Lec 26 Friday March 27

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

Assume $\{E_i\}_{i=0}^{D}$ is Q-poly

Fix $x \in X$, write $T = T(x)$ etc

For time being assume Γ is bipartite

Let W denote an irred T-module with endpt r

dual endpt t, diam $d \geq 1$. Recall W is thin by L125.

Obs $a_i(W) = 0$ $(0 \leq i \leq d)$

LEM 127 We have

(i) $\varphi_1 = \theta_t(\theta^*_{r+1} - \theta^*_r)$

(ii) $\theta_{t+1}(\theta^*_{r+d} - \theta^*_r) = \theta_t(\theta^*_{r+d-1} - \theta^*_{r+1})$

pf (i) by L122 $\theta_t + \dfrac{\varphi_1}{\theta^*_r - \theta^*_{r+1}} = a_0(W)$

$= 0$

(ii) By L123(i) and since $b_d(W) = 0$

$$c_d(W) = \frac{(\theta_{t+1} - \theta_t)(\theta^*_{r+d} - \theta^*_r) + \varphi_1}{\theta^*_{r+d-1} - \theta^*_{r+d}}$$

Also

$$\theta_t = c_d(w) + \underbrace{a_d(w)}_{0} + \underbrace{b_d(w)}_{0}$$

$$= c_d(w)$$

Comparing the two expressions for $c_d(w)$ and using (*) above

we get the result. □

L128 We have

(i) $c_0(w) = 0$

(ii) $$c_i(w) = \frac{\theta_t(\theta^*_{r+i+1} - \theta^*_{r+1}) - \theta_{t+1}(\theta^*_{r+i} - \theta^*_{r})}{\theta^*_{r+i+1} - \theta^*_{r+i-1}} \qquad (1 \le i \le d-1)$$

(iii) $c_d(w) = \theta_t$

(iv) $b_0(w) = \theta_t$

(v) $$b_i(w) = \frac{\theta_t(\theta^*_{r+1} - \theta^*_{r+i-1}) - \theta_{t+1}(\theta^*_{r} - \theta^*_{r+i})}{\theta^*_{r+i+1} - \theta^*_{r+i-1}} \qquad (1 \le i \le d-1)$$

(vi) $b_d(w) = 0$

Pf A linear equation involving $c_i(w)$, $b_i(w)$ is given in L123 (i).

We also have $c_i(w) + b_i(w) = \theta_t$

Solve this system for $c_i(w)$, $b_i(w)$ and elim φ_1 using L127 (i)

LEM 129

$$\theta_i = -\theta_{D-i} \qquad (0 \le i \le D)$$

[We saw this earlier for the natural ordering of the eigenvalues]

pf (For Case I) $\beta \neq \mp 2$

$$\theta_i = a + b q^{2i-D} + c q^{D-2i} \qquad 0 \le i \le D$$

$$\theta^*_i = a^* + b^* q^{2i-D} + c^* q^{D-2i}$$

$$\beta = q^2 + q^{-2}$$

By L74(i)

$$0 = \gamma$$

$$= \theta_{i-1} - \beta\theta_i + \theta_{i+1} \qquad 1 \le i \le D-1$$

$$= a(2-\beta)$$

So $a = 0$

Show $b = -c$

Apply L 127 (ii) to the primary module to get

$$\frac{\theta_1}{\theta_0} = \frac{\theta^*_{D-1} - \theta^*_D}{\theta^*_D - \theta^*_0} \qquad (*)$$

Note $q^{2i} \neq 1$ $(1 \le i \le D)$ since $\{\theta_i\}_{i=0}^{D}$ are dist

Eval $(*)$ to get

$$\frac{b q^{2-D} + c q^{D-2}}{b q^{-D} + c q^{D}} = \frac{q^{2-D} - q^{D-2}}{q^{-D} - q^{D}}$$

Solve to find $b = -c$

The pf for cases II, III is sim. □

LEM 130 For our module W

(i) $2t+d=D$

(ii) $D-d$ is even

Pf (i) (For Case I) keep notation of L129

By L127 (ii)

$$\frac{\theta_{t+1}}{\theta_t} = \frac{\theta^*_{r+d+1} - \theta^*_{r+1}}{\theta^*_{r+d} - \theta^*_r}$$

Gives

$$\frac{q^{2t+2-D} - q^{D-2t-2}}{q^{2t-D} - q^{D-2t}} = \frac{q^{2-d} - q^{d-2}}{q^{-d} - q^{d}}$$

this yields

$$q^{(2t+d-D)2} = 1$$

and then

$$2t+d=D$$

using $q^{2i} \neq 1$ $(1 \leq i \leq D)$ □

(ii) ✓

LEM 131 As a T-module the iso class of W is determined by r and d.

Pf let W' denote an irred T-module with endpt r and diam d. Show T-modules W, W' are iso.

Obs $t' = t$ by L130

Also by L128

$$c_i(W') = c_i(W), \qquad b_i(W') = b_i(W) \quad (0 \le i \le d)$$

Let $\{w_i\}_{i=0}^{d}$ be a st. basis for W

Let $\{w_i'\}_{i=0}^{d}$ be a st. basis for W'

Define a linear trans $\sigma : W \to W'$ s.t.

$$\sigma(w_i) = w_i' \qquad 0 \le i \le d$$

So σ is iso of vector spaces.

Matrix rep A rel $\{w_i\}_{i=0}^{d}$ equals the matrix rep A rel $\{w_i'\}_{i=0}^{d}$

So $A\sigma = \sigma A$

By cnst

$$A^* w_i = \theta^*_{r-i} w_i \qquad 0 \le i \le d$$

$$A^* w_i' = \theta^*_{r-i} w_i \qquad 0 \le i \le d$$

So

$$A^* \sigma = \sigma A^*$$

A, A^* gen T so σ is T-module iso. □

For our bipartite Γ

the iso classes of irred T-modules are indexed by the

endpt r and diam d

What are the possible r, d ?

Constraints:

(i) $\quad 0 \leq d \leq D$

(ii) $\quad D-d$ even

(iii) $\quad r \leq D-d \qquad$ (by constr)

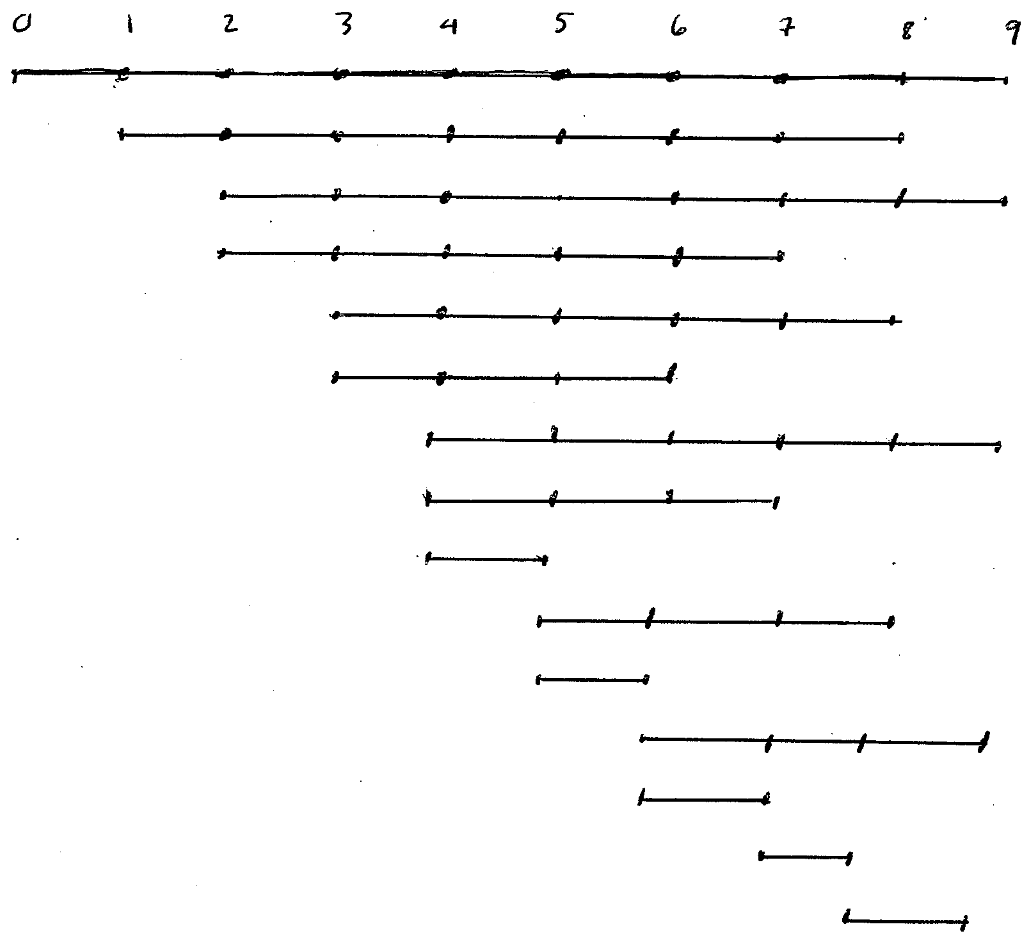
(iv) $\quad \frac{1}{2}(D-d) \leq r \qquad$ (by Cor 92(i))

Let $\quad \Delta = \{ (r,d) \in \mathbb{Z}^2 \mid r, d \text{ sat } (i)-(iv) \}$

Δ looks as follows (to illustrate take $D=9$)

In the diagram each (r,d) is rep by a line

segment from col r to col $r+d$



Caution: For some (r,d) the corresp module might not exist.

No longer assume Γ is bipartite

Instead assume $\{E_i\}_{i=0}^{D}$ is dual bip.

Then LEM 127–131 still hold if we replace

$\theta_i \leftrightarrow \theta_i^*$, $\quad r \leftrightarrow t$, $\quad c_i(W) \leftrightarrow c_i^*(W)$

$b_i(W) \leftrightarrow b_i^*(W)$ etc.

Indeed the proofs are purely algebraic.

Until further notice assume both

Γ is bipartite

$\{E_i\}_{i=0}^{d}$ is dual bipartite.

