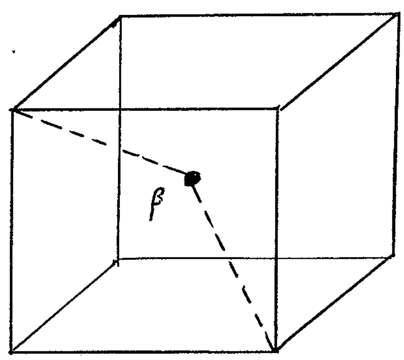
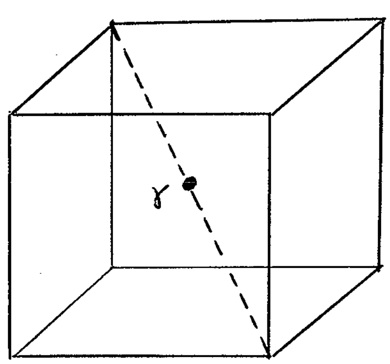


$\cos\alpha = \frac{1}{3}$



$\cos\beta = -1/3$



$\cos\gamma = -1$

$\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ $\quad \Gamma = (X, R)$ is any DRG diam D.

DEF 64 A <u>representation</u> of Γ is a pair

(ρ, H) where H is a nonzero Hermitean space

(with inner product $\langle , \rangle$) and $\rho : X \to H$ is

a map such that

(R1) $\quad H = \mathrm{Span}\{ \rho(x) \mid x \in X \}$

(R2) $\quad \forall \; x, y \in X$

$\langle \rho(x), \rho(y) \rangle$ depends only on $\partial(x, y)$

(R3) $\quad \forall \; x \in X$

$$\sum_{y \in \Gamma(x)} \rho(y) \in \mathrm{Span}(\rho(x))$$

The above representation is <u>nondegenerate</u> whenever

$\{ \rho(x) \mid x \in X \}$ are mutually distinct

Ex 65 For the D-cube $\Gamma = H(D,2)$ view

$$X = \left\{ \alpha_1 \alpha_2 \cdots \alpha_D \;\middle|\; \alpha_i \in \{1,-1\} \quad 1 \le i \le D \right\}$$

Take

$$H = \mathbb{F}^D \qquad (\text{column vectors}) \qquad \mathbb{F} = \mathbb{C} \text{ or } \mathbb{R}$$

$$\langle u, v \rangle = u^t \bar{v} \qquad (u, v \in H)$$

For a vertex $x = \alpha_1 \alpha_2 \cdots \alpha_D$ of Γ define

$$\rho(x) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_D \end{pmatrix} \in H$$

Then (ρ, H) is a representation of Γ.

Given a representation (ρ, H) of a DRG $\Gamma = (X, R)$

the corresponding <u>Gram matrix</u> is the matrix of inner products

$$E = \Big(\langle \rho(x), \rho(y) \rangle \Big)_{x,y \in X}$$

$$\in \mathrm{Mat}_X(\mathbb{F})$$

Representations (ρ, H) and (ρ', H') of Γ are called

<u>equivalent</u> whenever their Gram matrices E, E' satisfy

$$E' \in \mathrm{Span}(E)$$

Usually we do not distinguish between equivalent reps of Γ.

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

LEM 66 For a DRG $\Gamma = (X, R)$ with diameter D,

and given a primitive idempotent E_j of Γ

(i) (ρ, H) is a rep of Γ where

$$H = E_j V \qquad \text{with inner product inherited from } V$$

and

$$\rho : \begin{array}{ccc} X & \to & H \\ x & \to & E_j \hat{x} \end{array}$$

(ii) For this rep. the Gram matrix is E_j.

(iii) For $0 \le i \le D$ and $x \in X$

$$\sum_{y \in \Gamma_i(x)} \rho(y) = v_i(\theta_j)\,\rho(x) \qquad (*)$$

(iv) The rep is nondegenerate iff $u_i(\theta_j) \ne 1$ $(1 \le j \le D)$

pf (i)–(iii) Check axioms R1–R3 in Def 64

R1: $\mathrm{Span}(\rho(x) \mid x \in X)$ = col space of E_j

$= H$

R2: Gram matrix is E_j by LEM 58 (i)

R3: To verify * consider col x in

$$E_j A_i = v_i(\theta_j) E_j$$

Note R3 follows from *.

(iv) Use Cor 61 $\square$

thm 67 Given a DRG $\Gamma = (X, R)$ with diam D

Given a representation (ρ, H) of Γ.

then this rep is equivalent to a rep of Γ from LEM 67, for a unique j.

pf Consider Gram matrix

$$E = \left(\langle \rho(x), \rho(y) \rangle \right)_{x,y \in X}$$

Show E is a nonzero scalar multiple of a prim idempotent of Γ.

Obs $E \neq 0$ by R1

By R2 $\exists$ $\alpha_i \in \mathbb{F}$ $(0 \le i \le D)$ s.t

$$E = \sum_{i=0}^{D} \alpha_i A_i$$

So

$$E \in M$$

Pick $x \in X$. By R3 $\exists$ $\theta \in \mathbb{F}$ s.t.

$$\sum_{y \in \Gamma(x)} \rho(y) = \theta \rho(x) \qquad *$$

In (*) take $\langle , \rangle$ of each side with $\rho(x)$ to find θ is indep of x

Now $(*)$ implies

$$EA = \theta E$$

So θ is an eigenvalue of Γ and

$$E \in \mathrm{Span}(E_j) \qquad \text{where} \qquad \theta = \theta_j. \qquad \square$$

Given a DRG $\Gamma = (X, R)$ with diameter D

Given a primitive idempotent E_j of Γ

We call E_j <u>nondegenerate</u> whenever

$$u_i(\theta_j) \neq 1 \qquad (1 \le i \le D)$$

[ie whenever the corresp representation from L66 is nondegenerate]

$\square$

Lecture 8 Friday, Feb 6

Recall the Hamming graph $H(D, N)$

$$c_i = i \qquad b_i = (N-1)(D-i)$$

$$k_i = m_i = (N-1)^i \binom{D}{i} \qquad \theta_i = (D-i)(N-1) - i$$

$$|X| = N^D$$

We now explain how the corresp polynomials u_i are

<u>Krawtchouk</u> polynomials.

Notation $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

For an integer $i \geq 0$ and $a \in \mathbb{F}$

$$(a)_i = \underbrace{a(a+1)(a+2)\cdots(a+i-1)}_{i \text{ terms}}$$

We interpret

$$(a)_0 = 1$$

Obs for integers $i, j \geq 0$

$$(-j)_i = \begin{cases} \neq 0 & \text{if } \quad 0 \leq i \leq j \\ 0 & \text{if } \quad i \geq j+1 \end{cases}$$

For integers $r, s \geq 0$ and scalars

$$\alpha_1, \alpha_2 \ldots \alpha_r, \quad \beta_1, \beta_2 \ldots, \beta_s \in \mathbb{F}$$

the corresponding <u>hypergeometric series</u> in a variable z is

$$ {}_rF_s\left[\begin{matrix} \alpha_1 & \alpha_2 & \cdots & \alpha_r \\ \beta_1 & \beta_2 & \cdots & \beta_s \end{matrix} \;\middle|\; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_r)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_s)_n} \frac{z^n}{n!}$$

If at least one of $\alpha_1, \alpha_2 \ldots \alpha_r$ is an integer ≤ 0 then

the series has finitely many nonzero terms and questions of convergence

do not arise. In this course we consider this situation only.

