1. In general, we can't say anything about $f(3,1)$! $\lim_{(x,y)\to(3,1)} f(x,y) = 6$ means that the values of $f(x,y)$ approach 6 as $(x,y)$ approaches, but is not equal to, $(3,1)$. If $f$ is continuous, we know that $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, so $\lim_{(x,y)\to(3,1)} f(x,y) = f(3,1) = 6$.

2. (a) The outdoor temperature as a function of longitude, latitude, and time is continuous. Small changes in longitude, latitude, or time can produce only small changes in temperature, as the temperature doesn't jump abruptly from one value to another.

   (b) Elevation is not necessarily continuous. If we think of a cliff with a sudden drop-off, a very small change in longitude or latitude can produce a comparatively large change in elevation, without all the intermediate values being attained. Elevation *can* jump from one value to another.

   (c) The cost of a taxi ride is usually discontinuous. The cost normally increases in jumps, so small changes in distance traveled or time can produce a jump in cost. A graph of the function would show breaks in the surface.

3. We make a table of values of $f(x,y) = \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$ for a set of $(x,y)$ points near the origin.

| $x$ \ $y$ | $-0.2$ | $-0.1$ | $-0.05$ | $0$ | $0.05$ | $0.1$ | $0.2$ |
|---|---|---|---|---|---|---|---|
| $-0.2$ | $-2.551$ | $-2.525$ | $-2.513$ | $-2.500$ | $-2.488$ | $-2.475$ | $-2.451$ |
| $-0.1$ | $-2.525$ | $-2.513$ | $-2.506$ | $-2.500$ | $-2.494$ | $-2.488$ | $-2.475$ |
| $-0.05$ | $-2.513$ | $-2.506$ | $-2.503$ | $-2.500$ | $-2.497$ | $-2.494$ | $-2.488$ |
| $0$ | $-2.500$ | $-2.500$ | $-2.500$ | | $-2.500$ | $-2.500$ | $-2.500$ |
| $0.05$ | $-2.488$ | $-2.494$ | $-2.497$ | $-2.500$ | $-2.503$ | $-2.506$ | $-2.513$ |
| $0.1$ | $-2.475$ | $-2.488$ | $-2.494$ | $-2.500$ | $-2.506$ | $-2.513$ | $-2.525$ |
| $0.2$ | $-2.451$ | $-2.475$ | $-2.488$ | $-2.500$ | $-2.513$ | $-2.525$ | $-2.551$ |

As the table shows, the values of $f(x,y)$ seem to approach $-2.5$ as $(x,y)$ approaches the origin from a variety of different directions. This suggests that $\lim_{(x,y)\to(0,0)} f(x,y) = -2.5$. Since $f$ is a rational function, it is continuous on its domain. $f$ is defined at $(0,0)$, so we can use direct substitution to establish that $\lim_{(x,y)\to(0,0)} f(x,y) = \dfrac{0^2 0^3 + 0^3 0^2 - 5}{2 - 0 \cdot 0} = -\dfrac{5}{2}$, verifying our guess.

4. We make a table of values of $f(x,y) = \dfrac{2xy}{x^2+2y^2}$ for a set of $(x,y)$ points near the origin.

| $x$ \ $y$ | $-0.3$ | $-0.2$ | $-0.1$ | $0$ | $0.1$ | $0.2$ | $0.3$ |
|---|---|---|---|---|---|---|---|
| $-0.3$ | 0.667 | 0.706 | 0.545 | 0.000 | $-0.545$ | $-0.706$ | $-0.667$ |
| $-0.2$ | 0.545 | 0.667 | 0.667 | 0.000 | $-0.667$ | $-0.667$ | $-0.545$ |
| $-0.1$ | 0.316 | 0.444 | 0.667 | 0.000 | $-0.667$ | $-0.444$ | $-0.316$ |
| $0$ | 0.000 | 0.000 | 0.000 |  | 0.000 | 0.000 | 0.000 |
| $0.1$ | $-0.316$ | $-0.444$ | $-0.667$ | 0.000 | 0.667 | 0.444 | 0.316 |
| $0.2$ | $-0.545$ | $-0.667$ | $-0.667$ | 0.000 | 0.667 | 0.667 | 0.545 |
| $0.3$ | $-0.667$ | $-0.706$ | $-0.545$ | 0.000 | 0.545 | 0.706 | 0.667 |

