Vertex-Transitive and Cayley Graphs

Mingyao Xu
Peking University

January 18, 2011

DEFINITIONS

- Graphs $\Gamma = (V(\Gamma), E(\Gamma))$.

- Graphs $\Gamma = (V(\Gamma), E(\Gamma))$.
- Automorphisms

- Graphs $\Gamma = (V(\Gamma), E(\Gamma))$.
- Automorphisms
- Automorphism group $G = \mathrm{Aut}(\Gamma)$.

- Graphs $\Gamma = (V(\Gamma), E(\Gamma))$.
- Automorphisms
- Automorphism group $G = \mathrm{Aut}(\Gamma)$.
- Transitivity of graphs: vertex-, edge-, arc-transitive graphs.

- Graphs $\Gamma = (V(\Gamma), E(\Gamma))$.
- Automorphisms
- Automorphism group $G = \mathrm{Aut}(\Gamma)$.
- Transitivity of graphs: vertex-, edge-, arc-transitive graphs.
- Cayley graphs: Let G be a finite group and S a subset of G not containing the identity element 1. Assume $S^{-1} = S$. We define the Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ on G with respect to S by

$$\begin{aligned} V(\Gamma) &= G, \\ E(\Gamma) &= \{(g, sg) \mid g \in G, s \in S\}. \end{aligned}$$

- Graphs $\Gamma = (V(\Gamma), E(\Gamma))$.
- Automorphisms
- Automorphism group $G = \mathrm{Aut}(\Gamma)$.
- Transitivity of graphs: vertex-, edge-, arc-transitive graphs.
- Cayley graphs: Let G be a finite group and S a subset of G not containing the identity element 1. Assume $S^{-1} = S$. We define the Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ on G with respect to S by

$$\begin{aligned} V(\Gamma) &= G, \\ E(\Gamma) &= \{(g, sg) \mid g \in G, s \in S\}. \end{aligned}$$

Properties of Cayley graphs

- Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph on G with respect to S.
  (1) $\mathrm{Aut}(\Gamma)$ contains the right regular representation $R(G)$ of G, so Γ is vertex-transitive.
  (2) Γ is connected if and only if $G = \langle S \rangle$.

Properties of Cayley graphs

- Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph on G with respect to S.
  (1) $\mathrm{Aut}(\Gamma)$ contains the right regular representation $R(G)$ of G, so Γ is vertex-transitive.
  (2) Γ is connected if and only if $G = \langle S \rangle$.
- A graph $\Gamma = (V, E)$ is a Cayley graph of a group G if and only if $\mathrm{Aut}(\Gamma)$ contains a regular subgroup isomorphic to G.
  Above Proposition $\Longrightarrow$ Cayley graphs are just those vertex-transitive graphs whose full automorphism groups have a regular subgroup.

Properties of Cayley graphs

- Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph on G with respect to S.
  (1) $\mathrm{Aut}(\Gamma)$ contains the right regular representation $R(G)$ of G, so Γ is vertex-transitive.
  (2) Γ is connected if and only if $G = \langle S \rangle$.
- A graph $\Gamma = (V, E)$ is a Cayley graph of a group G if and only if $\mathrm{Aut}(\Gamma)$ contains a regular subgroup isomorphic to G.
  Above Proposition $\Longrightarrow$ Cayley graphs are just those vertex-transitive graphs whose full automorphism groups have a regular subgroup.

The Petersen graph is not Cayley

- Outline of a proof:
  If it is Cayley, then the group has order 10.
  There are two non-isomorphic groups: cyclic and dihedral.
  The girth of Cayley graphs on abelian groups are 3 or 4. So $G = D_{10}$.
  $G = \langle a, b \mid a^5 = b^2 = 1, bab = a^{-1} \rangle$.
  $S = \{a, a^{-1}, b\}$ or three involutions.
  For the former, since *abab* is a cyclic of size 4, this is not the case.
  For the latter, the product of any two involutions is 1 or of order 5, a contradiction.