For an integer $D \geq 0$ and $p \in \mathbb{F} \setminus \{0, 1\}$,

For $0 \leq i \leq D$ define $k_i = k_i(\lambda, p, D) \in \mathbb{F}[\lambda]$ by

$$k_i = {}_2F_1\left[\begin{matrix} -i, \ -\lambda \\ -D \end{matrix} \,\middle|\, \frac{1}{p}\right]$$

So

$$k_0 = 1$$

$$k_1 = 1 - \frac{\lambda}{D}\frac{1}{p}$$

$$k_2 = 1 - \frac{2\lambda}{D}\frac{1}{p} + \frac{\lambda(\lambda-1)}{D(D-1)}\frac{1}{p^2}$$

$$\vdots$$

Call $\{k_i\}_{i=0}^{D}$ the <u>Krawtchouk</u> <u>polynomials</u> with parameters D, p

One checks that $\{k_i\}_{i=0}^{D}$ satisfy the 3-term recurrence

$$-\lambda k_i = p(D-i)k_{i+1} - (p(D-i) + i(1-p))k_i + i(1-p)k_{i-1}$$
$$(0 \leq i \leq D)$$

Comparing this with our 3-term rec for $\{u_i\}_{i=0}^{D}$ we find

LEM 68 For the Hamming graph $H(D,N)$

$$u_i(\theta_j) = K_i(j) \qquad (0 \le i, j \le D)$$

where

$$K_i = K_i(\lambda, p; D), \qquad p = 1 - \frac{1}{N} \qquad \square$$

As an aside

We mention a few facts about the Krawtchouk polynomials.

For arb D, p

- the (row) orthogonality is

$$\sum_{r=0}^{D} K_i(r) K_j(r) \binom{D}{r} p^r (1-p)^{D-r} = \delta_{ij} \binom{D}{i}^{-1} \left(\frac{1-p}{p}\right)^D \qquad (0 \le i, j \le D)$$

[For $p = 1 - \frac{1}{N}$ $N = 2, 3 \ldots$ this is just thm 54(i)]

- Duality relation:

$$K_i(j) = K_j(i) \qquad (0 \le i, j \le D)$$

- Difference equation

$$-i K_i(j) = p(D-j) K_i(j+1) - \big(p(D-j) + j(1-p)\big) K_i(j) + j(1-p) K_i(j-1) \qquad (0 \le i, j \le D)$$

- "Forward shift operator"

$$K_i(j+1, p, D) - K_i(j, p, D) = -\frac{i}{Dp} K_{i-1}(j, p, D-1)$$

- "Backward shift operator"

$$(D+1-j)\, K_i(j, p, D) - j\,\frac{1-p}{p}\, K_i(j-1, p, D)$$
$$= (D+1)\, K_{i+1}(j, p, D+1)$$

- Generating function: For $0 \le i, j \le D$

$$\left(1 - \frac{1-p}{p} t\right)^j (1+t)^{D-j} = \sum_{i=0}^{D} \binom{D}{i} K_i(j, p, D)\, t^i$$

t = indeterminate

—

The Krawtchouk polynomials are members of a very general class of polynomials called the <u>Askey-scheme</u>

We will discuss the Askey scheme a bit later.

Next goal: Krein parameters

$\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ Given DRG $\Gamma = (X, R)$ diam D

For $B, C \in Mat_X(\mathbb{F})$ define

$$B \circ C \in Mat_X(\mathbb{F})$$

by

$$(B \circ C)_{xy} = B_{xy} C_{xy} \qquad \forall x, y \in X \qquad \text{"entry-wise mult"}$$

Obs

$$A_i \circ A_j = \delta_{ij} A_i \qquad (0 \le i, j \le D)$$

So Bose-Mesner alg M is closed under $\circ$

So $\exists$ $q^h_{ij} \in \mathbb{F}$ $(0 \le h, i, j \le D)$ s.t

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{ij} E_h \qquad (0 \le i, j \le D)$$

By constr

$$q^h_{ij} = q^h_{ji} \qquad (0 \le h, i, j \le D)$$

the primitive idempotents are real so $q^h_{ij} \in \mathbb{R}$ $(0 \le h, i, j \le D)$.

As we will see

$$q^h_{ij} \ge 0 \qquad (0 \le h, i, j \le D)$$

the q^h_{ij} are called the Krein parameters of Γ.

In order to avoid dealing directly with the entry-wise product we do the following.

Until further notice Fix $x \in X$. We call x the "base vertex"

For $0 \le i \le D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{F})$ with (y,y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{if } \partial(x,y) \ne i \end{cases} \qquad y \in X$$

Call E_i^* the ith <u>dual idempotent of Γ with respect to x</u>

For $0 \le i \le D$ and $y \in X$

$$E_i^* \hat{y} = \begin{cases} \hat{y} & \text{if } \partial(x,y) = i \\ 0 & \text{if } \partial(x,y) \ne i \end{cases}$$

So

$$E_i^* V = \mathrm{Span}\{ \hat{y} \mid y \in \Gamma_i(x) \}$$

We call $E_i^* V$ the <u>ith subconstituent of Γ with respect to x</u>

Obs

$$V = \sum_{i=0}^{D} E_i^* V \qquad (\text{orthog dir sum})$$

By construction

$$\{E_i^*\}_{i=0}^{D} \quad \text{are lin indep}$$

$$tr(E_i^*) = rank(E_i^*) \qquad (0 \le i \le D)$$
$$= k_i$$

LEM 69 With above notation

(i) $I = \sum_{i=0}^{D} E_i^*$

(ii) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \le i, j \le D)$

(iii) $\overline{E_i^*} = E_i^*$ $(0 \le i \le D)$

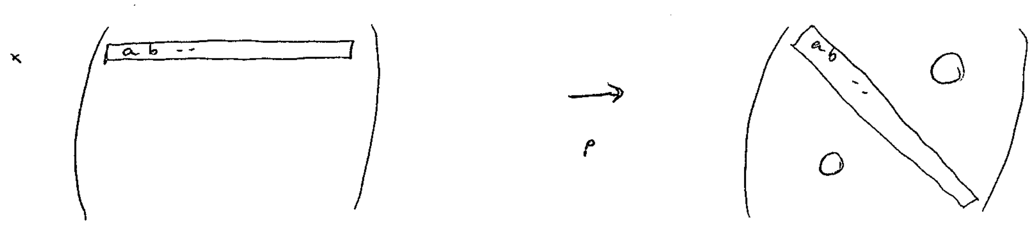
(iv) $(E_i^*)^t = E_i^*$ $(0 \le i \le D)$

pf clear.