LEM 132 For each irred T-module W

(i) $2r+d=D$

(ii) $2t+d=D$

(iii) $D-d$ even

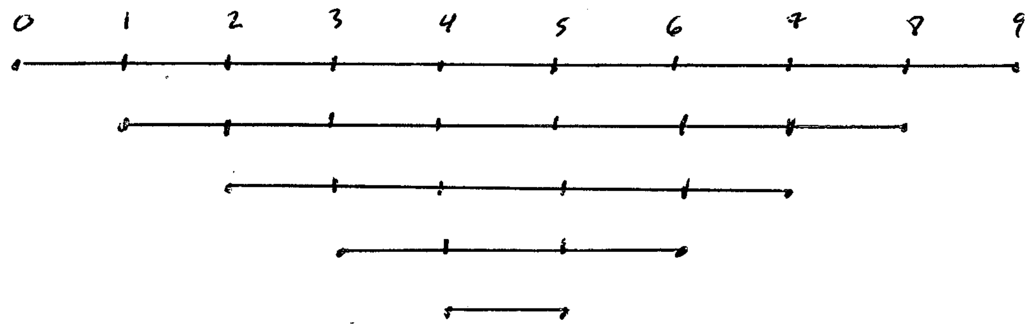
(iv) iso class of W is det by d

Pf By L130 and its dual, and L131

Referring to our set Δ from below L131

the set of (r, d) in Δ that satisfy L132 looks

as follows



LEM 133 Let W denote an irred T-module with endpt r. Then W appears in the std module V with mult

$$\begin{cases} 1 & \text{if } r=0 \\ k_r - k_{r-1} & \text{if } r \geq 1 \end{cases}$$

pf Denote the mult by μ_r.

Obs

$$\mu_0 + \mu_1 + \cdots + \mu_r = k_r$$

Result follows. $\square$

Ex 134 $\Gamma = H(D,2)$ hypercube

Γ bip ✓

Γ has dual bip Q-poly structure with

$\theta_i = D - 2i \qquad 0 \le i \le D$

$\theta_i^* = D - 2i \qquad 0 \le i \le D$

Let W denote an irred T-module diam d.

By L132 $\quad D-d$ even, and

$r = \frac{D-d}{2}, \qquad t = \frac{D-d}{2}$

Using L128

$c_i(W) = i, \qquad 0 \le i \le d$

$b_i(W) = d-i \qquad 0 \le i \le d$

Using the dual of L128

$c_i^*(W) = i \qquad 0 \le i \le d$

$b_i^*(W) = d-i \qquad 0 \le i \le d$

By L133.
W appears in V with mult $\begin{cases} 1 & \text{if } r=0 \\ \binom{D}{r} - \binom{D}{r-1} & \text{if } r \ge 1 \end{cases}$ □

Ex 135 Assume Γ is bipartite and $\{E_i\}_{i=0}^{D}$ is dual bipartite. Further assume $\beta \neq \mp 2$, and let q be as in L80. Recall

$$\beta = q^2 + q^{-2}$$

$$\theta_i = \theta_i^* = (q^{D-2} + q^{2-D}) \frac{q^{D-2i} - q^{2i-D}}{q^2 - q^{-2}} \qquad 0 \le i \le D$$

Let W denote an irred T-module diam d. By L132 $D-d$ is even and

$$r = \frac{D-d}{2}, \qquad t = \frac{D-d}{2}$$

Using L128 and its dual

$$c_i(W) = c_i^*(W) = \frac{q^{D-2} + q^{2-D}}{q^{d-2i} + q^{2i-d}} \frac{q^{2i} - q^{-2i}}{q^2 - q^{-2}}$$

$$b_i(W) = b_i^*(W) = \frac{q^{D-2} + q^{2-D}}{q^{d-2i} + q^{2i-d}} \frac{q^{2d-2i} - q^{2i-2d}}{q^2 - q^{-2}}$$

for $0 \le i \le d$.

$\square$

Lec 27 Monday March 30

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

Assume $\{E_i\}_{i=0}^{D}$ is Q-poly

Fix $x \in X$ write $T = T(x)$ etc

Fix a thin irred T-module W with endpt r, dual endpt t, diam $d \geq 1$

In L128 we found

$$c_i(W), \; b_i(W)$$

when Γ is bip. What if Γ not bip?

Similar formulae can be obtained by similar methods but it is slightly complicated, so I will skip the details and just state the result.

thm 136 Let the T-module W be as above and let

$\varphi_1 = \varphi_1(W)$ be as in L 121. Then

$$c_0(W) = 0$$

$$c_i(W) = \frac{\varphi_1 f_i^- + g_i^-}{(\theta^*_{i+r-1} - \theta^*_{i+r})(\theta^*_{r+i-1} - \theta^*_{r+i+1})} \qquad (1 \le i \le d-1)$$

$$c_d(W) = \frac{\varphi_1 + (\theta_{t+1} - \theta_t)(\theta^*_{r+d} - \theta^*_r)}{\theta^*_{r+d-1} - \theta^*_{r+d}}$$

$$b_0(W) = \frac{\varphi_1}{\theta^*_{r+1} - \theta^*_r}$$

$$b_i(W) = \frac{\varphi_1 f_i^+ + g_i^+}{(\theta^*_{r+i+1} - \theta^*_{r+i})(\theta^*_{r+i+1} - \theta^*_{r+i-1})} \qquad (1 \le i \le d-1)$$

$$b_d(W) = 0$$

where

$$f_i^{\pm} = \theta^*_{r+1} - \theta^*_{r+i\mp 1} - \frac{(\theta^*_{r+i} - \theta^*_r)(\theta^*_{r+1} - \theta^*_{r+d-1})}{\theta^*_{r+d} - \theta^*_r}$$

$$g_i^{\pm} = (\theta^*_{r+i} - \theta^*_r)\Big((\theta_{t+2} - \theta_{t+1})(\theta^*_{r+i} - \theta^*_{r+d}) - (\theta_{t+1} - \theta_t)(\theta^*_{r+i\mp 1} - \theta^*_{r+d-1})\Big)$$

if $d \ge 2$. To get $c_i^*(W)$, $b_i^*(W)$ interchange $r \leftrightarrow t$ and $\theta_j \leftrightarrow \theta^*_j$ $(0 \le j \le D)$ □

COR 137 With ref to Th 136

the isomorphism class of the T-module W is determined by the 4-tuple

(r, t, d, φ)

Pf this 4-tuple determines $c_i(W), b_i(W), a_i(W)$ etc. $\square$

Earlier we saw that A, A^* satisfy the tridiagonal relations TD1, TD2. When we restrict the actions of A, A^* to a thin irred T-module, these relations take a nicer form often called the Askey-Wilson relations.

We now explain the details.

In what follows $\beta, \gamma, \gamma^*, \delta, \delta^*$ are from Thm 57.

Our next goal is to show:

Thm 138 Let W denote a thin irred T-module.

Then $\exists$ scalars in $\mathbb{F}$:

$$\omega = \omega(W), \qquad \eta = \eta(W), \qquad \eta^* = \eta^*(W)$$

such that on W

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^* = \gamma^* A^2 + \omega A + \eta I \qquad \text{AW1}$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) - \delta^* A = \gamma A^{*2} + \omega A^* + \eta^* I \qquad \text{AW2}$$

AW1, AW2 are called the Askey-Wilson relations

We will prove Thm 138 after a few lemmas.

Recall the polynomials

$$P(\lambda,\mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma(\lambda+\mu) - \delta$$

$$P^*(\lambda,\mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma^*(\lambda+\mu) - \delta^*$$

LEM 139 Let W denote a thin irred T-module with dual endpt t and diam d_0

Then for all $\omega, \eta \in \mathbb{F}$

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^* = \gamma^* A^2 + \omega A + \eta I \quad \text{on } W \quad (*)$$

iff

$$a_i^*(W)\, P(\theta_{t+i}, \theta_{t+i}) = \gamma^* \theta_{t+i}^2 + \omega\, \theta_{t+i} + \eta \quad (0 \le i \le d) \quad (**)$$

pf Let ℓ, $\mathcal{R}$ denote LHS, RHS of $(*)$

Obs

$$\ell = \sum_{i=0}^{d} \sum_{j=0}^{d} E_{t+i}\, \ell\, E_{t+j}$$

$$\mathcal{R} = \sum_{i=0}^{d} \sum_{j=0}^{d} E_{t+i}\, \mathcal{R}\, E_{t+j}$$

Obs for $0 \le i, j \le d$

$$E_{t+i} \mathcal{L} E_{t+j} = P(\theta_{t+i}, \theta_{t+j}) E_{t+i} A^* E_{t+j}$$

$$E_{t+i} \mathcal{R} E_{t+j} = \delta_{ij} \left(\gamma^* \theta^2_{t+i} + \omega \theta_{t+i} + \eta \right) E_{t+i}$$

First assume $*$ so

$$\mathcal{L} = \mathcal{R} \quad \text{on } W$$

For $0 \le i \le d$

$$E_{t+i} \mathcal{L} E_{t+i} \overset{\text{on } W}{=} E_{t+i} \mathcal{R} E_{t+i}$$

$$P(\theta_{t+i}, \theta_{t+i}) \underbrace{E_{t+i} A^* E_{t+i}}_{a_i^*(W) E_{t+i}} \qquad \left(\gamma^* \theta^2_{t+i} + \omega \theta_{t+i} + \eta \right) E_{t+i}$$

By constr $E_{t+i} W \ne 0$ so

$$a_i^*(W) \, P(\theta_{t+i}, \theta_{t+i}) = \gamma^* \theta^2_{t+i} + \omega \theta_{t+i} + \eta$$

We have $(**)$.

Next assume $(**)$ show $\mathcal{L} = \mathcal{R}$ on W

Suf to show

$$E_{t+i} \mathcal{L} E_{t+j} \overset{\text{on } W}{=} E_{t+i} \mathcal{R} E_{t+j} \qquad 0 \le i, j \le d$$

For $0 \le i \le d$

$$E_{t+i} \ell E_{t+i} \overset{\text{on } W}{=} a_i^*(w) \, P(\theta_{t+i}, \theta_{t+i}) \, E_{t+i}$$

$$E_{t+i} R \, E_{t+i} \overset{\text{on } W}{=} (\gamma^* \theta_{t+i}^2 + \omega \theta_{t+i} + \eta) \, E_{t+i}$$

So

$$E_{t+i} \ell E_{t+i} \overset{\text{on } W}{=} E_{t+i} R \, E_{t+i}$$

For $0 \le i, j \le d$ with $|i-j| = 1$

Recall

$$P(\theta_{t+i}, \theta_{t+j}) = 0$$

by Prop 64 (iv) so

$$E_{t+i} \ell E_{t+j} = 0$$

Also

$$E_{t+i} R \, E_{t+j} = 0 \qquad \text{since } i \neq j.$$

so

$$E_{t+i} \ell E_{t+j} = E_{t+i} R \, E_{t+j}$$

For $0 \le i, j \le d$ s.t $|i-j| > 1$,

$$E_{t+i} A^* E_{t+j} = 0$$

so

$$E_{t+i}\, \ell\, E_{t+j} = 0$$

Also

$$E_{t+i}\, R\, E_{t+j} = 0 \qquad \text{since } i \neq j.$$

We have shown

$$E_{t+i}\, \ell\, E_{t+j} \overset{\text{on } W}{=} E_{t+i}\, R\, E_{t+j} \qquad 0 \le i, j \le d$$

so $\ell = R$ on W. $\square$

Pf of Thm 138: Set $d = \text{diam of } W$. Assume $d \geq 1$ else triv.