It appears from the table that the values of $f(x,y)$ are not approaching a single value as $(x,y)$ approaches the origin. For verification, if we first approach $(0,0)$ along the $x$-axis, we have $f(x,0)=0$, so $f(x,y)\to 0$. But if we approach $(0,0)$ along the line $y=x$, $f(x,x) = \dfrac{2x^2}{x^2+2x^2} = \dfrac{2}{3}$ $(x \neq 0)$, so $f(x,y)\to \frac{2}{3}$. Since $f$ approaches different values along different paths to the origin, this limit does not exist.

5. $f(x,y) = 5x^3 - x^2y^2$ is a polynomial, and hence continuous, so $\lim\limits_{(x,y)\to(1,2)} f(x,y) = f(1,2) = 5(1)^3 - (1)^2(2)^2 = 1$.

6. $-xy$ is a polynomial and therefore continuous. Since $e^t$ is a continuous function, the composition $e^{-xy}$ is also continuous. Similarly, $x+y$ is a polynomial and $\cos t$ is a continuous function, so the composition $\cos(x+y)$ is continuous. The product of continuous functions is continuous, so $f(x,y) = e^{-xy}\cos(x+y)$ is a continuous function and $\lim\limits_{(x,y)\to(1,-1)} f(x,y) = f(1,-1) = e^{-(1)(-1)}\cos(1+(-1)) = e^1\cos 0 = e$.

7. $f(x,y) = \dfrac{4-xy}{x^2+3y^2}$ is a rational function and hence continuous on its domain.

$(2,1)$ is in the domain of $f$, so $f$ is continuous there and $\lim\limits_{(x,y)\to(2,1)} f(x,y) = f(2,1) = \dfrac{4-(2)(1)}{(2)^2+3(1)^2} = \dfrac{2}{7}$.

8. $\dfrac{1+y^2}{x^2+xy}$ is a rational function and hence continuous on its domain, which includes $(1,0)$. $\ln t$ is a continuous function for $t>0$, so the composition $f(x,y) = \ln\left(\dfrac{1+y^2}{x^2+xy}\right)$ is continuous wherever $\dfrac{1+y^2}{x^2+xy} > 0$. In particular, $f$ is continuous at $(1,0)$ and so $\lim\limits_{(x,y)\to(1,0)} f(x,y) = f(1,0) = \ln\left(\dfrac{1+0^2}{1^2+1\cdot 0}\right) = \ln\dfrac{1}{1} = 0$.

9. $f(x,y) = y^4/(x^4+3y^4)$. First approach $(0,0)$ along the $x$-axis. Then $f(x,0) = 0/x^4 = 0$ for $x\neq 0$, so $f(x,y)\to 0$. Now approach $(0,0)$ along the $y$-axis. Then for $y\neq 0$, $f(0,y) = y^4/3y^4 = 1/3$, so $f(x,y)\to 1/3$. Since $f$ has two different limits along two different lines, the limit does not exist.

**10.** $f(x,y) = (x^2 + \sin^2 y)/(2x^2 + y^2)$. First approach $(0,0)$ along the $x$-axis. Then $f(x,0) = x^2/2x^2 = \frac{1}{2}$ for $x \neq 0$, so $f(x,y) \to \frac{1}{2}$. Next approach $(0,0)$ along the $y$-axis. For $y \neq 0$, $f(0,y) = \dfrac{\sin^2 y}{y^2} = \left(\dfrac{\sin y}{y}\right)^2$ and $\lim\limits_{y\to 0} \dfrac{\sin y}{y} = 1$, so $f(x,y) \to 1$. Since $f$ has two different limits along two different lines, the limit does not exist.