DEF 70 With above notation, obs $\{E_i^*\}_{i=0}^{D}$ is a basis for a commutative subalgebra of $\mathrm{Mat}_X(\mathbb{F})$ denoted $M^* = M^*(x)$. We call M^* the <u>dual Bose-Mesner algebra</u> of Γ with respect to x. Obs $\{E_i^*\}_{i=0}^{D}$ are the primitive idempotents of M^*

DEF 71 With above notation $\forall B \in \mathrm{Mat}_X(\mathbb{F})$ let B^ρ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{F})$ with (y,y)-entry

$$(B^\rho)_{yy} = B_{xy} \qquad \forall y \in X$$



B $\qquad\qquad$ B^ρ

LEM 72 With above notation

$$(B \circ C)^\rho = B^\rho C^\rho \qquad \forall B, C \in \mathrm{Mat}_X(\mathbb{F})$$

pf clear □

Lecture 9 Monday February 9

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Fix a DRG $\Gamma = (X, R)$ with diameter D and Bose-Mesner algebra M

Fix $x \in X$, let $M^* = M^*(x)$ be the dual Bose-Mesner algebra of Γ

with respect to x. Recall map $\rho : \mathrm{Mat}_X(\mathbb{F}) \to \mathrm{Mat}_X(\mathbb{F})$ from Def 71.

LEM 73 With above notation

(i) $A_i^{\rho} = E_i^*$ $\quad (0 \le i \le D)$

(ii) $I^{\rho} = E_0^*$

(iii) $J^{\rho} = I$

pf (i) $\forall y \in X$

$$(A_i^{\rho})_{yy} = (A_i)_{xy} = (E_i^*)_{yy}$$

(ii) Set $i=0$ in (i)

(iii) From def of ρ. $\square$

LEM 74 With above notation

the restriction

$$\rho|_M : M \longrightarrow M^*$$

is an isomorphism of vector spaces [Caution: not iso of algebras]

pf ρ sends the basis $\{A_i\}_{i=0}^D$ to the basis $\{E_i^*\}_{i=0}^D$ $\square$

LEM 75 With above notation,

$\forall\ B, C \in M$ we have

$$\langle B^\rho, C^\rho \rangle = |X|^{-1} \langle B, C \rangle$$

Moreover this quantity equals the (x,x)-entry of $B\overline{C}^t$

pf
$$\begin{aligned}
\langle B^\rho, C^\rho \rangle &= tr\left(B^\rho (\overline{C^\rho})^t \right) \\
&= tr\left(B^\rho \overline{C^\rho} \right) \\
&= \sum_{y \in X} (B^\rho)_{yy} (\overline{C^\rho})_{yy} \\
&= \sum_{y \in X} B_{xy} \overline{C}_{xy} \\
&= (B\overline{C}^t)_{xx}
\end{aligned}$$

Also

$$|X|^{-1}\langle B, C\rangle = |X|^{-1}\,\mathrm{tr}(B\bar{C}^t)$$

$$= |X|^{-1}\sum_{y\in X}(B\bar{C}^t)_{yy}$$

$$= (B\bar{C}^t)_{xx}$$

Since diagonal entries in $B\bar{C}^t$ are all equal $\square$

Here is a useful fact about ρ

LEM 76 With above notation,

$$E_0E_0^*B = E_0B^\rho \qquad \forall B\in M$$

pf Recall $E_0 = |X|^{-1}J$ and

$$E_0^* = \begin{array}{cc} & \begin{array}{cc} x & \end{array} \\ x & \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right) \end{array}$$

$\forall y, z\in X$

$$(E_0E_0^*B)_{yz} = \underbrace{(E_0)_{yx}}_{|X|^{-1}}\underbrace{(E_0^*)_{xx}}_{1}B_{xz}$$

$$= |X|^{-1}B_{xz}$$

Also since B^ρ is diagonal,

$$(E_0 B^\rho)_{yz} = (E_0)_{yz} \; (B^\rho)_{zz}$$

$$= |X|^{-1} B_{xz}$$

$\square$

The situation so far:

$$\begin{tikzcd}
A_i \arrow[r] & E_i^* \\
E_i \arrow[r] & ? \\
M \arrow[r, "\rho"'] & M^*
\end{tikzcd}$$

Def 77 With above notation,

for $0 \le i \le D$ define

$$A_i^* = |X| \, E_i^\rho$$

We obs A_i^* is diagonal with (y,y)-entry

$$(A_i^*)_{yy} = |X| \, (E_i)_{xy}$$

$$= m_i \, u_h(\theta_i) \qquad h = \partial(x,y)$$

Call A_i^* the ith <u>dual</u> <u>distance-matrix</u>

LEM 78 With above notation

(i) $\{A_i^*\}_{i=0}^D$ is a basis for M^*

(ii) For $0 \le i \le D$ and $y \in X$

$$A_i^* \hat{y} = m_i u_h(\theta_i) \hat{y} \qquad h = \partial(x,y)$$

Pf (i) By LEM 74 and since $\{E_i\}_{i=0}^D$ is a basis for M

(ii) By Def 77 $\square$

LEM 79 With above notation

(i) $A_0^* = I$

(ii) $\sum_{i=0}^D A_i^* = |X| E_0^*$

(iii) $\overline{A_i^*} = A_i^*$ $(0 \le i \le D)$

(iv) $(A_i^*)^t = A_i^*$ $(0 \le i \le D)$

(v) $A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^*$ $(0 \le i,j \le D)$

pf (i) $A_0^* = |X| E_0^\rho$ and $E_0 = |X|^{-1} J$

(ii) Apply ρ to

$$\sum_{i=0}^{D} E_i = I$$

(iii), (iv) clear

(v) Apply ρ to

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \qquad \square$$

LEM 80 With above notation

(i) $$A_j^* = m_j \sum_{i=0}^{D} u_i(\theta_j) \quad E_i^* \qquad (0 \le j \le D)$$

(ii) $$E_j^* = |X|^{-1} \sum_{i=0}^{D} v_j(\theta_i) A_i^* \qquad (0 \le j \le D)$$

pf Apply ρ to the equations in LEM 33 $\square$

LEM 81 With above notation

(i) $\langle E_i^*, E_j^* \rangle = \delta_{ij} k_i \qquad (0 \le i,j \le D)$

(ii) $\langle A_i^*, A_j^* \rangle = \delta_{ij} m_i |X| \qquad (0 \le i,j \le D)$

pf Combine LEM 52 and LEM 75 $\square$

Algebraically the p_{ij}^h, q_{ij}^h are very similar, as next few results show

LEM 82 With above notation, and

for $0 \le h, i, j \le D$

$$q_{ij}^h = |X|^{-1} m_h^{-1} \langle A_i^* A_j^*, A_h^* \rangle$$

$$= |X|^{-1} m_h^{-1} \langle A_h^*, A_i^* A_j^* \rangle$$

pf Expand $A_i^* A_j^*$ using

$$A_i^* A_j^* = \sum_{\ell=0}^{D} q_{ij}^{\ell} A_\ell^*$$

and use LEM 81 $\square$

LEM 83 With the above notation.

