51. $f(x,y)=x^3y^5+2x^4y \quad\Rightarrow\quad f_x(x,y)=3x^2y^5+8x^3y$, $f_y(x,y)=5x^3y^4+2x^4$. Then $f_{xx}(x,y)=6xy^5+24x^2y$, $f_{xy}(x,y)=15x^2y^4+8x^3$, $f_{yx}(x,y)=15x^2y^4+8x^3$, and $f_{yy}(x,y)=20x^3y^3$.

52. $f(x,y)=\sin^2(mx+ny) \quad\Rightarrow\quad f_x(x,y)=2\sin(mx+ny)\cos(mx+ny)\cdot m=m\sin(2mx+2ny)$ [using the identity $\sin 2\theta=2\sin\theta\cos\theta$], $f_y(x,y)=2\sin(mx+ny)\cos(mx+ny)\cdot n=n\sin(2mx+2ny)$.

Then $f_{xx}(x,y)=m\cos(2mx+2ny)\cdot 2m=2m^2\cos(2mx+2ny)$,

$f_{xy}(x,y)=m\cos(2mx+2ny)\cdot 2n=2mn\cos(2mx+2ny)$,

$f_{yx}(x,y)=n\cos(2mx+2ny)\cdot 2m=2mn\cos(2mx+2ny)$, and

$f_{yy}(x,y)=n\cos(2mx+2ny)\cdot 2n=2n^2\cos(2mx+2ny)$.

53. $w=\sqrt{u^2+v^2} \quad\Rightarrow\quad w_u=\frac{1}{2}(u^2+v^2)^{-1/2}\cdot 2u=\dfrac{u}{\sqrt{u^2+v^2}}$, $w_v=\frac{1}{2}(u^2+v^2)^{-1/2}\cdot 2v=\dfrac{v}{\sqrt{u^2+v^2}}$. Then

$$w_{uu}=\frac{1\cdot\sqrt{u^2+v^2}-u\cdot\frac{1}{2}(u^2+v^2)^{-1/2}(2u)}{\left(\sqrt{u^2+v^2}\right)^2}=\frac{\sqrt{u^2+v^2}-u^2/\sqrt{u^2+v^2}}{u^2+v^2}=\frac{u^2+v^2-u^2}{(u^2+v^2)^{3/2}}=\frac{v^2}{(u^2+v^2)^{3/2}},$$

$$w_{uv}=u\left(-\tfrac{1}{2}\right)\left(u^2+v^2\right)^{-3/2}(2v)=-\frac{uv}{(u^2+v^2)^{3/2}},\ w_{vu}=v\left(-\tfrac{1}{2}\right)\left(u^2+v^2\right)^{-3/2}(2u)=-\frac{uv}{(u^2+v^2)^{3/2}},$$

$$w_{vv}=\frac{1\cdot\sqrt{u^2+v^2}-v\cdot\frac{1}{2}(u^2+v^2)^{-1/2}(2v)}{\left(\sqrt{u^2+v^2}\right)^2}=\frac{\sqrt{u^2+v^2}-v^2/\sqrt{u^2+v^2}}{u^2+v^2}=\frac{u^2+v^2-v^2}{(u^2+v^2)^{3/2}}=\frac{u^2}{(u^2+v^2)^{3/2}}.$$

54. $v=\dfrac{xy}{x-y} \quad\Rightarrow\quad v_x=\dfrac{y(x-y)-xy(1)}{(x-y)^2}=-\dfrac{y^2}{(x-y)^2}$,

$v_y=\dfrac{x(x-y)-xy(-1)}{(x-y)^2}=\dfrac{x^2}{(x-y)^2}$. Then $v_{xx}=-y^2(-2)(x-y)^{-3}(1)=\dfrac{2y^2}{(x-y)^3}$,

$$v_{xy}=-\frac{2y(x-y)^2-y^2\cdot 2(x-y)(-1)}{[(x-y)^2]^2}=-\frac{2y(x-y)+2y^2}{(x-y)^3}=-\frac{2xy}{(x-y)^3},$$

$$v_{yx}=\frac{2x(x-y)^2-x^2\cdot 2(x-y)(1)}{[(x-y)^2]^2}=\frac{2x(x-y)-2x^2}{(x-y)^3}=-\frac{2xy}{(x-y)^3},\quad v_{yy}=x^2(-2)(x-y)^{-3}(-1)=\frac{2x^2}{(x-y)^3}.$$