**11.** $f(x,y) = (xy\cos y)/(3x^2 + y^2)$. On the $x$-axis, $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the $x$-axis. Approaching $(0,0)$ along the line $y = x$, $f(x,x) = (x^2\cos x)/4x^2 = \frac{1}{4}\cos x$ for $x \neq 0$, so $f(x,y) \to \frac{1}{4}$ along this line. Thus the limit does not exist.

**12.** $f(x,y) = 6x^3y/(2x^4 + y^4)$. On the $x$-axis, $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the $x$-axis. Approaching $(0,0)$ along the line $y = x$ gives $f(x,x) = 6x^4/(3x^4) = 2$ for $x \neq 0$, so along this line $f(x,y) \to 2$ as $(x,y) \to (0,0)$. Thus the limit does not exist.

**13.** $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$. We can see that the limit along any line through $(0,0)$ is 0, as well as along other paths through $(0,0)$ such as $x = y^2$ and $y = x^2$. So we suspect that the limit exists and equals 0; we use the Squeeze Theorem to prove our assertion. $0 \leq \left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| \leq |x|$ since $|y| \leq \sqrt{x^2+y^2}$, and $|x| \to 0$ as $(x,y) \to (0,0)$. So $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

**14.** $f(x,y) = \dfrac{x^4 - y^4}{x^2+y^2} = \dfrac{(x^2+y^2)(x^2-y^2)}{x^2+y^2} = x^2 - y^2$ for $(x,y) \neq (0,0)$. Thus the limit as $(x,y) \to (0,0)$ is 0.

**15.** Let $f(x,y) = \dfrac{x^2ye^y}{x^4+4y^2}$. Then $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the $x$-axis. Approaching $(0,0)$ along the $y$-axis or the line $y = x$ also gives a limit of 0. But $f(x,x^2) = \dfrac{x^2x^2e^{x^2}}{x^4+4(x^2)^2} = \dfrac{x^4e^{x^2}}{5x^4} = \dfrac{e^{x^2}}{5}$ for $x \neq 0$, so $f(x,y) \to e^0/5 = \frac{1}{5}$ as $(x,y) \to (0,0)$ along the parabola $y = x^2$. Thus the limit doesn't exist.

**16.** We can use the Squeeze Theorem to show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2\sin^2 y}{x^2+2y^2} = 0$:

$0 \leq \dfrac{x^2\sin^2 y}{x^2+2y^2} \leq \sin^2 y$ since $\dfrac{x^2}{x^2+2y^2} \leq 1$, and $\sin^2 y \to 0$ as $(x,y) \to (0,0)$, so $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2\sin^2 y}{x^2+2y^2} = 0$.

**17.**

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \cdot \frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \left(\sqrt{x^2+y^2+1}+1\right) = 2$$

**18.** $f(x,y) = xy^4/(x^2+y^8)$. On the $x$-axis, $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the $x$-axis. Approaching $(0,0)$ along the curve $x = y^4$ gives $f(y^4,y) = y^8/2y^8 = \frac{1}{2}$ for $y \neq 0$, so along this path $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$. Thus the limit does not exist.

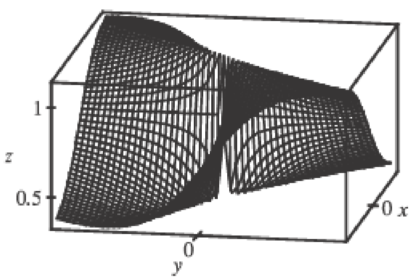
**19.** $e^{-xy}$ and $\sin(\pi z/2)$ are each compositions of continuous functions, and hence continuous, so their product $f(x,y,z) = e^{-xy}\sin(\pi z/2)$ is a continuous function. Then $\lim\limits_{(x,y,z)\to(3,0,1)} f(x,y,z) = f(3,0,1) = e^{-(3)(0)}\sin(\pi\cdot 1/2) = 1$.