$$m_h q^h_{ij} = m_i q^i_{hj} = m_j q^j_{ih} \qquad (0 \le h, i, j \le D)$$

pf To obtain $m_h q^h_{ij} = m_i q^i_{hj}$ observe

$$|X| \, m_h q^h_{ij} = \langle A_i^* A_j^*, A_h^* \rangle$$
$$= \langle A_i^*, A_h^* \overline{(A_j^*)}^t \rangle$$
$$= \langle A_i^*, A_h^* A_j^* \rangle$$
$$= |X| m_i q^i_{hj}$$

The rest is similarly obtained. $\square$

LEM 84 With the above notation and for $0 \leq h, i, j, r \leq D$

$$\sum_{\alpha=0}^{D} q^r_{h\alpha}\, q^\alpha_{ij} = \sum_{\beta=0}^{D} q^r_{j\beta}\, q^\beta_{hi}$$

pf Expand each side of

$$A_h^* (A_i^* A_j^*) = (A_h^* A_i^*) A_j^*$$

as a linear combination of the dual distance matrices, and compare coeffs. $\square$

LEM 85 With above notation

(i) $q^h_{0j} = \delta_{hj} \qquad (0 \le h, j \le D)$

(ii) $q^h_{iD} = \delta_{hi} \qquad (0 \le h, i \le D)$

(iii) $q^0_{ij} = \delta_{ij} \, m_i \qquad (0 \le i, j \le D)$

(iv) $\displaystyle\sum_{i=0}^{D} q^h_{ij} = m_j \qquad (0 \le h, j \le D)$

pf (i) obs by L 82

$$q^h_{0j} = |X|^{-1} m_h^{-1} \langle \underbrace{A_0^*}_{I} A_j^*, A_h^* \rangle$$

By L 81

$$\langle A_j^*, A_h^* \rangle = \delta_{jh} \, m_h \, |X|$$

(ii) Sim

(iii) Use LEM 83

(iv) $\displaystyle\sum_{i=0}^{D} q^h_{ij} = |X|^{-1} m_h^{-1} \sum_{i=0}^{D} \langle A_i^* A_j^*, A_h^* \rangle$ (by L 82)

$= m_h^{-1} \langle E_0^* A_j^*, A_h^* \rangle$ (by L 79 (ii))

$= m_h^{-1} m_j \langle E_0^*, A_h^* \rangle$ (by L 80 (i))

$= m_j$ (by L 80, L 81) $\square$

the following result shows how to compute the Krein parameters from the intersection numbers

theorem 86 For a DRG $\Gamma = (X, R)$ and for $0 \le h, i, j \le D$

$$q_{ij}^h = |X|^{-1} m_i m_j \sum_{r=0}^{D} u_r(\theta_i)\, u_r(\theta_j)\, v_r(\theta_h)$$

Pf Obs

$$\langle A_i^* A_j^*, A_h^* \rangle = \left\langle m_i m_j \sum_{r=0}^{D} u_r(\theta_i)\, u_r(\theta_j)\, E_r^*,\ m_h \sum_{s=0}^{D} u_s(\theta_h)\, E_s^* \right\rangle$$

$$= m_h m_i m_j \sum_{r=0}^{D} u_r(\theta_i)\, u_r(\theta_j)\, u_r(\theta_h)\, k_r$$

$$= m_h m_i m_j \sum_{r=0}^{D} u_r(\theta_i)\, u_r(\theta_j)\, v_r(\theta_h)$$

Now evaluate the above equation using L 82 $\square$

Next goal: show Krein parameters are nonneg.

LEM 87 With above notation and for $0 \le h, i, j, r, s, t \le D$

(i) $\langle E_i^* A_j E_h^*, E_r^* A_s E_t^* \rangle = \delta_{ir} \delta_{js} \delta_{ht} k_h p^h_{ij}$

(ii) $\langle E_i A_j^* E_h, E_r A_s^* E_t \rangle = \delta_{ir} \delta_{js} \delta_{ht} m_h q^h_{ij}$

pf (i) $\langle E_i^* A_j E_h^*, E_r^* A_s E_t^* \rangle$

$$= \mathrm{tr}\left(E_i^* A_j E_h^* \overline{(E_r^* A_s E_t^*)}^t \right)$$

$$= \mathrm{tr}\left(E_i^* A_j E_h^* E_t^* A_s E_r^* \right)$$

$$= \delta_{ir} \delta_{ht} \mathrm{tr}\left(E_i^* A_j E_h^* A_s \right)$$

$$\mathrm{tr}(E_i^* A_j E_h^* A_s) = \sum_{y \in X} \sum_{z \in X} (E_i^*)_{yy} (A_j)_{yz} (E_h^*)_{zz} (A_s)_{zy}$$

$$= \sum_{y \in X} \sum_{z \in X} (E_i^*)_{yy} \underbrace{(A_j \circ A_s)_{yz}}_{\delta_{js} A_j} (E_h^*)_{zz}$$

$$= \delta_{js} \sum_{y \in X} \sum_{z \in X} (E_i^*)_{yy} (A_j)_{yz} (E_h^*)_{zz}$$

$$= \delta_{js} \sum_{\substack{y \in \Gamma_i(x) \\ z \in \Gamma_h(x) \\ \partial(y,z) = j}} 1$$

$$= \delta_{js} k_h p^h_{ij}$$

(ii) $\langle E_i A_j^* E_h,\ E_r A_s^* E_t \rangle$

$$= \operatorname{tr}\left(E_i A_j^* E_h \ \left(\overline{E_r A_s^* E_t}\right)^t \right)$$

$$= \operatorname{tr}\left(E_i A_j^* E_h \ E_t A_s^* E_r \right) \qquad \operatorname{tr}(uv) = \operatorname{tr}(vu)$$

$$= \delta_{ir}\,\delta_{ht}\ \operatorname{tr}\left(E_i A_j^* E_h A_s^* \right)$$

$$\operatorname{tr}\left(E_i A_j^* E_h A_s^* \right) = \sum_{y\in X} \sum_{z\in X} (E_i)_{yz} \underbrace{(A_j^*)_{zz}}_{|X|(E_j)_{xz}} (E_h)_{zy} \underbrace{(A_s^*)_{yy}}_{|X|(E_s)_{xy}}$$

$$= |X|^2 \sum_{y\in X} \sum_{z\in X} (E_s)_{xy} \ (E_i \circ E_h)_{yz} \ (E_j)_{zx}$$

$$= |X|^2 \left((x,x)\text{-entry of } \quad E_s \left(E_i \circ E_h \right) E_j \right)$$