55. $z=\arctan\dfrac{x+y}{1-xy} \quad\Rightarrow$

$$z_x=\frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2}\cdot\frac{(1)(1-xy)-(x+y)(-y)}{(1-xy)^2}=\frac{1+y^2}{(1-xy)^2+(x+y)^2}=\frac{1+y^2}{1+x^2+y^2+x^2y^2}$$

$$=\frac{1+y^2}{(1+x^2)(1+y^2)}=\frac{1}{1+x^2},$$

$$z_y=\frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2}\cdot\frac{(1)(1-xy)-(x+y)(-x)}{(1-xy)^2}=\frac{1+x^2}{(1-xy)^2+(x+y)^2}=\frac{1+x^2}{(1+x^2)(1+y^2)}=\frac{1}{1+y^2}.$$

Then $z_{xx}=-(1+x^2)^{-2}\cdot 2x=-\dfrac{2x}{(1+x^2)^2}$, $z_{xy}=0$, $z_{yx}=0$, $z_{yy}=-(1+y^2)^{-2}\cdot 2y=-\dfrac{2y}{(1+y^2)^2}$.

56. $v = e^{xe^y} \Rightarrow v_x = e^{xe^y} \cdot e^y = e^{y+xe^y}$, $v_y = e^{xe^y} \cdot xe^y = xe^{y+xe^y}$. Then $v_{xx} = e^{y+xe^y} \cdot e^y = e^{2y+xe^y}$,

$v_{xy} = e^{y+xe^y}(1+xe^y)$, $v_{yx} = xe^{y+xe^y}(e^y) + e^{y+xe^y}(1) = e^{y+xe^y}(1+xe^y)$,

$v_{yy} = xe^{y+xe^y}(1+xe^y) = e^{y+xe^y}(x+x^2e^y)$.

57. $u = x\sin(x+2y) \Rightarrow u_x = x \cdot \cos(x+2y)(1) + \sin(x+2y) \cdot 1 = x\cos(x+2y) + \sin(x+2y)$,

$u_{xy} = x(-\sin(x+2y)(2)) + \cos(x+2y)(2) = 2\cos(x+2y) - 2x\sin(x+2y)$,

$u_y = x\cos(x+2y)(2) = 2x\cos(x+2y)$,

$u_{yx} = 2x \cdot (-\sin(x+2y)(1)) + \cos(x+2y) \cdot 2 = 2\cos(x+2y) - 2x\sin(x+2y)$. Thus $u_{xy} = u_{yx}$.

58. $u = x^4y^2 - 2xy^5 \Rightarrow u_x = 4x^3y^2 - 2y^5$, $u_{xy} = 8x^3y - 10y^4$ and $u_y = 2x^4y - 10xy^4$, $u_{yx} = 8x^3y - 10y^4$.

Thus $u_{xy} = u_{yx}$.

59. $u = \ln\sqrt{x^2+y^2} = \ln(x^2+y^2)^{1/2} = \frac{1}{2}\ln(x^2+y^2) \Rightarrow u_x = \dfrac{1}{2}\dfrac{1}{x^2+y^2} \cdot 2x = \dfrac{x}{x^2+y^2}$,

$u_{xy} = x(-1)(x^2+y^2)^{-2}(2y) = -\dfrac{2xy}{(x^2+y^2)^2}$ and $u_y = \dfrac{1}{2}\dfrac{1}{x^2+y^2} \cdot 2y = \dfrac{y}{x^2+y^2}$,

$u_{yx} = y(-1)(x^2+y^2)^{-2}(2x) = -\dfrac{2xy}{(x^2+y^2)^2}$. Thus $u_{xy} = u_{yx}$.

60. $u = xye^y \Rightarrow u_x = ye^y$, $u_{xy} = ye^y + e^y = (y+1)e^y$ and $u_y = x(ye^y + e^y) = x(y+1)e^y$, $u_{yx} = (y+1)e^y$.

Thus $u_{xy} = u_{yx}$.

61. $f(x,y) = 3xy^4 + x^3y^2 \Rightarrow f_x = 3y^4 + 3x^2y^2$, $f_{xx} = 6xy^2$, $f_{xxy} = 12xy$ and

$f_y = 12xy^3 + 2x^3y$, $f_{yy} = 36xy^2 + 2x^3$, $f_{yyy} = 72xy$.

62. $f(x,t) = x^2e^{-ct} \Rightarrow f_t = x^2(-ce^{-ct})$, $f_{tt} = x^2(c^2e^{-ct})$, $f_{ttt} = x^2(-c^3e^{-ct}) = -c^3x^2e^{-ct}$ and

$f_{tx} = 2x(-ce^{-ct})$, $f_{txx} = 2(-ce^{-ct}) = -2ce^{-ct}$.