As before def

$$\ell = A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*$$

By TD1

$$0 = [\ell, A]$$

Aside on lin alg: for any linear trans $\sigma: W \to W$

that is diagonalizable and has all eigenspaces dim 1

any linear trans $W \to W$ that commutes with σ is a poly in σ (ex)

By this aside $\exists$ $f \in \mathbb{F}[\lambda]$ s.t.

$$\ell \overset{\text{on } W}{=} f(A)$$

Minimal poly of A on W has degree $d+1$ so WLOG

$$\deg f \leq d.$$

Let h = degree of f.

<u>claim</u> $h \le 2$ Suppose $h > 2$

Obs

$$\underbrace{E^*_{r+h} \, \ell \, E^*_r}_{\substack{\| \\ 0 \text{ by form of } \ell}} \overset{\text{on } W}{=} \underbrace{E^*_{r+h} f(A) E^*_r}_{\substack{\| \\ \alpha \underbrace{E^*_{r+h} A^h E^*_0}_{\substack{\| \\ \underbrace{E^*_{r+h} A E^*_{r+h-1} A E^*_{r+h-2} A \cdots E^*_1 A E^*_0}_{\substack{\ne \\ 0 \text{ on } W}}}}}}$$

α = leading coeff of f

Cont.

claim proved.

Write

$$\ell = \varepsilon A^2 + \omega A + \eta I \qquad \varepsilon, \omega, \eta \in \mathbb{F}$$ //

For moment let $d = 1$. Then I, A, A^2 lin dep on W so ε

can be chosen so $\varepsilon = \gamma^*$.

Next assume $d \ge 2$ show $\varepsilon = \gamma^*$. Obs

$$\underbrace{E^*_{r+2} \quad \ell \quad E^*_r}_{E^*_{r+2} A^2 E^*_0 \underbrace{(\theta^*_r - \beta \theta^*_{r+1} + \theta^*_{r+2})}_{\gamma^*}} \overset{\text{on } W}{=} \underbrace{E^*_{r+2} (\varepsilon A^2 + \omega A + \eta I) E^*_0}_{\varepsilon E^*_{r+2} A^2 E^*_r}$$

$E^*_{r+2} A^2 E^*_r \neq 0$ on W since W is thin, so $\gamma^* = \varepsilon$

We have obtained AW1. Interchanging the roles of A, A^* in the argument so far, we find $\exists\ \omega^*, \eta^* \in \mathbb{F}$ s.t.

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) - \delta^* A \overset{\text{on } W}{=} \gamma A^{*2} + \omega^* A^* + \eta^* I \qquad (AW2')$$

Show $\omega = \omega^*$. Find the commutator of TD1 with A^*. Get

$$A^2A^{*2} - \beta AA^*AA^* + \beta A^*AA^*A - A^{*2}A^2 - \gamma(AA^{*2} - A^{*2}A)$$
$$\overset{\text{on } W}{=} \gamma^*(A^2A^* - A^*A^2) + \omega(AA^* - A^*A)$$

Now find the commutator of AW2' with A. Get

$$A^{*2}A^2 - \beta A^*AA^*A + \beta AA^*AA^* - A^2A^{*2} - \gamma^*(A^*A^2 - A^2A^*)$$
$$\overset{\text{on } W}{=} \gamma(A^{*2}A - AA^{*2}) + \omega^*(A^*A - AA^*)$$

Adding the above eqs get

$$0 \overset{\text{on } W}{=} (\omega - \omega^*)(AA^* - A^*A)$$

Obs $AA^* \neq A^*A$ on W since $d \geq 1$, so

$$\omega = \omega^* \qquad \square$$

We now consider how to find ω, η, η^*

Notation: Recall

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = \gamma \qquad 1 \le i \le D-1$$

$$\theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1} = \gamma^* \qquad 1 \le i \le D-1$$

Define θ_{-1}, θ_{D+1}, θ^*_{-1}, θ^*_{D+1} so that the above eqs hold for $i=0$ and $i=D$.

Thm 140 Let W denote a thin irred T-module with endpt r, dual endpt t, diam d. Then the scalars ω, η, η^* from thm 138 satisfy:

(i) $$\omega = a^*_i(W)(\theta_{t+i} - \theta_{t+i+1}) + a^*_{i-1}(W)(\theta_{t+i-1} - \theta_{t+i-2}) - \gamma^*(\theta_{t+i} + \theta_{t+i-1}) \qquad 1 \le i \le d$$

(ii) $$\omega = a_i(W)(\theta^*_{r+i} - \theta^*_{r+i+1}) + a_{i-1}(W)(\theta^*_{r+i-1} - \theta^*_{r+i-2}) - \gamma(\theta^*_{r+i} + \theta^*_{r+i-1}) \qquad 1 \le i \le d$$

(iii) $$\eta = a^*_i(W)(\theta_{t+i} - \theta_{t+i-1})(\theta_{t+i} - \theta_{t+i+1}) - \gamma^*\theta_{t+i}^2 - \omega\theta_{t+i} \qquad 0 \le i \le d$$

(iv) $\eta^* = a_i(w)(\theta^*_{r+i} - \theta^*_{r+i-1})(\theta^*_{r+i} - \theta^*_{r+i+1}) - \gamma\theta^{*2}_{r+i} - w\theta^*_{r+i}$

$0 \le i \le d$

Pf We start with (iii)

(iii) Let i be given. Claim

$$P(\theta_j, \theta_j) = (\theta_j - \theta_{j-1})(\theta_j - \theta_{j+1}) \qquad 0 \le j \le D$$

To see this for $1 \le j \le D$ elim θ_{j+1} using

$$\theta_{j-1} - \beta\theta_j + \theta_{j+1} = \gamma$$

and eval the result using

$$P(\theta_j, \theta_j) = (2-\beta)\theta_j^2 - 2\gamma\theta_j - \delta$$

and

$$P(\theta_{j-1}, \theta_j) = 0$$

To see the claim for $0 \le j \le D-1$, elim θ_{j-1} using

$$\theta_{j-1} - \beta\theta_j + \theta_{j+1} = \gamma$$

and proceed in a similar manner. Claim proved.

Now use the claim and L139. ☐

(iv) Sim. (i) Subtract (iii) (at i) from (iii) (at $i-1$) ☐

(ii) Sim.

Lecture 28 Wednesday April 1

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

Assume $\{E_i\}_{i=0}^{D}$ is Q-poly

Fix $x \in X$, write $T = T(x)$ etc.

Def 141 For $0 \leq i \leq D$, Γ called <u>i-thin</u>

(with respect to x) whenever each irred T-module with endpt

at most i is thin. (So Γ is 0-thin)

Γ called <u>thin</u> (with respect to x) whenever every irred

T-module is thin.

In what follows we fix scalars

$\beta, \gamma, \gamma^*, \delta, \delta^*$

from th 57

LEM 142 Assume $\mathbb{F} = \mathbb{C}$ and Γ is thin rel x.

Then $\exists$ central elements

$$\Omega, \quad Z, \quad Z^*$$

in T such that

(i) $A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^* = \gamma^* A^2 + \Omega A + Z$

(ii) $A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta^* A = \gamma A^{*2} + \Omega A^* + Z^*$

pf Decompose the st. module V into the homogeneous components

$$V = \sum_{\phi \in \Psi} V_\phi \qquad \text{(orthog ds of } T\text{-modules)}$$

Recall Ψ = set of iso classes of irred T-modules

For $\phi \in \Psi$ we define Ω, Z, Z^* on V_ϕ.

Let W denote an irred T-module in V_ϕ.

Define Ω, Z, Z^* on V_ϕ s.t

$$(\Omega - \omega I) \,|\, V_\phi = 0, \quad (Z - \eta I) \,|\, V_\phi = 0, \quad (Z^* - \eta^* I) \,|\, V_\phi = 0$$

where $\omega = \omega(W)$, $\eta = \eta(W)$, $\eta^* = \eta^*(W)$ are from Th 138

By construction Ω, Z, Z^* sat (i), (ii) and commute with each element of T.

Show Ω, Z, Z^* are in T (sketch):

The Standard module V is direct sum of irred T-modules, so T is semi-simple. We give a basis for the center of T.

$\forall \phi \in \Psi$ define $e_\phi \in Mat_X(\mathbb{C})$ s.t

$$(e_\phi - I) V_\phi = 0,$$

$$e_\phi V_\varphi = 0 \quad \text{if } \varphi \neq \phi \qquad (\forall \varphi \in \Psi).$$

So e_ϕ is the projection $V \to V_\phi$.

By the Wedderburn theory

$$\{ e_\phi \mid \phi \in \Psi \}$$

is a basis for the center of T. Each of Ω, Z, Z^* is a linear combination of this basis and hence in T. □

Note Referring to L142,

Given Ω we can get Z, Z^* using (i), (ii)

Open Problem 143 Find the combinatorial meaning of Ω

In a moment we will give a conjecture about Ω

First some motivation.

Since Ω is central

$$E_i^* \Omega E_j^* = 0 \qquad \forall\ i \neq j \qquad (0 \leq i, j \leq D)$$

So

$$\Omega = \sum_{i=0}^{D} E_i^* \Omega E_i^*$$

So $\forall\ y, z \in X$

$$\Omega_{yz} \neq 0 \quad \rightarrow \quad \partial(y,x) = \partial(z,x)$$

Conjecture 144 $\forall\ y, z \in X$

$$\Omega_{yz} \neq 0 \quad \rightarrow \quad \partial(y,z) \leq 1$$

Project 145 Compute Ω, Z, Z^* for some examples of Q-poly DRG's. See if Conj 144 holds and try to find the combin meaning.