**20.** $f(x,y,z) = \dfrac{x^2+2y^2+3z^2}{x^2+y^2+z^2}$. Then $f(x,0,0) = \dfrac{x^2+0+0}{x^2+0+0} = 1$ for $x \neq 0$, so $f(x,y,z)\to 1$ as $(x,y,z)\to(0,0,0)$ along the $x$-axis. But $f(0,y,0) = \dfrac{0+2y^2+0}{0+y^2+0} = 2$ for $y\neq 0$, so $f(x,y,z)\to 2$ as $(x,y,z)\to(0,0,0)$ along the $y$-axis. Thus, the limit doesn't exist.

**21.** $f(x,y,z) = \dfrac{xy+yz^2+xz^2}{x^2+y^2+z^4}$. Then $f(x,0,0) = 0/x^2 = 0$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the $x$-axis, $f(x,y,z)\to 0$. But $f(x,x,0) = x^2/(2x^2) = \frac{1}{2}$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the line $y=x$, $z=0$, $f(x,y,z)\to\frac{1}{2}$. Thus the limit doesn't exist.

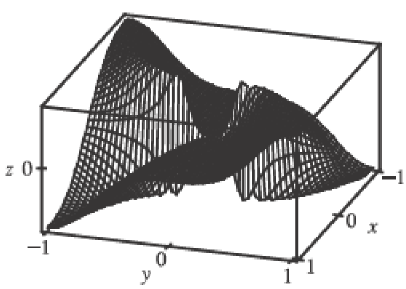
**22.** $f(x,y,z) = \dfrac{yz}{x^2+4y^2+9z^2}$. Then $f(x,0,0) = 0$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the $x$-axis, $f(x,y,z)\to 0$. But $f(0,y,y) = y^2/(13y^2) = \frac{1}{13}$ for $y\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the line $z=y$, $x=0$, $f(x,y,z)\to\frac{1}{13}$. Thus the limit doesn't exist.

**23.**




From the ridges on the graph, we see that as $(x,y)\to(0,0)$ along the lines under the two ridges, $f(x,y)$ approaches different values. So the limit does not exist.

**24.**

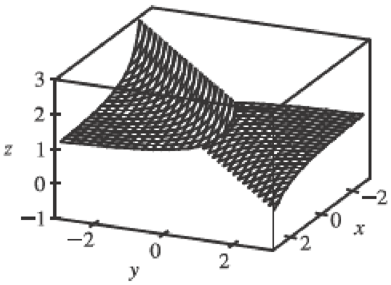



From the graph, it appears that as we approach the origin along the lines $x=0$ or $y=0$, the function is everywhere 0, whereas if we approach the origin along a certain curve it has a constant value of about $\frac{1}{2}$. [In fact, $f(y^3,y) = y^6/(2y^6) = \frac{1}{2}$ for $y\neq 0$, so $f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,0)$ along the curve $x=y^3$.] Since the function approaches different values depending on the path of approach, the limit does not exist.

**25.** $h(x,y) = g(f(x,y)) = (2x+3y-6)^2 + \sqrt{2x+3y-6}$. Since $f$ is a polynomial, it is continuous on $\mathbb{R}^2$ and $g$ is continuous on its domain $\{t \mid t\geq 0\}$. Thus $h$ is continuous on its domain. $D = \{(x,y)\mid 2x+3y-6\geq 0\} = \left\{(x,y)\mid y\geq -\frac{2}{3}x+2\right\}$, which consists of all points on or above the line $y = -\frac{2}{3}x+2$.