63. $f(x,y,z) = \cos(4x+3y+2z) \Rightarrow$

$f_x = -\sin(4x+3y+2z)(4) = -4\sin(4x+3y+2z)$, $f_{xy} = -4\cos(4x+3y+2z)(3) = -12\cos(4x+3y+2z)$,

$f_{xyz} = -12(-\sin(4x+3y+2z))(2) = 24\sin(4x+3y+2z)$ and

$f_y = -\sin(4x+3y+2z)(3) = -3\sin(4x+3y+2z)$,

$f_{yz} = -3\cos(4x+3y+2z)(2) = -6\cos(4x+3y+2z)$, $f_{yzz} = -6(-\sin(4x+3y+2z))(2) = 12\sin(4x+3y+2z)$.

64. $f(r,s,t) = r\ln(rs^2t^3) \Rightarrow f_r = r \cdot \dfrac{1}{rs^2t^3}(s^2t^3) + \ln(rs^2t^3) \cdot 1 = \dfrac{rs^2t^3}{rs^2t^3} + \ln(rs^2t^3) = 1 + \ln(rs^2t^3)$,

$f_{rs} = \dfrac{1}{rs^2t^3}(2rst^3) = \dfrac{2}{s} = 2s^{-1}$, $f_{rss} = -2s^{-2} = -\dfrac{2}{s^2}$ and $f_{rst} = 0$.

65. $u = e^{r\theta}\sin\theta \quad\Rightarrow\quad \dfrac{\partial u}{\partial\theta} = e^{r\theta}\cos\theta + \sin\theta\cdot e^{r\theta}\,(r) = e^{r\theta}\,(\cos\theta + r\sin\theta)$,

$\dfrac{\partial^2 u}{\partial r\,\partial\theta} = e^{r\theta}\,(\sin\theta) + (\cos\theta + r\sin\theta)\,e^{r\theta}\,(\theta) = e^{r\theta}\,(\sin\theta + \theta\cos\theta + r\theta\sin\theta)$,

$\dfrac{\partial^3 u}{\partial r^2\,\partial\theta} = e^{r\theta}\,(\theta\sin\theta) + (\sin\theta + \theta\cos\theta + r\theta\sin\theta)\cdot e^{r\theta}\,\,(\theta) = \theta e^{r\theta}\,(2\sin\theta + \theta\cos\theta + r\theta\sin\theta)$.

66. $z = u\sqrt{v-w} = u(v-w)^{1/2} \quad\Rightarrow\quad \dfrac{\partial z}{\partial w} = u\left[\tfrac{1}{2}(v-w)^{-1/2}(-1)\right] = -\tfrac{1}{2}u(v-w)^{-1/2}$,

$\dfrac{\partial^2 z}{\partial v\,\partial w} = -\tfrac{1}{2}u\left(-\tfrac{1}{2}(v-w)^{-3/2}(1)\right) = \tfrac{1}{4}u(v-w)^{-3/2}$, $\dfrac{\partial^3 z}{\partial u\,\partial v\,\partial w} = \tfrac{1}{4}(v-w)^{-3/2}$.

67. $w = \dfrac{x}{y+2z} = x(y+2z)^{-1} \quad\Rightarrow\quad \dfrac{\partial w}{\partial x} = (y+2z)^{-1}$, $\dfrac{\partial^2 w}{\partial y\,\partial x} = -(y+2z)^{-2}(1) = -(y+2z)^{-2}$,

$\dfrac{\partial^3 w}{\partial z\,\partial y\,\partial x} = -(-2)(y+2z)^{-3}(2) = 4(y+2z)^{-3} = \dfrac{4}{(y+2z)^3}$ and $\dfrac{\partial w}{\partial y} = x(-1)(y+2z)^{-2}(1) = -x(y+2z)^{-2}$,

$\dfrac{\partial^2 w}{\partial x\,\partial y} = -(y+2z)^{-2}$, $\dfrac{\partial^3 w}{\partial x^2\,\partial y} = 0$.

68. $u = x^a y^b z^c$. If $a = 0$, or if $b = 0$ or 1, or if $c = 0, 1$, or 2, then $\dfrac{\partial^6 u}{\partial x\,\partial y^2\,\partial z^3} = 0$. Otherwise $\dfrac{\partial u}{\partial z} = cx^a y^b z^{c-1}$,

$\dfrac{\partial^2 u}{\partial z^2} = c(c-1)x^a y^b z^{c-2}$, $\dfrac{\partial^3 u}{\partial z^3} = c(c-1)(c-2)x^a y^b z^{c-3}$, $\dfrac{\partial^4 u}{\partial y\,\partial z^3} = bc(c-1)(c-2)x^a y^{b-1} z^{c-3}$,

$\dfrac{\partial^5 u}{\partial y^2\,\partial z^3} = b(b-1)c(c-1)(c-2)x^a y^{b-2} z^{c-3}$, and $\dfrac{\partial^6 u}{\partial x\,\partial y^2\,\partial z^3} = ab(b-1)c(c-1)(c-2)x^{a-1} y^{b-2} z^{c-3}$.