The Petersen graph is not Cayley

- Outline of a proof:
  If it is Cayley, then the group has order 10.
  There are two non-isomorphic groups: cyclic and dihedral.
  The girth of Cayley graphs on abelian groups are 3 or 4. So $G = D_{10}$.
  $G = \langle a, b \mid a^5 = b^2 = 1, bab = a^{-1} \rangle$.
  $S = \{a, a^{-1}, b\}$ or three involutions.
  For the former, since *abab* is a cyclic of size 4, this is not the case.
  For the latter, the product of any two involutions is 1 or of order 5, a contradiction.
- Note: For a positive integer n, if every transitive group has a regular subgroup then every vertex-transitive graph is Cayley.
  (Example: $n = p$, a prime.)

The Petersen graph is not Cayley

- Outline of a proof:
  If it is Cayley, then the group has order 10.
  There are two non-isomorphic groups: cyclic and dihedral.
  The girth of Cayley graphs on abelian groups are 3 or 4. So $G = D_{10}$.
  $G = \langle a, b \mid a^5 = b^2 = 1, bab = a^{-1} \rangle$.
  $S = \{a, a^{-1}, b\}$ or three involutions.
  For the former, since *abab* is a cyclic of size 4, this is not the case.
  For the latter, the product of any two involutions is 1 or of order 5, a contradiction.
- Note: For a positive integer n, if every transitive group has a regular subgroup then every vertex-transitive graph is Cayley.
  (Example: $n = p$, a prime.)

A Question

Let

$$\begin{aligned}\mathcal{NR} &= \{n \in \mathbb{N} \mid \text{there is a transitive group} \\ &\quad \text{of degree } n \text{ without a regular subgroup}\} \\ \mathcal{NC} &= \{n \in \mathbb{N} \mid \text{there is a vertex-transitive graph} \\ &\quad \text{of order } n \text{ which is non-Cayley}\}\end{aligned}$$

Then $\mathcal{NR} \supseteqq \mathcal{NC}$.

- Question: $\mathcal{NR} = \mathcal{NC}$?.

Let

$$\mathcal{NR} = \left\{ n \in \mathbb{N} \,\middle|\, \text{there is a transitive group of degree } n \text{ without a regular subgroup} \right\}$$

$$\mathcal{NC} = \left\{ n \in \mathbb{N} \,\middle|\, \text{there is a vertex-transitive graph of order } n \text{ which is non-Cayley} \right\}$$

Then $\mathcal{NR} \supseteq \mathcal{NC}$.

- Question: $\mathcal{NR} = \mathcal{NC}$?.
- Answer: $\mathcal{NR} \supsetneq \mathcal{NC}$. For example, $12 \notin \mathcal{NC}$, but $12 \in \mathcal{NR}$ since M_{11}, acting on 12 points, has no regular subgroup.

Let

$$\mathcal{NR} = \{n \in \mathbb{N} \mid \text{there is a transitive group of degree } n \text{ without a regular subgroup}\}$$

$$\mathcal{NC} = \{n \in \mathbb{N} \mid \text{there is a vertex-transitive graph of order } n \text{ which is non-Cayley}\}$$

Then $\mathcal{NR} \supseteq \mathcal{NC}$.

- Question: $\mathcal{NR} = \mathcal{NC}$?.
- Answer: $\mathcal{NR} \supsetneq \mathcal{NC}$. For example, $12 \notin \mathcal{NC}$, but $12 \in \mathcal{NR}$ since M_{11}, acting on 12 points, has no regular subgroup.
- Exercise: 6 is the smallest number in $\mathcal{NR} \setminus \mathcal{NC}$ since A_6 has no regular subgroups.

Let

$$\mathcal{NR} = \{n \in \mathbb{N} \mid \text{there is a transitive group of degree } n \text{ without a regular subgroup}\}$$

$$\mathcal{NC} = \{n \in \mathbb{N} \mid \text{there is a vertex-transitive graph of order } n \text{ which is non-Cayley}\}$$

Then $\mathcal{NR} \supseteq \mathcal{NC}$.

- Question: $\mathcal{NR} = \mathcal{NC}$?.
- Answer: $\mathcal{NR} \supsetneqq \mathcal{NC}$. For example, $12 \notin \mathcal{NC}$, but $12 \in \mathcal{NR}$ since M_{11}, acting on 12 points, has no regular subgroup.
- Exercise: 6 is the smallest number in $\mathcal{NR} \setminus \mathcal{NC}$ since A_6 has no regular subgroups.