$$= |X| \ \operatorname{trace}\left(E_s \left(E_i \circ E_h \right) E_j \right)$$

$$= |X| \ \operatorname{tr}\left(\left(E_i \circ E_h\right) E_j E_s \right)$$

$$= \delta_{js}\ |X|\ \operatorname{tr}\left(\underbrace{\left(E_i \circ E_h \right)}_{|X|^{-1} \sum_{\ell=0}^{D} q_{ih}^{\ell} E_\ell} E_j \right)$$

$$= \delta_{js}\ q_{ih}^{j}\ \operatorname{tr}(E_j)$$

$$= \delta_{js}\ q_{ih}^{j}\ m_j$$

$$= \delta_{js}\ m_h\, q_{ij}^{h} \qquad \square$$

Ref to LEM 87 setting $r=i,\ s=j,\ t=h$ gives

$$\| E_i^* A_j E_h^* \|^2 = k_h\, p^h_{ij} \qquad (0 \le h,i,j \le D)$$

$$\| E_i A_j^* E_h \|^2 = m_h\, q^h_{ij} \qquad (0 \le h,i,j \le D)$$

thm 88 (Krein condition) We have

$$q^h_{ij} \ge 0 \qquad (0 \le h,i,j \le D)$$

pf Recall $\|B\|^2 \ge 0 \quad \forall\ B \in \mathrm{Mat}_X(\mathbb{F})$ $\square$

thm 89 (triple product relations) For $0 \le h,i,j \le D$

(i) $E_i^* A_j E_h^* = 0$ iff $p^h_{ij} = 0$

(ii) $E_i A_j^* E_h = 0$ iff $q^h_{ij} = 0$

pf Recall $\|B\|^2 = 0$ iff $B = 0$, $\forall B \in \mathrm{Mat}_X(\mathbb{F})$ $\square$

Notation For subspaces Y, Z of $\mathrm{Mat}_X(\mathbb{F})$ define

$$YZ = \mathrm{Span}\{ yz \mid y \in Y, \ z \in Z \}$$

thm 90 With above notation

(i) the space $M^* M M^*$ has an orthogonal basis

$$\{ E_i^* A_j E_h^* \mid 0 \le h,i,j \le D, \ p^h_{ij} \ne 0 \}$$

(ii) the space $M M^* M$ has an orthogonal basis

$$\{ E_i A_j^* E_h \mid 0 \le h,i,j \le D, \ q^h_{ij} \ne 0 \}$$

pf Routine using LEM 87, thm 89 $\square$

$\square$

Lecture 10 Wednesday, February 11

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Fix a DRG $\Gamma = (X, R)$ with diameter D and Bose-Mesner algebra M

Fix $x \in X$ and let $M^* = M^*(x)$ be the dual Bose-Mesner algebra of Γ wrt x.

Next goal: an interpretation of the Krein parameters.

Consider standard module $V = \mathbb{F}^X$

Pick $v \in V$

$$v = \sum_{y \in X} \alpha_y \hat{y} \qquad \alpha_y \in \mathbb{F}$$

View v as function

$$v : \quad X \longrightarrow \mathbb{F}$$
$$y \rightarrow \alpha_y$$

View V as set of all functions $X \to \mathbb{F}$

Vector space V, together with product of functions is

$\mathbb{F}$-algebra. For

$$v = \sum_{y \in X} \alpha_y \hat{y} \qquad w = \sum_{y \in X} \beta_y \hat{y}$$

in V, write

$$v \circ w = \sum_{y \in X} \alpha_y \beta_y \hat{y}$$

to represent the product of v, w viewed as functions.

LEM 91 With above notation

for $0 \le i \le D$ and $v \in V$

$$A_i^* v = |X| \, E_i \hat{x} \circ v$$

pf For $y \in X$ coordinate y of $A_i^* v$ is

$$(A_i^* v)_y = (A_i^*)_{yy} v_y$$

$$= |X| (E_i)_{xy} v_y$$

Also

$$(E_i \hat{x} \circ v)_y = (E_i \hat{x})_y v_y$$

$$= (E_i)_{yx} v_y$$

$$= (E_i)_{xy} v_y$$

Result follows. $\square$

For subspaces Y, Z of V define

$$Y \circ Z = \mathrm{Span}\{ y \circ z \,/\, y \in Y,\ z \in Z \}$$

thm 92 With above notation and for $0 \le i, j \le D$

$$E_i V \circ E_j V = \sum_{\substack{0 \le h \le D \\ q_{ij}^h \ne 0}} E_h V$$

Pf $\subseteq$: Given h $(0 \le h \le D)$ such that $q_{ij}^h = 0$ show

$$E_h (E_i V \circ E_j V) = 0$$

Suffices to show: $\forall y, z \in X$

$$E_h (E_i \hat{y} \circ E_j \hat{z}) = 0$$

Since our base vertex x was arbitrary, WLOG $x = y$

Show

$$E_h (E_i \hat{x} \circ E_j \hat{z}) = 0 \qquad (*)$$

But

$$E_h (E_i \hat{x} \circ E_j \hat{z}) = E_h \left(|X|^{-1} A_i^* E_j \hat{z} \right)$$

$$= |X|^{-1} (E_h A_i^* E_j) \hat{z}$$

But $q_{ij}^h = 0$ so $E_h A_i^* E_j = 0$ which gives $(*)$.

$\supseteq$: For $0 \le h \le D$ such that $q^h_{ij} \neq 0$ show

$$E_iV \circ E_jV \supseteq E_hV.$$

Obs

$$E_iV \circ E_jV = \operatorname{Span}\{E_i\hat{y} \circ E_j\hat{z} \mid y, z \in X\}$$

$$\supseteq \operatorname{Span}\{\underbrace{E_i\hat{y} \circ E_j\hat{y}}_{(E_i \circ E_j)\hat{y}} \mid y \in X\}$$

$$= (E_i \circ E_j)\operatorname{Span}\{\hat{y} \mid y \in X\}$$

$$= (E_i \circ E_j)V$$

$$\supseteq (E_i \circ E_j)E_hV$$

$$= \left(|X|^{-1}\sum_{\ell=0}^{D} q^{\ell}_{ij}E_\ell\right)E_hV$$

$$= \underset{\neq 0}{|X|^{-1}}\,\underset{\neq 0}{q^h_{ij}}\,E_hV$$

$$= E_hV$$

$\square$

COR 93 With above notation and for $0 \le i,j \le D$

$$m_i m_j \ge \sum_{\substack{0 \le h \le D \\ q^h_{ij}}} m_h$$

pf Recall

$$m_r = \dim E_r V \qquad (0 \le r \le D)$$

Obs

$$\dim (E_i V \circ E_j V) \le (\dim E_i V)(\dim E_j V).$$

Result follows. $\square$

COR 93 Gives another "Feasibility condition" on the intersection numbers.