We now put the equations of L142 in cyclic form.

For simplicity we assume $\beta \neq \mp 2$.

Thm 146 Assume $\mathbb{F} = \mathbb{C}$ and $\beta \neq \mp 2$ and Γ is then rel x.

Pick $0 \neq t^\varepsilon \in \mathbb{F}$ and $r^\varepsilon \in \mathbb{F}$. Then $\exists$ $A^\varepsilon \in T$ and

central $\Phi, \Phi^*, \Phi^\varepsilon \in T$ such that

$$qAA^* - q^{-1}A^*A + r^*A + rA^* + t^\varepsilon A^\varepsilon = \Phi^\varepsilon,$$

$$qA^*A^\varepsilon - q^{-1}A^\varepsilon A^* + r^\varepsilon A^* + r^*A^\varepsilon + tA = \Phi$$

$$qA^\varepsilon A - q^{-1}AA^\varepsilon + rA^\varepsilon + r^\varepsilon A + t^*A^* = \Phi^*$$

where

$$\beta = q^2 + q^{-2}$$

$$\gamma = r(q - q^{-1}), \qquad \gamma^* = r^*(q - q^{-1})$$

$$\delta = r^2 - t^* t^\varepsilon \qquad \delta^* = r^{*2} - t t^\varepsilon$$

$$\Omega = (rr^* - r^\varepsilon t^\varepsilon) I - \Phi^\varepsilon (q - q^{-1})$$

$$Z = \Phi^* t^\varepsilon - \Phi^\varepsilon r$$

$$Z^* = \Phi t^\varepsilon - \Phi^\varepsilon r^*$$

Pf Using $\beta, \gamma, \gamma^*, \delta, \delta^*, \Omega, Z, Z^*$

and the last 8 equations, define in order

$q, r, r^*, t, t^*, \Phi^\varepsilon, \Phi, \Phi^*$

Now define A^ε using the 1st eq.

It remains to verify the 2nd and 3d eqs. In these eqs we elim A^ε using eq 1 and find the result matches the equations in L142. Result follows. □

Ex 147 Assume $\mathbb{F} = \mathbb{C}$ and $\beta \neq \mp 2$ and Γ bipartite.

Using th 74,

$$\gamma = 0$$

$$\Omega = 0$$

$$Z^* = 0$$

So by L 142

$$Z = A^2A^* - \beta AA^*A + A^*A^2 - \delta A^* - \gamma^* A^2$$

Now in th 146

$$r = 0$$

$$\Phi^\varepsilon = -\frac{r^\varepsilon t^\varepsilon}{q - q^{-1}} I$$

$$\Phi = -\frac{r^* r^\varepsilon}{q - q^{-1}} I$$

$$\Phi^* = \frac{1}{t^\varepsilon} Z$$

$$A^\varepsilon = \frac{-1}{t^\varepsilon}\left(qAA^* - q^{-1}A^*A + \frac{\gamma^*}{q - q^{-1}} A \right) - \frac{r^\varepsilon}{q - q^{-1}} I$$

Since $r^\varepsilon, t^\varepsilon$ are free choose

$$r^\varepsilon = 0, \quad t^\varepsilon = 1$$

so

$$\Phi^\varepsilon = 0 \qquad \Phi = 0$$

$$\Phi^* = Z$$

$$A^\varepsilon = qAA^* - q^{-1}A^*A + \frac{\gamma^*}{q-q^{-1}} A$$

Problem 148 Assume $\mathbb{F} = \mathbb{C}$ and $\beta \neq \mp 2$ and

Γ is thin rel x.

Find combinatorical meaning of A^ε, Φ, Φ^*, Φ^ε

Algebraically A^ε should resemble A, A^*

What are the eigenvalues of A^ε?

Is A^ε an imag adj matrix in sense of Def 55?

Try Γ bip as a special case.

Problem 149 Find analogs of Th 146 that

apply to $\beta = \mp 2$.

□

Lecture 29 Friday April 3

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

Next goal:

Given the intersection numbers of Γ

find an easy way to determine if

Γ has a Q-poly structure.

Motivation. Assume $\{E_i\}_{i=0}^{D}$ is a Q-poly ordering

of the prim idempotents of Γ.

Applying thm 140 to the primary module we find

the intersection number a_i satisfies

$$a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) = \gamma\, \theta_i^{*2} + \omega\, \theta_i^* + \eta^* \qquad 0 \leq i \leq D$$

for some $\omega, \eta^* \in \mathbb{F}$, where $\theta_{-1}^*, \theta_{D+1}^*$ are defined

as above thm 140.

Next thm due to Arlene Pascasio

Thm 150 Given a nontriv prim idempotent

$$E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$$

of Γ. Then E is Q-poly iff

(i) $\theta_i^* \neq \theta_0^*$ $\quad 1 \le i \le D$

(ii) $\exists$ $\beta, \gamma^* \in \mathbb{F}$ s.t

$$\gamma^* = \theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^* \qquad 1 \le i \le D-1 \qquad (*)$$

(iii) $\exists$ $\gamma, \omega, \eta^* \in \mathbb{F}$ s.t

$$a_i (\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) = \gamma \theta_i^{*2} + \omega \theta_i^* + \eta^* \qquad 0 \le i \le D$$

where θ_{-1}^*, θ_{D+1}^* are such that $(*)$ holds for $i=0$, $i=D$

Pf We saw earlier that E Q-poly implies (i)-(iii).

Next assume (i)-(iii)

Fix $x \in X$, write $T = T(x)$ etc.

Write $E = E_1$, $A^* = A_1^*$

For the time being let $\{E_i\}_{i=2}^{D}$ be any ordering of the remaining nontriv prim idempotents of Γ. Let

θ_i = eigval of E_i for $0 \le i \le D$.

Define a graph Δ_E with vertex set

$$\{0, 1, \ldots, D\}$$

Vertices i, j are adj in Δ_E iff

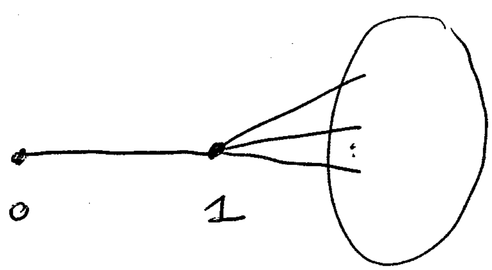
$$i \neq j \quad \text{and} \quad q^{l}_{ij} \neq 0 \qquad (0 \le i, j \le D)$$

Recall

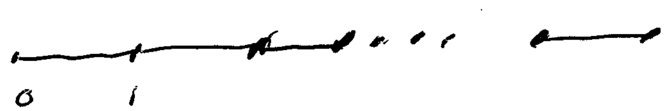
$$q^{1}_{0j} = \delta_{1j} \qquad (0 \le j \le D)$$

So vertex 0 is adjacent to vertex 1 and no other vertex

Δ_E :



We will show Δ_E is a path



claim 1 Δ_E is connected

pf cl let $S \subseteq \{0,1,\ldots,D\}$ be the connected component of Δ_E that contains 0, 1.

Define

$$U = \sum_{i \in S} E_i V \qquad V = \text{std module}$$

Obs

$$E_1 V \subseteq U$$

recall For $0 \le i \le D$

$$E_1 V \circ E_i V = \sum_{\substack{0 \le h \le D \\ q^1_{ih} \ne 0}} E_h V$$

by th 92. So

$$E_1 V \circ U \subseteq U$$

But $E_1 V$ generates V in the function algebra $V, \circ$ by Cor 117

So

$$U = V$$

So

$$S = \{0,1,\ldots,D\}$$

cl proved.

Applying L63 to $\{\theta_i^*\}_{i=-1}^{D+1}$ $\exists\, \delta^* \in \mathbb{F}$ s.t.

$$\theta_{i-1}^{*2} - \beta \theta_{i-1}^* \theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*) = \delta^* \qquad 0 \le i \le D+1$$

Using this one checks

$$(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) = (2-\beta)\theta_i^{*2} - 2\gamma^*\theta_i^* - \delta^* \qquad 0 \le i \le D$$

claim 2 On the primary T-module W

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) - \delta^* A = \gamma A^{*2} + \omega A^* + \eta I.$$

Pf (Sim to L 13 9) let ℓ = LHS in above eq. On W

$$\ell = \left(\sum_{i=0}^{D} E_i^*\right) \ell \left(\sum_{j=0}^{D} E_j^*\right)$$

$$= \sum_{i=0}^{D} \sum_{j=0}^{D} E_i^* A E_j^* \left(\theta_i^{*2} - \beta\theta_i^*\theta_j^* + \theta_j^{*2} - \gamma^*(\theta_i^* + \theta_j^*) - \delta^*\right)$$

$$= \sum_{i=0}^{D} E_i^* A E_i^* (\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*)$$

$$= \sum_{i=0}^{D} E_i^* a_i (\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*)$$

$$= \sum_{i=0}^{D} E_i^* (\gamma\theta_i^{*2} + \omega\theta_i^* + \eta^*)$$

$$= \gamma A^{*2} + \omega A^* + \eta^* I \quad \checkmark$$

claim proved.

<u>claim 3</u> Given vertices i, j in Δ_E at $\partial(i,j) = 2$. Assume there exists a unique vertex h in Δ_E that is adjacent both i, j. Then

$$\gamma = \theta_i - \beta\theta_h + \theta_j$$

<u>pf cl 3</u> In the equation of cl 2 mult each term on left by E_i and right by E_j, and simplify. To help in this simplification note

$$E_i A^{*2} E_j = E_i A^* \left(\sum_{r=0}^{D} E_r \right) A^* E_j$$

$$= E_i A^* E_h A^* E_j$$

and

$$E_i A^* A A^* E_j = E_i A^* \left(\sum_{r=0}^{D} \theta_r E_r \right) A^* E_j$$

$$= \theta_h E_i A^* E_h A^* E_j$$

By above comments

$$0 \overset{W}{=} E_i A^* E_h A^* E_j \left(\theta_i - \beta\theta_h + \theta_j - \gamma \right)$$

But

$$0 \overset{W}{\neq} E_i A^* E_h A^* E_j$$

so

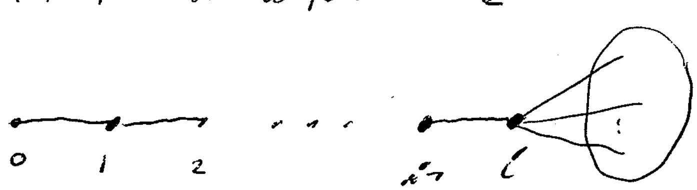
$$r = \theta_i - \beta\theta_h + \theta_j$$

claim proved.