A Question

$p^2 \notin \mathcal{NC}$

- (Marušič ıč) Any transitive group G of degree p^2 on Ω has a regular subgroup, i.e., $p^2 \notin \mathcal{NR}$.

$p^2 \notin \mathcal{NC}$

- (Marušič ıč) Any transitive group G of degree p^2 on Ω has a regular subgroup, i.e., $p^2 \notin \mathcal{NR}$.
- Outline of a proof: Take a minimal transitive subgroup P of G. Then P is a p-group and every maximal subgroup M of P is intransitive. For any $\alpha \in \Omega$, we have $|P_\alpha| = |P|/p^2$ and $|M_\alpha| > |M|/p^2$, so $M_\alpha = P_\alpha$. It follows that $P_\alpha \leq M$ and hence $P_\alpha \leq \Phi(P)$. If $|P : \Phi(P)| = p$, then P is cyclic and is regular. If $|P : \Phi(P)| = p^2$, then $P_\alpha = \Phi(P)$. Since $\Phi(P)$ is normal in P and P_α is core-free, we have $P_\alpha = 1$ and hence $P \cong \mathbb{Z}_p^2$ is regular.

$p^2 \notin \mathcal{NC}$

- (Marušič ıč) Any transitive group G of degree p^2 on Ω has a regular subgroup, i.e., $p^2 \notin \mathcal{NR}$.
- Outline of a proof: Take a minimal transitive subgroup P of G. Then P is a p-group and every maximal subgroup M of P is intransitive. For any $\alpha \in \Omega$, we have $|P_\alpha| = |P|/p^2$ and $|M_\alpha| > |M|/p^2$, so $M_\alpha = P_\alpha$. It follows that $P_\alpha \leq M$ and hence $P_\alpha \leq \Phi(P)$. If $|P : \Phi(P)| = p$, then P is cyclic and is regular. If $|P : \Phi(P)| = p^2$, then $P_\alpha = \Phi(P)$. Since $\Phi(P)$ is normal in P and P_α is core-free, we have $P_\alpha = 1$ and hence $P \cong \mathbb{Z}_p^2$ is regular.

$p^3 \in \mathcal{NR} \setminus \mathcal{NC}$

- (Marušič ıč) $p^3 \notin \mathcal{NC}$.

$p^3 \in \mathcal{NR} \setminus \mathcal{NC}$

- (Marušič ıč) $p^3 \notin \mathcal{NC}$.
- $p^3 \in \mathcal{NR}$ (For $p > 2$).
  Let G be the following group of order p^4

$$G = \langle a, b \mid a^{p^2} = b^p = c^p = 1, [a, b] = c, [c, a] = a^p, [c, b] = 1 \rangle.$$

  Let $H = \langle c \rangle$. Consider the transitive permutation representation φ of G acting on the coset space $[G : H]$.
  Then $\varphi(G)$ is a transitive group of degree p^3, and $\varphi(G)$ has no regular subgroups.

$p^3 \in \mathcal{NR} \setminus \mathcal{NC}$

- (Marušič ıč) $p^3 \notin \mathcal{NC}$.
- $p^3 \in \mathcal{NR}$ (For $p > 2$).
  Let G be the following group of order p^4

  $$G = \left\langle a, b \;\middle|\; a^{p^2} = b^p = c^p = 1, [a, b] = c, [c, a] = a^p, [c, b] = 1 \right\rangle.$$

  Let $H = \langle c \rangle$. Consider the transitive permutation representation φ of G acting on the coset space $[G : H]$.
  Then $\varphi(G)$ is a transitive group of degree p^3, and $\varphi(G)$ has no regular subgroups.

- $p^3 \in \mathcal{NR}$ (For $p = 2$).
  Let

$$G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^4 = 1, [a, b] = [b, c] = [c, a] = 1, a^d = ab, b^d = bc, c^d = c \rangle.$$

Then $G \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$ has order 2^5. Let $H = \langle b, d^2 \rangle$ and φ be the transitive permutation representation of G acting on the coset space $[G : H]$.
Then $\varphi(G)$ is a transitive group of degree 2^3 and has no regular subgroup.