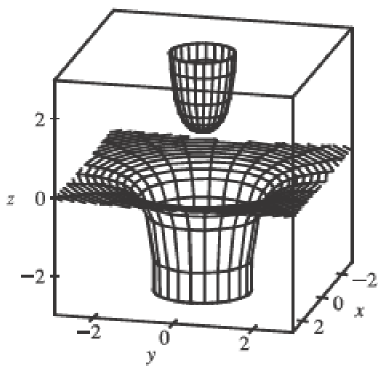
**26.** $h(x,y) = g(f(x,y)) = \dfrac{1-xy}{1+x^2y^2} + \ln\left(\dfrac{1-xy}{1+x^2y^2}\right)$. $f$ is a rational function, so it is continuous on its domain. Because $1 + x^2y^2 > 0$, the domain of $f$ is $\mathbb{R}^2$, so $f$ is continuous everywhere. $g$ is continuous on its domain $\{t \mid t > 0\}$. Thus $h$ is continuous on its domain $\left\{(x,y) \;\middle|\; \dfrac{1-xy}{1+x^2y^2} > 0\right\} = \{(x,y) \mid xy < 1\}$ which consists of all points between (but not on) the two branches of the hyperbola $y = 1/x$.

**27.**




From the graph, it appears that $f$ is discontinuous along the line $y = x$. If we consider $f(x,y) = e^{1/(x-y)}$ as a composition of functions, $g(x,y) = 1/(x-y)$ is a rational function and therefore continuous except where $x - y = 0 \;\Rightarrow\; y = x$. Since the function $h(t) = e^t$ is continuous everywhere, the composition $h(g(x,y)) = e^{1/(x-y)} = f(x,y)$ is continuous except along the line $y = x$, as we suspected.

**28.**




We can see a circular break in the graph, corresponding approximately to the unit circle, where $f$ is discontinuous. Since $f(x,y) = \dfrac{1}{1-x^2-y^2}$ is a rational function, it is continuous except where $1 - x^2 - y^2 = 0 \;\Rightarrow\; x^2 + y^2 = 1$, confirming our observation that $f$ is discontinuous on the circle $x^2 + y^2 = 1$.

**29.** The functions $\sin(xy)$ and $e^x - y^2$ are continuous everywhere, so $F(x,y) = \dfrac{\sin(xy)}{e^x - y^2}$ is continuous except where $e^x - y^2 = 0 \;\Rightarrow\; y^2 = e^x \;\Rightarrow\; y = \pm\sqrt{e^x} = \pm e^{\frac{1}{2}x}$. Thus $F$ is continuous on its domain $\{(x,y) \mid y \neq \pm e^{x/2}\}$.

**30.** $F(x,y) = \dfrac{x-y}{1+x^2+y^2}$ is a rational function and thus is continuous on its domain $\mathbb{R}^2$ (since the denominator is never zero).

**31.** $F(x,y) = \arctan\left(x + \sqrt{y}\right) = g(f(x,y))$ where $f(x,y) = x + \sqrt{y}$, continuous on its domain $\{(x,y) \mid y \geq 0\}$, and $g(t) = \arctan t$ is continuous everywhere. Thus $F$ is continuous on its domain $\{(x,y) \mid y \geq 0\}$.

**32.** $e^{x^2y}$ is continuous on $\mathbb{R}^2$ and $\sqrt{x+y^2}$ is continuous on its domain $\{(x,y) \mid x + y^2 \geq 0\} = \{(x,y) \mid x \geq -y^2\}$, so $F(x,y) = e^{x^2y} + \sqrt{x+y^2}$ is continuous on the set $\{(x,y) \mid x \geq -y^2\}$.

**33.** $G(x,y) = \ln\left(x^2 + y^2 - 4\right) = g(f(x,y))$ where $f(x,y) = x^2 + y^2 - 4$, continuous on $\mathbb{R}^2$, and $g(t) = \ln t$, continuous on its domain $\{t \mid t > 0\}$. Thus $G$ is continuous on its domain $\{(x,y) \mid x^2 + y^2 - 4 > 0\} = \{(x,y) \mid x^2 + y^2 > 4\}$, the exterior of the circle $x^2 + y^2 = 4$.