LEM 94 With above notation and for $0 \le j \le D$,

define a binary operation

$$* : \begin{array}{ccc} E_j V \times E_j V & \longrightarrow & E_j V \\ u \quad\quad v & \longrightarrow & u * v \end{array}$$

where $u * v = E_j (u \circ v)$

Then

(i) $u * v = v * u \quad \forall\, u, v \in E_j V$

(ii) $u * (v + v') = u * v + u * v' \quad \forall\, u, v, v' \in E_j V$

(iii) $(\alpha u) * v = \alpha (u * v) \quad \forall\, \alpha \in \mathbb{F} \ \forall\, u, v \in E_j V$

(iv) $u * v = 0 \ \forall\, u, v \in E_j V$ iff $q^j_{jj} = 0$

pf (i) – (iii) Clear

(iv) By th 92 $\square$

Note 95 With reference to LEM 94 we call the vector space $E_j V$ together with $*$ the <u>Norton algebra</u> on $E_j V$. We denote this algebra by N_j. N_j is <u>commutative</u>, <u>nonassociative</u>, <u>no mult identity</u>.

By an <u>automorphism</u> of $N_{\mathfrak{z}}$ we mean an iso of vector spaces $\sigma : E_{\mathfrak{z}}V \to E_{\mathfrak{z}}V$ such that

$$\sigma(u * v) = \sigma(u) * \sigma(v) \qquad \forall\, u, v \in E_{\mathfrak{z}}V$$

The set of automorphisms of $N_{\mathfrak{z}}$ forms a group under composition, denoted $\mathrm{Aut}(N_{\mathfrak{z}})$.

Next goal: How is $\mathrm{Aut}(N_{\mathfrak{z}})$ related to $\mathrm{Aut}(\Gamma)$?

LEM 96 Given a DRG $\Gamma = (X, R)$ with diameter D

then for $0 \le j \le D$

(i) $\sigma|_{E_j V} \in Aut(N_j) \quad \forall \sigma \in Aut(\Gamma)$

(ii) $Aut(\Gamma) \longrightarrow Aut(N_j)$ is hom of gps.

$\sigma \longrightarrow \sigma|_{E_j V}$

(iii) Suppose $u_i(\theta_j) \ne 1$ for $1 \le i \le D$. Then the hom in (ii)

is injective.

Pf (i) σ is invertible on V and $\sigma E_j V \subseteq E_j V$ so

$\sigma|_{E_j V} : E_j V \to E_j V$ is iso of v.s.

Also $\forall u, v \in E_j V$

$$\sigma(u * v) \overset{?}{=} \sigma(u) * \sigma(v)$$

$$\sigma(u * v) = \sigma(E_j(u \circ v)) = E_j \sigma(u \circ v)$$

$$\sigma(u) * \sigma(v) = E_j(\sigma(u) \circ \sigma(v))$$

Obs $\sigma(u \circ v) = \sigma(u) \circ \sigma(v)$ since σ permutes

the coordinates in V

(ii) By constr.

(iii) Each $\sigma \in \mathrm{Aut}(\Gamma)$ permutes the vectors

$$\{ E_y \hat{y} \mid y \in X \}$$

and these vectors are mutually distinct by L66 (iv)

Result follows. $\square$

- The monster finite simple group (Mon) was first constructed around 1980 by Robert Greiss.
- At that time the character table of Mon was known, but it was not known if that table corresponded with an actual group.
- Here is a summary of the construction of Mon, due to Greiss.
- There is a mild generalization of a PRG called a <u>commutative association scheme</u>.
- Any finite gp G gives an example, called the group scheme Γ_G
- There is a natural injection of groups $G \to \mathrm{Aut}(\Gamma_G)$.
- Using the character table of Mon, compute the intersection numbers p^h_{ij} and Krein parameters q^h_{ij} of Γ_{Mon}.
- Find j where m_j is small and $q^j_{jj} \neq 0$
- Guess abstract structure of Norton algebra N_j using p^h_{ij}, q^h_{ij}
- Compute $\mathrm{Aut}(N_j)$
- Using the injection $\mathrm{Mon} \to \mathrm{Aut}(\Gamma_{Mon}) \to \mathrm{Aut}(N_j)$ find Mon as a subgp of $\mathrm{Aut}(N_j)$ $\square$

$\square$

Lecture 11 Friday February 13

Given a DRG $\Gamma = (X, R)$ with diameter D and primitive idempotents $\{E_i\}_{i=0}^{D}$. So far we have assumed the E_i are in the natural order:

$$k = \theta_0 > \theta_1 > \cdots > \theta_D$$

This was for notational convenience only.

We could have used any order such that $\theta_0 = k$.

For the moment let $\{E_i\}_{i=0}^{D}$ be any ordering such that $\theta_0 = k$.

Def 97 With the above notation the ordering $\{E_i\}_{i=0}^{D}$ is called Q-polynomial whenever the following (i), (ii) hold for $0 \le h, i, j \le D$

(i) $q^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two

(ii) $q^h_{ij} \neq 0$ if one of h, i, j is equal to the sum of the other two.

—

Γ might have 0, 1, or ≥ 2 Q-poly orderings of its primitive idempotents.

Until further notice fix a Q-polynomial ordering $\{E_i\}_{i=0}^{D}$ of the primitive idempotents of Γ.

Define

$$c_i^* = q_{1\,i-1}^{i}, \qquad a_i^* = q_{1\,i}^{i}, \qquad b_i^* = q_{1\,i+1}^{i}$$

for $0 \le i \le D$. So

$$c_0^* = 0, \quad a_0^* = 0, \quad b_0^* = 0, \quad c_1^* = 1,$$

$$c_i^* > 0 \qquad 1 \le i \le D,$$

$$b_i^* > 0 \qquad 0 \le i \le D-1$$

LEM 98 With above notation

(i) $\quad m_i c_i^* = m_{i-1} b_{i-1}^* \qquad (1 \le i \le D)$

(ii) $\quad m_i = \dfrac{b_0^* b_1^* \cdots b_{i-1}^*}{c_1^* c_2^* \cdots c_i^*} \qquad (0 \le i \le D)$

Pf (i) This is

$$m_h q_{ij}^{h} = m_i q_{hj}^{i}$$

with $j = 1$ and $h = i-1$

(ii) Use (i) above and $m_0 = 1$ $\square$

Fix $x \in X$ and recall $A_i^* = A_i^*(x)$ $E_i^* = E_i^*(x)$ $(0 \le i \le D)$.

We abbreviate $A^* = A_1^*$ and call this the <u>dual adjacency matrix</u> wrt x (and the given Q-poly structure)

LEM 99 With above notation

$$A^* A_i^* = b_{i-1}^* A_{i-1}^* + a_i^* A_i^* + c_{i+1}^* A_{i+1}^* \quad (0 \le i \le D)$$

where $A_{-1}^* = 0$, $A_{D+1}^* = 0$, $b_{-1}^* = 1$, $c_{D+1}^* = 1$.

pf This is just

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*$$

with $j = 1$ $\square$

$\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$

DEF 100 With above notation define polynomials

$\{v_i^*\}_{i=0}^{D+1}$ in $\mathbb{F}[\lambda]$ by

$v_0^* = 1, \qquad v_1^* = \lambda,$

$\lambda v_i^* = c_{i+1}^* v_{i+1}^* + a_i^* v_i^* + b_{i-1}^* v_{i-1}^* \qquad 1 \le i \le D$

where $c_{D+1}^* = 1$.