- $p^3 \in \mathcal{NR}$ (For $p = 2$).
  Let

$$G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^4 = 1, [a, b] = [b, c] = [c, a] = 1, a^d = ab, b^d = bc, c^d = c \rangle.$$

Then $G \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$ has order 2^5. Let $H = \langle b, d^2 \rangle$ and φ be the transitive permutation representation of G acting on the coset space $[G : H]$.
Then $\varphi(G)$ is a transitive group of degree 2^3 and has no regular subgroup.

Determine the set $\mathcal{NR}$

- Let $p < q$ be two primes. Then $pq \in \mathcal{NR}$.

- Let $p < q$ be two primes. Then $pq \in \mathcal{NR}$.

- Let $p < q$ be two primes. Then $pq \in \mathcal{NR}$.

Theorem

Let n be a positive integer greater than 1. Then $n \in \mathcal{NR}$ unless $n = p$ or p^2 for a prime p.

- Let $p < q$ be two primes. Then $pq \in \mathcal{NR}$.

Theorem

Let n be a positive integer greater than 1. Then $n \in \mathcal{NR}$ unless $n = p$ or p^2 for a prime p.

Let

$$\mathcal{N}_2\mathcal{R} = \{n \in \mathbb{N} \mid \text{there is a 2-closed transitive group of degree } n \text{ without a regular subgroup}\}$$

$$\mathcal{ND} = \{n \in \mathbb{N} \mid \text{there is a vertex-transitive digraph of order } n \text{ which is non-Cayley}\}$$

- Question 1: Is $\mathcal{N}_2\mathcal{R} = \mathcal{NC}$?

Let

$$\mathcal{N}_2\mathcal{R} = \{n \in \mathbb{N} \mid \text{there is a 2-closed transitive group of degree } n \text{ without a regular subgroup}\}$$

$$\mathcal{ND} = \{n \in \mathbb{N} \mid \text{there is a vertex-transitive digraph of order } n \text{ which is non-Cayley}\}$$

- Question 1: Is $\mathcal{N}_2\mathcal{R} = \mathcal{NC}$?
- Question 2: Is $\mathcal{ND} = \mathcal{NC}$?

Let

$$\begin{aligned}\mathcal{N}_2\mathcal{R} &= \Big\{n \in \mathbb{N} \;\Big|\; \text{there is a 2-closed transitive group} \\ &\qquad \text{of degree } n \text{ without a regular subgroup}\Big\} \\ \mathcal{ND} &= \Big\{n \in \mathbb{N} \;\Big|\; \text{there is a vertex-transitive digraph} \\ &\qquad \text{of order } n \text{ which is non-Cayley}\Big\}\end{aligned}$$

- Question 1: Is $\mathcal{N}_2\mathcal{R} = \mathcal{NC}$?
- Question 2: Is $\mathcal{ND} = \mathcal{NC}$?

Let

$$\mathcal{PNR} = \{n \in \mathbb{N} \mid \text{there is a primitive group of degree } n \text{ without a regular subgroup}\}$$

- Determine the set $\mathcal{PNR}$.

Let

$$\mathcal{PNR} = \{n \in \mathbb{N} \mid \text{there is a primitive group of degree } n \text{ without a regular subgroup}\}$$

- Determine the set $\mathcal{PNR}$.
- Note: Different from the set $\mathcal{NR}$, we know that $p^n \notin \mathcal{PNR}$ for any prime p and any positive integer n. Thus, determining the set $\mathcal{PNR}$ should be much harder than $\mathcal{NR}$.

Let

$$\mathcal{PNR} = \Big\{n \in \mathbb{N} \;\Big|\; \text{there is a primitive group of degree } n \text{ without a regular subgroup}\Big\}$$

- Determine the set $\mathcal{PNR}$.
- Note: Different from the set $\mathcal{NR}$, we know that $p^n \notin \mathcal{PNR}$ for any prime p and any positive integer n. Thus, determining the set $\mathcal{PNR}$ should be much harder than $\mathcal{NR}$.