**34.** $G(x,y) = g(f(x,y))$ where $f(x,y) = (x+y)^{-2}$, a rational function that is continuous on $\mathbb{R}^2$ except where $x + y = 0$, and $g(t) = \tan^{-1} t$, continuous everywhere. Thus $G$ is continuous on its domain $\{(x,y) \mid x + y \neq 0\} = \{(x,y) \mid y \neq -x\}$.

**35.** $\sqrt{y}$ is continuous on its domain $\{y \mid y \geq 0\}$ and $x^2 - y^2 + z^2$ is continuous everywhere, so $f(x,y,z) = \dfrac{\sqrt{y}}{x^2 - y^2 + z^2}$ is continuous for $y \geq 0$ and $x^2 - y^2 + z^2 \neq 0 \quad \Rightarrow \quad y^2 \neq x^2 + z^2$, that is, $\left\{(x,y,z) \mid y \geq 0, y \neq \sqrt{x^2+z^2}\right\}$.

**36.** $f(x,y,z) = \sqrt{x+y+z} = h(g(x,y,z))$ where $g(x,y,z) = x + y + z$, continuous everywhere, and $h(t) = \sqrt{t}$ is continuous on its domain $\{t \mid t \geq 0\}$. Thus $f$ is continuous on its domain $\{(x,y,z) \mid x + y + z \geq 0\}$, so $f$ is continuous on and above the plane $z = -x - y$.

**37.** $f(x,y) = \begin{cases} \dfrac{x^2 y^3}{2x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ The first piece of $f$ is a rational function defined everywhere except at the origin, so $f$ is continuous on $\mathbb{R}^2$ except possibly at the origin. Since $x^2 \leq 2x^2 + y^2$, we have $\left|x^2y^3/(2x^2+y^2)\right| \leq \left|y^3\right|$. We know that $\left|y^3\right| \to 0$ as $(x,y) \to (0,0)$. So, by the Squeeze Theorem, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2} = 0$.
But $f(0,0) = 1$, so $f$ is discontinuous at $(0,0)$. Therefore, $f$ is continuous on the set $\{(x,y) \mid (x,y) \neq (0,0)\}$.

**38.** $f(x,y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ The first piece of $f$ is a rational function defined everywhere except at the origin, so $f$ is continuous on $\mathbb{R}^2$ except possibly at the origin. $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the $x$-axis. But $f(x,x) = x^2/(3x^2) = \frac{1}{3}$ for $x \neq 0$, so $f(x,y) \to \frac{1}{3}$ as $(x,y) \to (0,0)$ along the line $y = x$. Thus $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so $f$ is not continuous at $(0,0)$ and the largest set on which $f$ is continuous is $\{(x,y) \mid (x,y) \neq (0,0)\}$.

**39.** $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)^3 + (r\sin\theta)^3}{r^2} = \lim_{r\to 0^+} \left(r\cos^3\theta + r\sin^3\theta\right) = 0$$

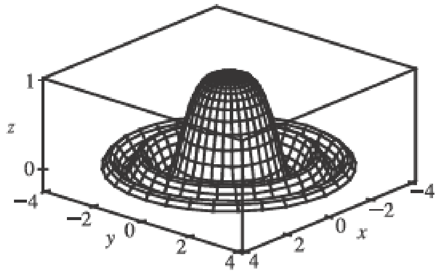
**40.** $$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim_{r\to 0^+} r^2 \ln r^2 = \lim_{r\to 0^+} \frac{\ln r^2}{1/r^2} = \lim_{r\to 0^+} \frac{(1/r^2)(2r)}{-2/r^3} \quad \text{[using l'Hospital's Rule]}$$
$$= \lim_{r\to 0^+} (-r^2) = 0$$

**41.** $$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2} - 1}{x^2+y^2} = \lim_{r\to 0^+} \frac{e^{-r^2} - 1}{r^2} = \lim_{r\to 0^+} \frac{e^{-r^2}(-2r)}{2r} \quad \text{[using l'Hospital's Rule]}$$
$$= \lim_{r\to 0^+} -e^{-r^2} = -e^0 = -1$$