We can now easily show Δ_E is a path.

Since Δ_E is connected, and since vertex 0 is adj only to vertex 1, it suf to show each vertex in Δ_E is adj at most 2 vertices in Δ_E. Suppose $\exists$ vertex i in Δ_E that is adj at least 3 vertices in Δ_E. Of all such vertices pick i such that $\partial(0,i)$ is minimal. WLOG the vertices of Δ_E are labelled s.t $\partial(0,i) = i$, and that

$0, 1, \ldots, i$ is a path in Δ_E



By constr $i \geq 1$

By assumption $\exists$ distinct vertices j, j' in $\Delta_E \setminus \{0, 1, 2, \ldots, i\}$

that are both adj i

Obs $\partial(i-1, j) = 2$, and i is unique vertex in Δ_E

adj both $i-1, j$. So by cl 3

$$\gamma = \theta_{i-1} - \beta\theta_i + \theta_j.$$

Replacing j by j' in above argument

$$\gamma = \theta_{i-1} - \beta\theta_i + \theta_{j'}$$

So $\theta_j = \theta_{j'}$ cont.

Therefore Δ_E is a path $\square$

Note 151 Referring to Thm 150, assume E is Q-poly and let $\{\theta_i\}_{i=0}^{D}$ be the eigenvalue ordering for the corresp Q-poly structure. These eigenvalues are found as follows:

$\theta_0 = k$ the valency

$\theta_1 =$ eigenvalue of E, found using

$$c_i \theta^*_{i-1} + a_i \theta^*_i + b_i \theta^*_{i+1} = \theta_1 \theta^*_i \qquad (0 \le i \le D)$$

For $1 \le i \le D-1$, θ_{i+1} is found recursively using

$$\theta_{i-1} - \beta \theta_i + \theta_{i+1} = \gamma$$

where β, γ is from Th 150.

Project Redo Th 150, Note 151 at level of Leonard systems

$\square$

Lecture 30 Monday April 6

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

We now define what it means for Γ to have

classical parameters

Notation $\forall\, b \in \mathbb{Z}$ write

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = 1 + b + b^2 + \cdots + b^{i-1}$$

$$= \begin{cases} \frac{b^i - 1}{b-1} & \text{if } b \neq 1 \\ i & \text{if } b = 1 \end{cases}$$

DEF 152 Γ has classical parameters

(D, b, α, σ) whenever the intersection numbers

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \qquad 0 \leq i \leq D$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\sigma - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \qquad 0 \leq i \leq D$$

Note in this case $b \neq 0$, $b \neq -1$

Th 153 (Brouwer, Cohen, Neumaier)

Assume Γ has classical parameters (D, b, α, σ). Then

(i) $\theta = \frac{b_1}{b} - 1$ is an eigenvalue of Γ with $\theta \neq k$

(ii) Let $E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$ be the associated primitive idempotent. Then

$$\theta_i^*/\theta_0^* = 1 + \left(\theta/k - 1\right)\begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \qquad 0 \le i \le D$$

(iii) E is Q-poly

Pf (i), (ii) To show that θ is an eigval of Γ suffices to check

$$c_i \theta_{i-1}^* + a_i \theta_i^* + b_i \theta_{i+1}^* = \theta \theta_i^* \qquad 0 \le i \le D$$

where $a_i = k - c_i - b_i$ $\qquad k = b_0$

this check is routine.

Show $\theta \neq k$: Suppose $\theta = k$. Then

$$\frac{b_1}{b} - 1 = k$$

so

$$b > 0$$

Using Def 152

$$b c_i - b_i - b(b c_{i-1} - b_{i-1}) = (k-\theta) b$$
$$= 0 \qquad 1 \le i \le D$$

Hence

$$b c_i - b_i = b^i (b c_0 - b_0)$$
$$= -b^i k \qquad 1 \le i \le D$$

Setting $i = D$

$$\underset{\vee_0}{b c_D} = \underset{\wedge_0}{-b^D k} \qquad \text{cont.}$$

So $\theta \ne k$.

(iii) Check the three conditions of Th 150 are satisfied! □

Need to check

$$a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) \quad \text{is quad poly in } \theta_i^*$$

$a_0 = 0$ so expect one factor is $\theta_i^* - \theta_0^*$

One checks

$$\frac{a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*)}{\theta_i^* - \theta_0^*} \quad \text{is linear poly in } \theta_i^* \qquad 1 \le i \le D.$$

Example 154 The following have classical parameters

(i) Hamming graph $H(D, N)$

$$b = 1 \qquad \alpha = 0 \qquad \sigma = N - 1$$

(ii) Johnson graph $J(D, N)$ $(N \geq 2D)$

$$b = 1 \qquad \alpha = 1 \qquad \sigma = N - D$$

(iii) q-Johnson graph $J_q(D, N)$ $(N \geq 2D)$

$$b = q \qquad \alpha = q \qquad \sigma = \frac{q^{N-D+1} - 1}{q - 1} - 1$$

Pf For these examples the b_i, c_i are given in Ex 17 of Ch I.

Compare these formula with Def 152 □

LEM 155 Assume Γ has classical params (D, b, α, σ), and let $\{E_i\}_{i=0}^{D}$ be the corresp Q-poly order of the prim ids. Then

$$\theta_i = \frac{b_{\mu}}{b^i} - \left[{i \atop 1} \right] \qquad 0 \le i \le D$$

Pf We saw earlier

$$\frac{a_i (\theta_i^* - \theta_{i+1}^*)(\theta_i^* - \theta_{i-1}^*)}{\theta_i^* - \theta_0^*}$$

is a linear poly in θ_i^* for $1 \le i \le D$. In this linear poly leading coeff is

$$\gamma = \frac{\alpha(b^D + 1) + (\sigma - 1)(b - 1)}{b}$$

Now get $\{\theta_i\}_{i=0}^{D}$ using $\theta_0 = k$, $\theta_1 = \theta$

$$\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1} \qquad 1 \le i \le D-1$$

where $\beta = b + b^{-1}$ $\square$

We mention a variation on Pb 150

DEF 156 Let E denote a nontrivial primitive idempotent of Γ. Call E a *tail* whenever

$E \circ E$ is a linear combination of E_0, E, and at most one other prim idempotent of Γ

Ref to Def 156 and writing $E = E_1$

E is a tail iff $q_{11}^i \neq 0$ for at most one i $(2 \leq i \leq D)$

Thm 157 (Jurisic + Zitnik) Given a nontrivial primitive idempotent

$$E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$$

of Γ. Then E is Q-poly iff

(i) $\theta_i^* \neq \theta_0^*$ $\quad 1 \leq i \leq D$

(ii) $\exists\ \beta, \gamma^* \in \mathbb{F}$ s.t

$$\gamma^* = \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* \qquad 1 \leq i \leq D-1$$

(iii) E is a tail.

Pf (Sketch)

$\Rightarrow$ clear.

$\Leftarrow$ Fix $x \in X$, write $T = T(x)$

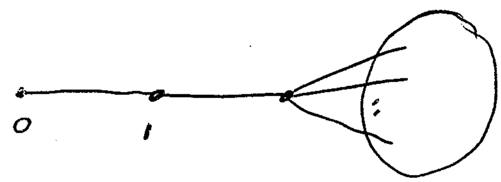
Write $E = E_1$, $A^* = A_1^*$

Let Δ_E diagram be as in pf of Th 150

Obs Δ_E is connected by same arg as in Th 150

Since E is a tail.

Δ_E :



We will show Δ_E is a path

By L63 $\exists\ \delta^* \in \mathbb{F}$ s.t

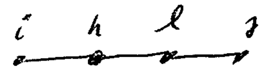
$$\theta_{i-1}^{*2} - \beta\theta_{i-1}^*\theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*) = \delta^* \qquad 1 \le i \le D$$

Invoking LEM 62 we get TD2 :

$$0 = \left[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) - \delta^* A \right]$$

<u>claim</u> Given verts i, j in Δ_E at $\partial(i,j) = 3$

Assume there exist a unique path of length 3 connecting i, j.

Call it 

then $\theta_i - (\beta+1)\theta_h + (\beta+1)\theta_l - \theta_j = 0$

Pf d (Sim to pf of d3 in Th 150)

In TD2 multiply each term on the left by E_i and right by E_j, and simplify

Note TD2 has form

$$0 = A^{*3}A - (\beta+1)A^{*2}AA^{*} + (\beta+1)A^{*}AA^{*2} - AA^{*3}$$

+ lower terms

Obs

$$E_i A^{*3} A E_j = E_i A^{*3} E_j \; \theta_j$$

$$E_i A^{*3} E_j = E_i A^{*}\left(\sum_{r=0}^{D} E_r\right) A^{*}\left(\sum_{s=0}^{D} E_s\right) A^{*} E_j$$

$$= E_i A^{*} E_h A^{*} E_\ell A^{*} E_j$$

Similarly

$$E_i A^{*2} A A^{*} E_j = E_i A^{*} E_h A^{*} E_\ell A^{*} E_j \; \theta_\ell$$

$$E_i A^{*} A A^{*2} E_j = E_i A^{*} E_h A^{*} E_\ell A^{*} E_j \; \theta_h$$

$$E_i A A^{*3} E_j = E_i A^{*} E_h A^{*} E_\ell A^{*} E_j \; \theta_i$$

Obs $E_i A^{*} E_h A^{*} E_\ell A^{*} E_j \neq 0$ since the restriction to the primary module is non 0.

claim follows.

We can now easily show Δ_E is a path

Since Δ_E is connected and vertex 0 is adj vertex 1 and no other vertex, it suffices to show each vertex in Δ_E is adj at most 2 verts in Δ_E.

Suppose $\exists$ vertex i in Δ_E that is adj at least 3 vertices in Δ_E. Of all such verts pick i s.t $\partial(0,i)$ minimal

WLOG vertices of Δ_E are labelled s.t $\partial(0,i) = i$ and $0,1,2,\ldots,i$ is a path in Δ_E:

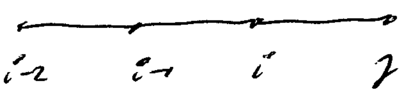
Since E is a tail, $i \geq 2$

By assumption $\exists$ dist verts j, j' in

$$\Delta_E \setminus \{0,1,2,\ldots,i\}$$

that are adj i

Obs $\partial(i-2, j) = 3$ and



is unique path of length 3 connecting $i-2$, j.