LEM 101 With above notation

(i) $\deg v_i^* = i \qquad (0 \le i \le D+1)$

(ii) the coeff of λ^i in v_i^* is $(c_1^* c_2^* \cdots c_i^*)^{-1} \qquad (0 \le i \le D+1)$

(iii) $v_i^*(A^*) = A_i^* \qquad (0 \le i \le D)$

(iv) $v_{D+1}^*(A^*) = 0$

Pf (i), (ii) Clear from Def 100

(iii), (iv) Compare Lem 99, Def 100 $\square$

In LEM 80 we saw

$$A^* = m_1 \sum_{i=0}^{D} u_i(\theta_1) \; E_i^*$$

Abbrev

$$\theta_i^* = m_1 u_i(\theta_1) \qquad (0 \le i \le D)$$

so that

$$A^* = \sum_{i=0}^{D} \theta_i^* E_i^*$$

We will see in a moment that $\{\theta_i^*\}_{i=0}^{D}$ are mutually distinct.

LEM 102 With above notation

(i) A^* generates M^*

(ii) $\{\theta_i^*\}_{i=0}^{D}$ are mutually distinct

(iii) $\{\theta_i^*\}_{i=0}^{D}$ are the zeros of v_{D+1}^*

pf (i) By LEM 101 (iii) and since $\{A_i^*\}_{i=0}^{D}$ is a basis for M^*.

(ii) By (i)

(iii) By L 101 (iv) v_{D+1}^* is a scalar mult of the min poly of A^* □

Def 103 With above notation def

$$B^* = \begin{pmatrix} a_0^* & b_0^* & & & 0 \\ c_1^* & a_1^* & b_1^* & & \\ & c_2^* & \ddots & \ddots & \\ & & \ddots & \ddots & b_{D-1}^* \\ 0 & & & c_D^* & a_D^* \end{pmatrix} \qquad D+1 \times D+1$$

Call B^* the <u>dual</u> <u>intersection</u> <u>matrix</u> of Γ (with respect to the given Q-poly structure)

LEM 104 With above notation the following coincide

(i) the min poly of A^*

(ii) the min poly of B^*

(iii) the char poly of B^*

Pf Sim to L27

COR 105 With above notation

the eigenvalues of B^* are $\{\theta_i^*\}_{i=0}^{D}$.

LEM 106 With above notation for any eigenvalue

$\theta^* = \theta_j^*$ of B^* define a row vector

$$v^* = (v_0^*(\theta^*), v_1^*(\theta^*), \ldots, v_D^*(\theta^*))$$

where the v_i^* are from Def 100. Then

$$v^* B^* = \theta^* v^*$$

Pf Sim to L29 $\square$

DEF 107 With above notation define polys $\{u_i^*\}_{i=0}^D$ in $\mathbb{F}[\lambda]$ by

$$u_0^* = 1, \qquad u_1^* = \frac{\lambda}{m_1}$$

$$\lambda u_i^* = c_i^* u_{i-1}^* + a_i^* u_i^* + b_i^* u_{i+1}^* \qquad (1 \le i \le D-1)$$

LEM 108 With above notation

$$u_i^* = \frac{v_i^*}{m_i} \qquad (0 \le i \le D)$$

pf Sim to L31. $\square$

LEM 109 With above notation for an eigenvalue $\theta^* = \theta_j^*$ of B^* define a col vector

$$u^* = \begin{pmatrix} u_0^*(\theta^*) \\ u_1^*(\theta^*) \\ \vdots \\ u_D^*(\theta^*) \end{pmatrix}$$

Then

$$B^* u^* = \theta^* u^*$$

pf Sim to L32

Thm 110 (Askey-Wilson duality) With above notation

$$u_i(\theta_j) = u_j^*(\theta_i^*) \qquad (0 \le i, j \le D)$$

Pf Obs

$$\begin{aligned} A_j^* &= A_j^* I \\ &= A_j^* \sum_{i=0}^{D} E_i^* \\ &= v_j^*(A^*) \sum_{i=0}^{D} E_i^* \\ &= \sum_{i=0}^{D} v_j^*(\theta_i^*) E_i^* \\ &= m_j \sum_{i=0}^{D} u_j^*(\theta_i^*) E_i^* \end{aligned}$$

Also by L80

$$A_j^* = m_j \sum_{i=0}^{D} u_i(\theta_j) E_i^*$$

Result follows. $\square$

LEM 111 With above notation

(i) $u_i^*(\theta_0^*) = 1 \qquad (0 \le i \le D)$

(ii) $v_i^*(\theta_0^*) = m_i \qquad (0 \le i \le D)$

pf (i) $u_i^*(\theta_0^*) = u_0(\theta_i)$

$= 1$

(ii) By (i) and since $v_i^* = m_i u_i^*$ $\square$

With above notation

earlier we obtained some "row" and "column" orthogonality relations for the polynomials u_i and v_i

Using the same methods we could get similar relations for the polynomials u_i^* and v_i^*. We could also use Askey-Wilson duality

thm 112 With above notation

(i) for $0 \le i, j \le D$

$$\sum_{r=0}^{D} u_i^*(\theta_r^*)\, u_j^*(\theta_r^*)\, k_r = \delta_{ij}\, m_i^{-1} |X|$$

(ii) for $0 \le r, s \le D$

$$\sum_{i=0}^{D} u_i^*(\theta_r^*)\, u_i^*(\theta_s^*)\, m_i = \delta_{rs}\, k_r^{-1} |X|$$

Pf Combine th 54 and th 110 $\square$

Thm 113 With above notation

(i) For $0 \le i, j \le D$

$$\sum_{r=0}^{D} v_i^*(\theta_r^*)\, v_j^*(\theta_r^*)\, k_r = \delta_{ij}\, m_i\, |X|$$

(ii) For $0 \le r, s \le D$

$$\sum_{i=0}^{D} v_i^*(\theta_r^*)\, v_i^*(\theta_s^*)\, m_i^{-1} = \delta_{rs}\, k_r^{-1}\, |X|$$

pf Eval Thm 112 using $v_i^* = m_i u_i^*$ $(0 \le i \le D)$ $\square$

We will see shortly that for the Hamming graph $H(D,N)$

and for the ordering $\theta_0 > \theta_1 > \cdots > \theta_D$

$$q_{ij}^h = p_{ij}^h \qquad (0 \le h, i, j \le D)$$
$$\theta_i^* = \theta_i \qquad (0 \le i \le D)$$

So $\{E_i\}_{i=0}^D$ is Q-polynomial and

$u_i^* = u_i$, $\quad v_i^* = v_i$ $\quad 0 \le i \le D$

$\square$

Lecture 12 Monday Feb 16

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Given a DRG $\Gamma = (X, R)$ with diameter D and standard module $V = \mathbb{F}^X$. Primitive idempotents $\{E_i\}_{i=0}^D$ in any order except $E_0 = |X|^{-1}J$. Fix $x \in X$ write $M^* = M^*(x)$ etc.