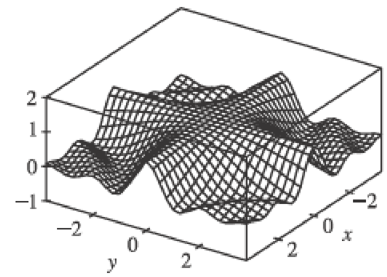
**42.** $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0^+} \dfrac{\sin(r^2)}{r^2}$, which is an indeterminate form of type $0/0$. Using l'Hospital's Rule, we get

$$\lim_{r\to 0^+} \frac{\sin(r^2)}{r^2} \overset{\text{H}}{=} \lim_{r\to 0^+} \frac{2r\cos(r^2)}{2r} = \lim_{r\to 0^+} \cos(r^2) = 1.$$

*Or:* Use the fact that $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.




**43.** $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{xy} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$




From the graph, it appears that $f$ is continuous everywhere. We know $xy$ is continuous on $\mathbb{R}^2$ and $\sin t$ is continuous everywhere, so $\sin(xy)$ is continuous on $\mathbb{R}^2$ and $\dfrac{\sin(xy)}{xy}$ is continuous on $\mathbb{R}^2$ except possibly where $xy = 0$. To show that $f$ is continuous at those points, consider any point $(a,b)$ in $\mathbb{R}^2$ where $ab = 0$. Because $xy$ is continuous, $xy \to ab = 0$ as $(x,y) \to (a,b)$. If we let $t = xy$, then $t \to 0$ as $(x,y) \to (a,b)$ and $\lim_{(x,y)\to(a,b)} \dfrac{\sin(xy)}{xy} = \lim_{t\to 0} \dfrac{\sin(t)}{t} = 1$ by Equation 3.4.2 [ET 3.3.2]. Thus $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ and $f$ is continuous on $\mathbb{R}^2$.

**44.** (a) $f(x,y) = \begin{cases} 0 & \text{if } y \le 0 \quad \text{or} \quad y \ge x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$ Consider the path $y = mx^a$, $0 < a < 4$. [The path does not pass through $(0,0)$ if $a \le 0$ except for the trivial case where $m = 0$.] If $mx^a \le 0$ then $f(x, mx^a) = 0$. If $mx^a > 0$ then $mx^a = |mx^a| = |m|\,|x^a|$ and $mx^a \ge x^4 \;\Leftrightarrow\; |m|\,|x^a| \ge x^4 \;\Leftrightarrow\; \dfrac{x^4}{|x^a|} \le |m| \;\Leftrightarrow\; |x|^{4-a} \le |m|$ whenever $x^a$ is defined. Then $mx^a \ge x^4 \;\Leftrightarrow\; |x| \le |m|^{1/(4-a)}$ so $f(x, mx^a) = 0$ for $|x| \le |m|^{1/(4-a)}$ and $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along this path.

(b) If we approach $(0,0)$ along the path $y = x^5$, $x > 0$ then we have $f(x, x^5) = 1$ for $0 < x < 1$ because $0 < x^5 < x^4$ there. Thus $f(x,y) \to 1$ as $(x,y) \to (0,0)$ along this path, but in part (a) we found a limit of 0 along other paths, so $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist and $f$ is discontinuous at $(0,0)$.

(c) First we show that $f$ is discontinuous at any point $(a,0)$ on the $x$-axis. If we approach $(a,0)$ along the path $x = a$, $y > 0$ then $f(a,y) = 1$ for $0 < y < a^4$, so $f(x,y) \to 1$ as $(x,y) \to (a,0)$ along this path. If we approach $(a,0)$ along the path $x = a$, $y < 0$ then $f(a,y) = 0$ since $y < 0$ and $f(x,y) \to 0$ as $(x,y) \to (a,0)$. Thus the limit does not exist and $f$ is discontinuous on the line $y = 0$. $f$ is also discontinuous on the curve $y = x^4$: For any point $(a, a^4)$ on this curve, approaching the point along the path $x = a$, $y > a^4$ gives $f(a,y) = 0$ since $y > a^4$, so $f(x,y) \to 0$ as $(x,y) \to (a,a^4)$. But approaching the point along the path $x = a$, $y < a^4$ gives $f(a,y) = 1$ for $y > 0$, so $f(x,y) \to 1$ as $(x,y) \to (a,a^4)$ and the limit does not exist there.