So by claim

$$0 = \theta_{i-2} - (\beta+1)\theta_{i-1} + (\beta+1)\theta_i - \theta_j.$$

Similarly

$$0 = \theta_{i-2} - (\beta+1)\theta_{i-1} + (\beta+1)\theta_i - \theta_{j'}$$

So $\theta_j = \theta_{j'}$ cont.

Therefore Δ_E is a path and E is Q-poly. □

Project: Redo Th 157 at level of Leonard systems

Next topic: the split decomposition

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

Assume $\{E_i\}_{i=0}^{D}$ is Q-polynomial

Fix $x \in X$, write $T = T(x)$ etc.

DEF 158 For $-1 \leq i, j \leq D$ define

$$V_{ij} = (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_0 V + E_1 V + \cdots + E_j V)$$

where V = st. module. Obs $V_{ij} = 0$ if $i = -1$ or $j = -1$

LEM 159 For $0 \leq i, j \leq D$ and $v \in V$ TFAE

(i) $v \in V_{ij}$

(ii) $E_h^* v = 0$ for $i < h \leq D$ and $E_h v = 0$ for $j < h \leq D$

Pf clear □

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LEM 160 For $0 \le i, j \le 0$

$$V_{i-1,j} \le V_{ij}$$

$$V_{i,j-1} \le V_{ij}$$

Pf clear □

□

Lecture 31 Wednesday April 8

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

Assume $\{E_i\}_{i=0}^{D}$ is Q-polynomial

Fix $x \in X$, write $T = T(x)$ etc.

Recall

$$V_{ij} = (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_0 V + E_1 V + \cdots + E_j V)$$

for $0 \leq i, j \leq D$

LEM 161

(i) $V_{iD} = E_0^* V + E_1^* V + \cdots + E_i^* V$ $(0 \leq i \leq D)$

(ii) $V_{0j} = E_0 V + E_1 V + \cdots + E_j V$ $(0 \leq j \leq D)$

(iii) $V_{DD} = V$

Pf clear. □

Note it will turn out that

$$V_{ij} = 0 \quad \text{if } i + j < D \qquad (0 \leq i, j \leq D)$$

DEF 162 For $0 \le i, j \le D$ obs that

$$V_{i-1,j} + V_{i,j-1} \subseteq V_{ij}$$

Let $\tilde{V}_{ij}$ denote the orthogonal complement of

$V_{i-1,j} + V_{i,j-1}$ in V_{ij}, so

$$V_{ij} = \tilde{V}_{ij} + (V_{i-1,j} + V_{i,j-1})$$

↑ orthog dir sum

For notational convenience define

$\tilde{V}_{ij} = 0$ if $i \notin \{0,1,\ldots,D\}$ or $j \notin \{0,1,\ldots,D\}$

Our next goal is to show

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{ij} \qquad \text{(d sum)}$$

LEM 163 For $0 \le i,j \le D$

$$\dim \tilde{V}_{ij} = \dim V_{ij} - \dim V_{i,j-1} - \dim V_{i-1,j} + \dim V_{i-1,j-1}$$

Pf Observe

$$\dim V_{ij} = \dim \tilde{V}_{ij} + \dim\left(V_{i-1,j} + V_{i,j-1}\right)$$

Also By linear algebra

$$\dim\left(V_{i-1,j} + V_{i,j-1}\right) = \dim V_{i-1,j} + \dim V_{i,j-1} - \dim\left(V_{i-1,j} \cap V_{i,j-1}\right)$$

Obs

$$V_{i-1,j} \cap V_{i,j-1} = V_{i-1,j-1}$$

Result follows. □

thm 164 For $0 \le r, s \le D$

$$V_{r,s} = \sum_{i=0}^{r} \sum_{j=0}^{s} \tilde{V}_{ij} \qquad \text{(d sum)}$$

Pf Show

$$V_{r,s} = \sum_{i=0}^{r} \sum_{j=0}^{s} \tilde{V}_{ij} \qquad (*)$$

Pf is by induction on $r+s$. (*) holds for $r+s=0$

Since $\tilde{V}_{00} = V_{00}$. Assume $r+s>0$. By constr

$$V_{rs} = \tilde{V}_{rs} + V_{r,s-1} + V_{r-1,s}$$

By ind

$$V_{r,s-1} = \sum_{i=0}^{r} \sum_{j=0}^{s-1} \tilde{V}_{ij}$$

$$V_{r-1,s} = \sum_{i=0}^{r-1} \sum_{j=0}^{s} \tilde{V}_{ij}$$

Combining these eqs get *.

Now show sum (*) is direct: Using L163 obtain

$$\dim V_{r,s} = \sum_{i=0}^{r} \sum_{j=0}^{s} \dim \tilde{V}_{ij}$$

Therefore the sum (*) is direct. □

COR 165 We have

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{ij} \qquad \text{(dsum)}$$

Pf Set $r = s = 0$ in Th 164. □

DEF 166 We call the sum in COR 165 the <u>split</u> <u>decomposition</u> of V with respect to x. Caution: this sum is not orthogonal in general

—

We will return the split dec shortly.

Another characterization of the Q-poly prop:

the balanced set condition

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

We do not assume a Q-poly structure

Fix $x \in X$ write $T = T(x)$ etc.

Given nontrivial prim idempotent E write

$$E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$$

Recall E is nondegenerate iff $\theta_i^* \neq \theta_0^*$ for $1 \leq i \leq D$

Thm 167 With above notation TFAE

(i) E is Qpolynomial

(ii) E is nondeg and $\forall\, i,j$ $(0 \le i, j \le D)$ and $\forall$ dist $y, z \in X$

$$\sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} E\hat{w} \quad - \quad \sum_{w \in \Gamma_i(z) \cap \Gamma_j(y)} E\hat{w} \quad = p^h_{ij} \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} \left(E\hat{y} - E\hat{z}\right)$$

$h = \partial(y, z)$

(iii) E is nondeg and $\forall\, y, z \in X$

$$\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} E\hat{w} \quad - \quad \sum_{w \in \Gamma_2(y) \cap \Gamma(z)} E\hat{w} \quad \in \text{Span}\left(E\hat{y} - E\hat{z}\right)$$

(Illustrate using models of platonic solids.

3-cube, Icosahedron, Octahedron are Qpoly

Dodecahedron is not)

Pf write $E = E_1$, $A^* = A_1^*$

(i) → (ii) E is nondeg since $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$

are mut distinct.

Recall L60:

$$\text{Span}\left(RA^*S - SA^*R \;/\; R, S \in M \right) \qquad M = \text{Bose-Mesner}$$

$$= \left(YA^* - A^*Y \;/\; Y \in M \right)$$

Take $R = A_i$ and $S = A_j$. Get

$$A_i A^* A_j - A_j A^* A_i = \sum_{h=1}^{D} r_{ij}^h \left(A^* A_h - A_h A^* \right) \qquad (*)$$

some $r_{ij}^h \in \mathbb{F}$

<u>claim</u>

$$r_{ij}^n = p_{ij}^n \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_n^*} \qquad (1 \le n \le D)$$

<u>pf cl</u> For $1 \le n \le D$ pick $z \in X$ st $\partial(x, z) = n$.

Compute (x,z)-entry in $(*)$

$$(A_i A^* A_j)_{xz} = \sum_{w \in X} (A_i)_{xw} A^*_{ww} (A_j)_{wz}$$

$$= \sum_{w \in \Gamma_i(x) \cap \Gamma_j(z)} \theta_i^*$$

$$= p^h_{ij} \theta_i^*$$

Sim

$$(A_j A^* A_i)_{xz} = p^h_{ij} \theta_j^*$$

Sim For $1 \le h \le D$

$$(A_h A^*)_{xz} = p^h_{hD} \theta_h^*$$

$$(A^* A_h)_{xz} = p^h_{Dh} \theta_D^*$$

So by (*)

$$p^h_{ij}(\theta_i^* - \theta_j^*) = \sum_{h=1}^{D} r^h_{ij} \Big(\underset{\delta_{hn}}{p^n_{Dh}} \theta_D^* - \underset{\delta_{hn}}{p^n_{hD}} \theta_h^* \Big)$$

so

$$p^n_{ij}(\theta_i^* - \theta_j^*) = r^n_{ij}(\theta_D^* - \theta_n^*)$$

claim follows.

Pick dist $y, z \in X$ write $h = \partial(y, z)$

We now show

$$\sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} E\hat{w} \quad - \sum_{w \in \Gamma_j(y) \cap \Gamma_i(z)} E\hat{w} \quad = \quad p^h_{ij} \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} \left(E\hat{y} - E\hat{z} \right)$$

Since our base vertex x is arb, WLOG it suffices to show

coord x of LHS = coord x of RHS

To obtain this, compute (y,z)-entry in $(*)$

$$(A_i A^* A_j)_{yz} = \sum_{w \in X} (A_i)_{yw} (A^*)_{ww} (A_j)_{wz}$$

$$= \sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} (A^*)_{ww}$$

$(A^*)_{ww} = |X| \, E_{xw} = |X| \left(x\text{-coord of } E\hat{w} \right)$

$$= x\text{-coord of } |X| \sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} E\hat{w}$$

other terms are similar.