Next goal: A characterization of the Q-poly property in terms of the function algebra $V, \circ$. We start with a handy fact

LEM 114 With above notation TFAE

(i) the ordering $\{E_i\}_{i=0}^D$ is Q-polynomial

(ii) For $0 \le h, j \le D$

$$q^h_{1j} = \begin{cases} 0 & \text{if } h > j+1 \\ \neq 0 & \text{if } h = j+1 \end{cases}$$

(iii) $\exists$ polynomials $\{v_i^*\}_{i=0}^D$ in $\mathbb{F}[\lambda]$ such that

$$\deg v_i^* = i \quad (0 \le i \le D)$$

$$A_i^* = v_i^*(A_1^*) \quad (0 \le i \le D)$$

Pf ex using the def of Q-poly $\square$

Referring to the function algebra $V, \circ$

for a subspace $W \subseteq V$ consider the subalgebra of V

generated by W. This subalgebra contains $\mathbb{I}$ by the

definition of subalgebra. To see what else

is in the subalgebra, define a binary relation

$\sim \; = \; \underset{W}{\sim}$ on X by

$y \sim z$ whenever for all $w \in W$

y-coord of w = z-coord of w.

Obs $\sim$ is an equivalence relation

For a subset $Y \subseteq X$ let

$$\hat{Y} = \sum_{y \in Y} \hat{y} \qquad \text{"characteristic vector of } Y\text{"}$$

LEM 115 With above notation the following are equal

(i) The subalgebra of V generated by W

(ii) $\mathrm{Span}\{\hat{Y} \mid Y \text{ an equiv class of } \sim\}$

Pf $\subseteq$: $\mathrm{Span}\{\hat{Y} \mid Y \text{ an equiv class of } \sim\}$

is a subalgebra of V that contains W

$\supseteq$ List the equiv classes of $\sim$

$Y_1, Y_2, \ldots, Y_t$

For $w \in W$ write

$$w = \sum_{i=1}^{t} \alpha_i(w)\, \hat{Y}_i \qquad \alpha_i(w) \in \mathbb{F}$$

For dist i, j $(1 \le i, j \le t)$ $\exists$ $w_{ij} \in W$

s.t

$$\alpha_i(w_{ij}) \neq \alpha_j(w_{ij})$$

Obs for $1 \le i \le t$

$$\hat{Y}_i = \prod_{\substack{1 \le j \le t \\ j \ne i}} \frac{w_{ij} - \alpha_j(w_{ij})\mathbb{1}}{\alpha_i(w_{ij}) - \alpha_j(w_{ij})}$$

where the product is with respect to $\circ$. Result follows $\square$

Given a primitive idempotent E of Γ

we now consider the previous discussion for $W = EV$

LEM 116 Let E denote a prim idempotent of Γ.

Then for $y, z \in X$ the following are equivalent

(i) $y \underset{EV}{\sim} z$

(ii) $E\hat{y} = E\hat{z}$

pf

$y \underset{EV}{\sim} z \iff$ y-coord of w $=$ z-coord of w $\forall w \in EV$

$\iff \underbrace{y\text{-coord of } E\hat{\jmath}}_{\substack{\| \\ \langle E\hat{y},\, E\hat{\jmath}\rangle \\ \| \\ \jmath\text{-coord of } E\hat{y}}} = z\text{-coord of } E\hat{\jmath} \quad \forall \jmath \in X$

$\iff \jmath$-cord of $E\hat{y}$ $=$ $\jmath$-coord of $E\hat{z}$ $\forall \jmath \in X$

$\iff E\hat{y} = E\hat{z}$

$\square$

COR 117 For a primitive idempotent E of Γ the following are equivalent:

(i) EV generates V in the function algebra $V, \circ$

(ii) E is nondegenerate

Pf EV generates V

$\Leftrightarrow$ $V = \mathrm{Span}\{\hat{\Psi} \mid \Psi$ an equiv class for $\underset{EV}{\sim}\}$

$\Leftrightarrow$ Each equiv class of $\underset{EV}{\sim}$ has single element

$\Leftrightarrow$ $\{E\hat{y} \mid y \in X\}$ mut distinct

$\Leftrightarrow$ E nondegenerate. $\square$

Thm 118 With above notation TFAE

(i) the ordering $\{E_i\}_{i=0}^{D}$ is Q-polynomial

(ii) E_1 is nondegenerate and

$$E_1V \circ E_iV \subseteq E_{i-1}V + E_iV + E_{i+1}V \qquad (0 \le i \le D)$$

where $E_{-1}=0$, $E_{D+1}=0$

(iii) For $0 \le i \le D$

$$E_0V + E_1V + \cdots + E_iV = (E_0V+E_1V) \circ \underbrace{(E_0V+E_1V) \circ \cdots \circ (E_0V+E_1V)}_{i \text{ factors}}$$

Pf (i) → (ii) Recall dual eigenvalues

$$\theta_i^* = m_1 u_i(\sigma) \qquad 0 \le i \le D.$$

$\{\theta_i^*\}_{i=0}^{D}$ are mut distinct

so

$$\theta_i^* \ne \theta_0^* \qquad 1 \le i \le D$$

so E_1 nondeg.

Recall

$$E_1V \circ E_iV = \sum_{\substack{0 \le h \le D \\ q_{1i}^h \ne 0}} E_hV$$

and

$$q_{1i}^{h} = 0 \quad \text{if} \quad |h-i| > 1$$

(ii) → (iii) Recall E_0V is spanned by the all 1's vector in V

so $E_0V \circ E_jV = E_jV$ for $0 \le j \le D$.

So $\underbrace{(E_0V + E_1V) \circ (\quad) \circ \cdots \circ (\quad)}_{i} = E_0V + E_1V + E_1V \circ E_1V + \cdots + \underbrace{E_1V \circ \cdots \circ E_1V}_{i}$

$\subseteq E_0V + \cdots + E_iV$

by ind on i

Using Cor 117 we get = in above line.

(iii) → (i) For $0 \le j \le D-1$

$$(E_0V + \cdots + E_jV) \circ (E_0V + E_1V) = E_0V + \cdots + E_{j+1}V$$

So by thm 92

$$q_{1j}^{h} = \begin{cases} 0 & \text{if} \quad h > j+1 \\ \neq 0 & \text{if} \quad h = j+1 \end{cases}$$

Now $\{E_i\}_{i=0}^{D}$ is Q-polynomial by LEM 114. $\square$