**45.** Since $|\mathbf{x} - \mathbf{a}|^2 = |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2\,|\mathbf{x}|\,|\mathbf{a}|\cos\theta \ge |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2\,|\mathbf{x}|\,|\mathbf{a}| = (|\mathbf{x}| - |\mathbf{a}|)^2$, we have $\big||\mathbf{x}| - |\mathbf{a}|\big| \le |\mathbf{x} - \mathbf{a}|$. Let $\epsilon > 0$ be given and set $\delta = \epsilon$. Then if $0 < |\mathbf{x} - \mathbf{a}| < \delta$, $\big||\mathbf{x}| - |\mathbf{a}|\big| \le |\mathbf{x} - \mathbf{a}| < \delta = \epsilon$. Hence $\lim_{\mathbf{x}\to\mathbf{a}} |\mathbf{x}| = |\mathbf{a}|$ and $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$.

**46.** Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that if $0 < |\mathbf{x} - \mathbf{a}| < \delta$ then $|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c}\cdot\mathbf{x} - \mathbf{c}\cdot\mathbf{a}| < \epsilon$. But $|\mathbf{c}\cdot\mathbf{x} - \mathbf{c}\cdot\mathbf{a}| = |\mathbf{c}\cdot(\mathbf{x} - \mathbf{a})|$ and $|\mathbf{c}\cdot(\mathbf{x} - \mathbf{a})| \le |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}|$ by Exercise 13.3.57 [ET 12.3.57] (the Cauchy-Schwartz Inequality). Set $\delta = \epsilon/|\mathbf{c}|$. Then if $0 < |\mathbf{x} - \mathbf{a}| < \delta$,
$|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c}\cdot\mathbf{x} - \mathbf{c}\cdot\mathbf{a}| \le |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}| < |\mathbf{c}|\,\delta = |\mathbf{c}|\,(\epsilon/|\mathbf{c}|) = \epsilon$. So $f$ is continuous on $\mathbb{R}^n$.

**2.** By Definition 4, $f_T(92, 60) = \lim_{h \to 0} \dfrac{f(92 + h, 60) - f(92, 60)}{h}$, which we can approximate by considering $h = 2$ and $h = -2$ and using the values given in Table 1: $f_T(92, 60) \approx \dfrac{f(94, 60) - f(92, 60)}{2} = \dfrac{111 - 105}{2} = 3$, $f_T(92, 60) \approx \dfrac{f(90, 60) - f(92, 60)}{-2} = \dfrac{100 - 105}{-2} = 2.5$. Averaging these values, we estimate $f_T\,(92, 60)$ to be approximately $2.75$. Thus, when the actual temperature is $92°\text{F}$ and the relative humidity is $60\%$, the apparent temperature rises by about $2.75°\text{F}$ for every degree that the actual temperature rises.

Similarly, $f_H(92, 60) = \lim_{h \to 0} \dfrac{f(92, 60 + h) - f(92, 60)}{h}$ which we can approximate by considering $h = 5$ and $h = -5$: $f_H(92, 60) \approx \dfrac{f(92, 65) - f(92, 60)}{5} = \dfrac{108 - 105}{5} = 0.6$, $f_H(92, 60) \approx \dfrac{f(92, 55) - f(92, 60)}{-5} = \dfrac{103 - 105}{-5} = 0.4$. Averaging these values, we estimate $f_H(92, 60)$ to be approximately $0.5$. Thus, when the actual temperature is $92°\text{F}$ and the relative humidity is $60\%$, the apparent temperature rises by about $0.5°\text{F}$ for every percent that the relative humidity increases.