(ii) → (iii) clear

□

Lecture 32 Friday April 10

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

Given nontrivial primitive idempotent

$$E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$$

We continue the proof of Th 167

(iii) → (i) We assume E is nondegenerate so

$$\theta_i^* \neq \theta_0^* \qquad 1 \leq i \leq D$$

Assume $D \geq 3$. Since for $D = 2$ E is Q-poly

<u>Claim 1</u> Pick h $(1 \leq h \leq D)$ and $y, z \in X$ at $\partial(y,z) = h$. Then

$$\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} E\hat{w} - \sum_{w \in \Gamma(z) \cap \Gamma_2(y)} E\hat{w} = r_{12}^h (E\hat{y} - E\hat{z})$$

where $$r_{12}^h = p_{12}^h \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}$$

<u>pf cl</u> By assumption $\exists \alpha \in \mathbb{F}$ s.t

$$\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} E\hat{w} - \sum_{w \in \Gamma(z) \cap \Gamma_2(y)} E\hat{w} = \alpha (E\hat{y} - E\hat{z})$$

In above equation take the inner product of each term with $E\hat{y}$

Recall $\forall w \in X$

$$\langle E\hat{w}, E\hat{y}\rangle = |X|^{-1}\theta_g^* \qquad g = \partial(w,y)$$

So

$$\sum_{w\in\Gamma(y)\cap\Gamma_2(z)} \langle E\hat{w}, E\hat{y}\rangle = |X|^{-1} p^h_{12}\,\theta_1^*$$

Other terms similar. Get

$$|X|^{-1} p^h_{12}(\theta_1^* - \theta_2^*) = \alpha |X|^{-1}(\theta_0^* - \theta_h^*)$$

so

$$\alpha = p^h_{12}\frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}$$

<u>claim 2</u> $$A A^* A_2 - A_2 A^* A = \sum_{h=1}^{D} r^h_{12}(A^* A_h - A_h A^*)$$

<u>pf cl</u> For $y,z\in X$ find (y,z)-entry of LHS $-$ RHS

For $y = z$ one checks this entry is 0. For $y \neq z$ this entry is

$$|X|\left\langle \sum_{w\in\Gamma(y)\cap\Gamma_2(z)} E\hat{w} - \sum_{w\in\Gamma(z)\cap\Gamma_2(y)} E\hat{w} - r^h_{12}(E\hat{y} - E\hat{z}),\ E\hat{x}\right\rangle$$

$$h = \partial(y,z)$$

this expression is 0 by claim 1

Conceivably $\theta_1^* = \theta_2^*$

In this case

$$r_{12}^h = 0 \quad 1 \le h \le D$$

So by claim 2

$$A_2 A^* A = A A^* A_2$$

Use

$$A_2 = \frac{A^2 - a_1 A - kI}{c_2}$$

to get

$$A^2 A^* A - A A^* A^2 = k(A^* A - A A^*)$$

We will return to this equation shortly.

<u>claim 3</u> Suppose $\theta_1^* \ne \theta_2^*$. Then $\exists\, \beta, \gamma, \delta \in \mathbb{F}$ s.t.

$$0 = \left[A,\ A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \delta A^* \right] \qquad \text{TD 1}$$

<u>pf cl</u> Recall

$$p_{12}^h = \begin{cases} 0 & \text{if } h > 3 \\ \ne 0 & \text{if } h = 3 \end{cases}$$

So

$$r_{12}^h = \begin{cases} 0 & \text{if } h > 3 \\ \ne 0 & \text{if } h = 3 \end{cases}$$

Also recall for $0 \le i \le D$ $\exists$ $f_i \in \mathbb{F}[\lambda]$ with

$$A_i = f_i(A) \qquad \deg f_i = i$$

In claim 2 we write A_2, A_3 as poly in A and simplify

to find

$$A^3A^* - A^*A^3 \in \text{Span}\left(A^2A^*A - AA^*A^2,\ A^2A^* - A^*A^2,\ AA^* - A^*A\right)$$

So $\exists$ $\beta, \gamma, \delta \in \mathbb{F}$ s.t.

$$0 = A^3A^* - A^*A^3 - (\beta+1)\left(A^2A^*A - AA^*A^2\right)$$

$$- \gamma\left(A^2A^* - A^*A^2\right) - \delta\left(AA^* - A^*A\right)$$

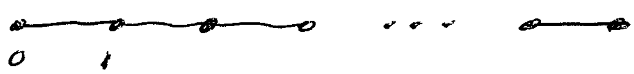
Rewrite this to get TD1

Recall diagram Δ_E as in pf of Th 15.0.

Recall

- Δ_E is connected since we assume E is nondeg.
- node 0 is adj node 1 and nothing else

Show Δ_E is a path



To do this we show that each node i of Δ_E is adjacent at most 2 nodes of Δ_E

<u>claim 4</u> Given distinct nodes i, j of Δ_E that are adj,

If $\theta_1^* = \theta_2^*$ then $\theta_i \theta_j = -k$

If $\theta_1^* \neq \theta_2^*$ then $P(\theta_i, \theta_j) = 0$

where

$$P(\lambda, \mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma(\lambda + \mu) - \delta$$

<u>pf cl</u> assume $\theta_1^* \neq \theta_2^*$. In TD1 mult each term on left by E_i and on right by E_j. Simplify to get

$$0 = \underbrace{E_i A^* E_j}_{\neq 0} \underbrace{(\theta_i - \theta_j)}_{\neq 0} \underbrace{P(\theta_i, \theta_j)}_{\text{must be } 0}$$

since i, j adj   since $i \neq j$

$$P(\theta_i, \theta_j) = 0$$

For $\theta_1^* = \theta_2^*$ the argument is similar using the equation above

claim 3.

<u>Claim 5</u> $\theta_1^* \neq \theta_2^*$.

<u>pf cl</u> Suppose $\theta_1^* = \theta_2^*$. By claim 4 and since node 0 is adj node 1 in Δ_E,

$$\theta_1 = -1$$

Δ_E is connected so node 1 is adj some node j $j \neq 0$

By cl 4

$$\theta_1 \theta_j = -k$$

($\theta_1 = -1$)

so

$$\theta_j = k$$

So $j = 0$ cont.

Claim 6 Each node i in Δ_E is adj at most 2 nodes in Δ_E

pf cl By claims 4, 5, for each node j in Δ_E that is adj i the eigval θ_j is the root of

$$p(\theta_i, \mu) = \theta_i^2 - \beta\theta_i + \mu^2 - \gamma(\theta_i + \mu) - \delta$$

This is a quadratic polynomial in μ so it has at most 2 distinct roots. The claim follows.

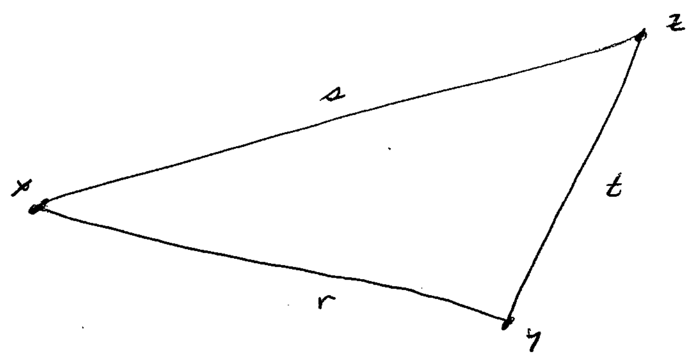
We have now shown Δ_E is a path so E is Q-polynomial. □

We now consider: What is the combinatorial meaning of the Q-poly property?

In addition to our base vertex x we fix 2 more vertices y, z. To avoid trivialities assume x, y, z are distinct.

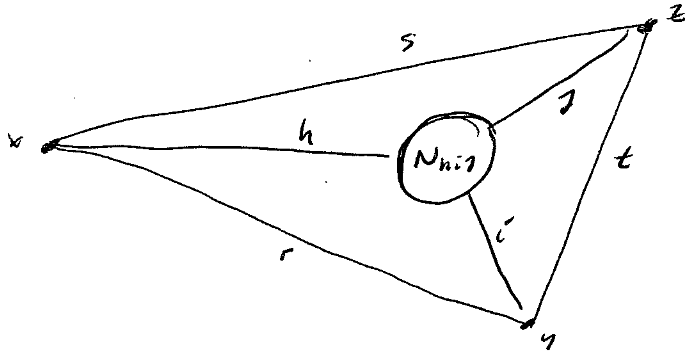
Set

$$r = \partial(x,y) \qquad s = \partial(x,z) \qquad t = \partial(y,z)$$



For $0 \leq h, i, j \leq D$ define

$$N_{hij} = \left| \Gamma_h(x) \cap \Gamma_i(y) \cap \Gamma_j(z) \right|$$



Caution: N_{hij} depends on choice of x, y, z not just

h, i, j, r, s, t.

Using the DRG axiom

$$\sum_{h=0}^{D} N_{hij} = |\Gamma_i(y) \cap \Gamma_j(z)|$$

$$= p^t_{ij}$$

Instead of summing over h, we could sum over i or j to get similar equations. But unless we assume more, thats about all we can say. If we assume a Q-poly structure we can say more.

Thm 168. Let E denote a Q-poly prim idempotent for Γ. Then for $0 \le i, j \le D$

$$\sum_{h=0}^{D} \theta_h^* \left(N_{hij} - N_{hji} \right) = p^t_{ij} \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_t^*} \left(\theta_r^* - \theta_s^* \right)$$

pf. In the eq of Th167 (ii) take the inner product of each term with $E\hat{x}$ and simplify. □

We now consider the combinatorial meaning of the Q-poly property from another point of view

Fix $x \in X$, write $T = T(x)$ etc.

Define

$$R = \sum_{i=0}^{D-1} E_{i+1}^* A E_i^* \qquad \text{"raise"}$$

$$F = \sum_{i=0}^{D} E_i^* A E_i^* \qquad \text{"flat"}$$

$$L = \sum_{i=1}^{D} E_{i-1}^* A E_i^* \qquad \text{"lower"}$$

and recall

$$A = R + F + L$$

Observe

$$R E_i^* V \subseteq E_{i+1}^* V \qquad 0 \le i \le D$$

$$F E_i^* V \subseteq E_i^* V \qquad 0 \le i \le D$$

$$L E_i^* V \subseteq E_{i-1}^* V \qquad 0 \le i \le D$$

where $E_{-1}^* = 0$, $E_{D+1}^* = 0$.

□

Lecture 33 Monday April 13

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

Fix $x \in X$ write $T = T(x)$ etc.

Recall $A = R + F + L$

$$R = \sum_{i=0}^{D-1} E_{i+1}^* A E_i^*$$

$$F = \sum_{i=0}^{D} E_i^* A E_i^*$$

$$L = \sum_{i=1}^{D} E_{i-1}^* A E_i^*$$

Recall $$V = \sum_{i=0}^{D} E_i^* V \qquad \text{(ods)}$$

Until further notice assume Γ is Q poly wrt $\{E_i\}_{i=0}^{D}$

let $\beta, \gamma, \gamma^*, \delta, \delta^*$ be as in thm 57.

thm 169 With above notation

(i) For $2 \le i \le D$

$$g_i^+ L^2 F + LFL + g_i^- FL^2 - \gamma L^2 = 0$$

on $E_i^* V$, where

$$g_i^+ = \frac{\theta_{i-2}^* - (\beta+1)\theta_{i-1}^* + \beta\theta_i^*}{\theta_{i-2}^* - \theta_i^*}$$

$$g_i^- = \frac{-\beta\theta_{i-2}^* + (\beta+1)\theta_{i-1}^* - \theta_i^*}{\theta_{i-2}^* - \theta_i^*}$$

(ii) For $0 \le i \le D$

$$[F, LR - h_i RL] = 0$$

on $E_i^* V$ where

$$h_i = \frac{\theta_{i-1}^* - \theta_i^*}{\theta_i^* - \theta_{i+1}^*} \qquad \forall\ 1 \le i \le D-1$$

and h_0, h_D indets.

(iii) For $1 \le i \le D$

$$e_i^+ L^2 R + (\beta+2) LRL + e_i^- RL^2$$

$$+ LF^2 - \beta FLF + F^2 L - \gamma(LF+FL) - \delta L$$

$$= 0$$

on $E_i^* V$ where

$$e_i^+ = \frac{\theta_{i-1}^* - (\beta+2)\theta_i^* + (\beta+1)\theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \qquad \text{if } 1 \le i \le D-1$$

$$e_i^- = \frac{-(\beta+1)\theta_{i-2}^* + (\beta+2)\theta_{i-1}^* - \theta_i^*}{\theta_{i-1}^* - \theta_i^*} \qquad \text{if } 2 \le i \le D$$

and e_D^+, e_1^- indets

Pf Recall TD1

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*]$$

$$= A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2)$$

$$- \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A)$$

Call above expression Δ.

then $0 = \Delta$

(i) Observe

$$0 = E^*_{i-2}\Delta E^*_i$$

Show that on E^*_iV

$$E^*_{i-2}\Delta E^*_i = g_i^+ L^2F + LFL + g_i^- FL^2 - \gamma L^2$$

to show this, for each term in Δ mult on left by E^*_{i-2}

and on right by E^*_i and simplify. For example

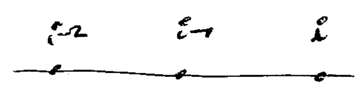
$$E^*_{i-2} A^3A^*E^*_i = \theta^*_i E^*_{i-2} A^3 E^*_i$$

and

$$E^*_{i-2} A^3 E^*_i = E^*_{i-2} A \left(\sum_{r=0}^{D} E^*_r \right) A \left(\sum_{s=0}^{D} E^*_s \right) A E^*_i$$

$$= \sum_{r,s} E^*_{i-2} A E^*_r A E^*_s A E^*_i$$

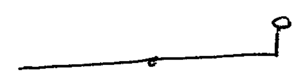
$$= \sum_{\substack{r,s \\ |i-2-r| \le 1 \\ |r-s| \le 1 \\ |s-i| \le 1}} E^*_{i-2} A E^*_r A E^*_s A E^*_i$$



$$= E^*_{i-2} A E^*_{i-2} A E^*_{i-1} A E^*_i$$

$$+ E^*_{i-2} A E^*_{i-1} A E^*_{i-1} A E^*_i$$

$$+ E^*_{i-2} A E^*_{i-1} A E^*_i A E^*_i$$



$$= \left(F L^2 + L F L + L^2 F \right) E^*_i$$

Similarly

$$E^*_{i-2} A^* A^3 E^*_i = \theta^*_{i-2} (FL^2 + LFL + L^2F) E^*_i$$

$$E^*_{i-2} A^2 A^* A E^*_i = (\theta^*_{i-1} (FL^2 + LFL) + \theta^*_i L^2 F) E^*_i$$

$$E^*_{i-2} A A^* A^2 E^*_i = (\theta^*_{i-2} FL^2 + \theta^*_{i-1} (LFL + L^2F)) E^*_i$$

$$E^*_{i-2} (A^2 A^* - A^* A^2) E^*_i = (\theta^*_i - \theta^*_{i-2}) L^2 E^*_i$$

$$E^*_{i-2} (A A^* - A^* A) E^*_i = 0$$

Result follows.

(ii) Evaluate

$$0 = E^*_i \Delta E^*_i$$

as in (i)

(iii) Evaluate

$$0 = E^*_{i-1} \Delta E^*_i$$

as in (i). $\square$

LEM 170 Referring to Thm 169

$$e_i^+ = \frac{\theta_i^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i+1}^*} \qquad 1 \le i \le d-2$$

$$e_i^- = \frac{\theta_{i-1}^* - \theta_{i-3}^*}{\theta_{i-1}^* - \theta_i^*} \qquad 3 \le i \le d$$

$$g_i^+ = \frac{\theta_i^* - \theta_{i+1}^*}{\theta_i^* - \theta_{i-2}^*} \qquad 2 \le i \le d-1$$

$$g_i^- = \frac{\theta_{i-2}^* - \theta_{i-3}^*}{\theta_{i-2}^* - \theta_i^*} \qquad 3 \le i \le d$$

In particular $e_i^\pm$, $g_i^\pm$ are non0 for the range of i given above.

Pf In each case, equate the above expression with the corresp expression in Thm 169. The resulting equation is just

$$\beta+1 = \frac{\theta_{j-2}^* - \theta_{j+1}^*}{\theta_{j-1}^* - \theta_j^*}$$

which we saw earlier.

For example consider e_i^+

$$\frac{\theta^*_{i-1} - (\beta+2)\theta^*_i + (\beta+1)\theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \overset{?}{=} \frac{\theta^*_i - \theta^*_{i+2}}{\theta^*_i - \theta^*_{i+1}}$$

$$\theta^*_{i-1} - (\beta+2)\theta^*_i + (\beta+1)\theta^*_{i+1} \overset{?}{=} \theta^*_{i+2} - \theta^*_i$$

$$\theta^*_{i-1} - (\beta+1)\theta^*_i + (\beta+1)\theta^*_{i+1} - \theta^*_{i+2} \overset{\checkmark}{=} 0$$

□

Note 171 With ref to thm 169 assume Γ is Bipartite

then $F=0$ by construction and $\gamma=0$ by thm 74

so (i) yields no info.

Sim (ii) yields no info.

(iii) implies for $1 \le i \le D$

$$e_i^+ L^2 R + (\beta+2) LRL + e_i^- RL^2 - \delta L = 0$$

on $E_i^* V$

No longer assume Γ has a Q-poly structure.

Open Problem 172. Assume Γ is Bipartite

and that for $1 \le i \le D$,

$$L^2R, \quad LRL, \quad RL^2, \quad L$$

are lin dep on E_i^*V (coeffs may depend on i)

Then does Γ have a Q-poly structure?

—

Aside. The above discussion suggests that we should

consider the following type of partially ordered sets.

X = fin. nonempty set

Consider a partial order $\leq$ on X

$\forall y, z \in X$ say <u>z covers y</u> whenever

$$y < z$$

and there does not exist $w \in X$ s.t.

$$y < w < z.$$

Hasse diagram: $y \leftarrow z$ whenever z covers y.

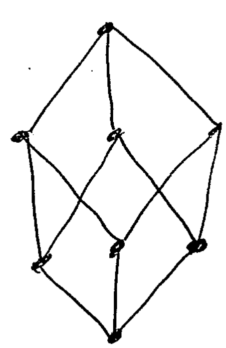
A function $r : X \to \{0, 1, 2 \ldots\}$

is a <u>rank function</u> whenever $\forall y, z \in X$

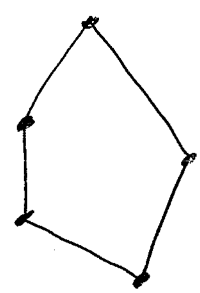
$$z \text{ covers } y \quad \to \quad r(z) - r(y) = 1$$

Poset X is <u>ranked</u> whenever it has a rank function.

Hasse diag



ranked



not ranked

↓

A *zero* of X is an element $0 \in X$ s.t.

$$0 \le y \quad \forall y \in X$$

0 is unique if it exists

Suppose X is ranked and has a 0. Then

$$\text{WLOG} \quad r(0) = 0$$

For $i \ge 0$ let $X_i = \{ y \in X \mid r(y) = i \}$

$$X_0 = \{0\}$$

$$X_1 = \{ y \in X \mid y \text{ covers } 0 \}$$

$$X_2 = \{ y \in X \mid y \text{ covers something in } X_1 \}$$

$$\vdots$$

Suppose X is ranked and has a 0

Consider st. module $V = \mathbb{F}^X$ as before

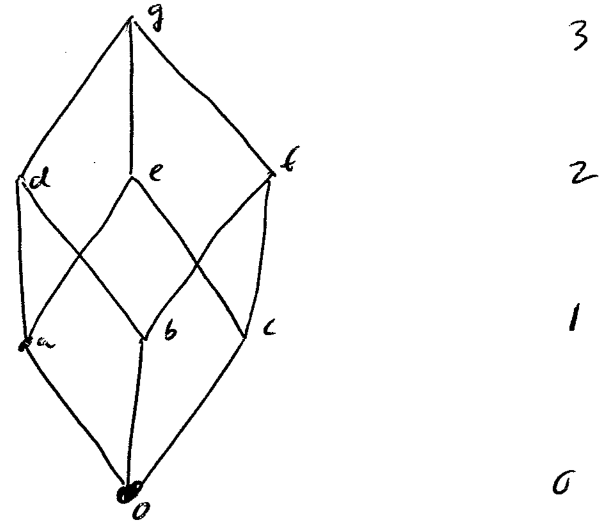
Define $R, L, E_i^* : V \to V$ by

$$R\hat{y} = \sum_{\substack{z \in X \\ z \text{ covers } y}} \hat{z} \qquad L\hat{y} = \sum_{\substack{z \in X \\ y \text{ covers } z}} \hat{z}$$

$$E_i^* \hat{y} = \begin{cases} \hat{y} & r(y) = i \\ 0 & r(y) \ne i \end{cases}$$

rank

Ex



$R\hat{a} = \hat{d} + \hat{e}$ $\qquad R\hat{g} = 0$

$L\hat{g} = \hat{d} + \hat{e} + \hat{f}$ $\qquad L\hat{c} = \hat{0}$ $\qquad L\hat{0} = 0$

Def 173 Assume poset X is ranked with 0

Call X uniform whenever For $i \geq 1$

$L^2R, \ LRL, \ RL^2, \ L$

are lin dep on $E_i^* V$ (coeff may depend on i)

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Many classical posets of interest in geometry, group theory etc are uniform. It is an open problem to systematically study these posets. □