69. By Definition 4, $f_x(3,2) = \lim_{h\to 0} \dfrac{f(3+h,2) - f(3,2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$:

$f_x(3,2) \approx \dfrac{f(3.5,2) - f(3,2)}{0.5} = \dfrac{22.4 - 17.5}{0.5} = 9.8$, $f_x(3,2) \approx \dfrac{f(2.5,2) - f(3,2)}{-0.5} = \dfrac{10.2 - 17.5}{-0.5} = 14.6$. Averaging these values, we estimate $f_x(3,2)$ to be approximately 12.2. Similarly, $f_x(3,2.2) = \lim_{h\to 0} \dfrac{f(3+h,2.2) - f(3,2.2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$: $f_x(3,2.2) \approx \dfrac{f(3.5,2.2) - f(3,2.2)}{0.5} = \dfrac{26.1 - 15.9}{0.5} = 20.4$, $f_x(3,2.2) \approx \dfrac{f(2.5,2.2) - f(3,2.2)}{-0.5} = \dfrac{9.3 - 15.9}{-0.5} = 13.2$. Averaging these values, we have $f_x(3,2.2) \approx 16.8$.

To estimate $f_{xy}(3,2)$, we first need an estimate for $f_x(3,1.8)$:

$f_x(3,1.8) \approx \dfrac{f(3.5,1.8) - f(3,1.8)}{0.5} = \dfrac{20.0 - 18.1}{0.5} = 3.8$, $f_x(3,1.8) \approx \dfrac{f(2.5,1.8) - f(3,1.8)}{-0.5} = \dfrac{12.5 - 18.1}{-0.5} = 11.2$.

Averaging these values, we get $f_x(3,1.8) \approx 7.5$. Now $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[f_x(x,y)\right]$ and $f_x(x,y)$ is itself a function of two variables, so Definition 4 says that $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[f_x(x,y)\right] = \lim_{h\to 0} \dfrac{f_x(x,y+h) - f_x(x,y)}{h} \quad \Rightarrow$

$f_{xy}(3,2) = \lim_{h\to 0} \dfrac{f_x(3,2+h) - f_x(3,2)}{h}$.

We can estimate this value using our previous work with $h = 0.2$ and $h = -0.2$:

$f_{xy}(3,2) \approx \dfrac{f_x(3,2.2) - f_x(3,2)}{0.2} = \dfrac{16.8 - 12.2}{0.2} = 23$, $f_{xy}(3,2) \approx \dfrac{f_x(3,1.8) - f_x(3,2)}{-0.2} = \dfrac{7.5 - 12.2}{-0.2} = 23.5$.

Averaging these values, we estimate $f_{xy}(3,2)$ to be approximately 23.25.

70. (a) If we fix y and allow x to vary, the level curves indicate that the value of f decreases as we move through P in the positive x-direction, so f_x is negative at P.

(b) If we fix x and allow y to vary, the level curves indicate that the value of f increases as we move through P in the positive y-direction, so f_y is positive at P.

(c) $f_{xx} = \dfrac{\partial}{\partial x}(f_x)$, so if we fix y and allow x to vary, f_{xx} is the rate of change of f_x as x increases. Note that at points to the right of P the level curves are spaced farther apart (in the x-direction) than at points to the left of P, demonstrating that f decreases less quickly with respect to x to the right of P. So as we move through P in the positive x-direction the (negative) value of f_x increases, hence $\dfrac{\partial}{\partial x}(f_x) = f_{xx}$ is positive at P.

(d) $f_{xy} = \dfrac{\partial}{\partial y}(f_x)$, so if we fix x and allow y to vary, f_{xy} is the rate of change of f_x as y increases. The level curves are closer together (in the x-direction) at points above P than at those below P, demonstrating that f decreases more quickly with respect to x for y-values above P. So as we move through P in the positive y-direction, the (negative) value of f_x decreases, hence f_{xy} is negative.

(e) $f_{yy} = \dfrac{\partial}{\partial y}(f_y)$, so if we fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. The level curves are closer together (in the y-direction) at points above P than at those below P, demonstrating that f increases more quickly with respect to y above P. So as we move through P in the positive y-direction the (positive) value of f_y increases, hence $\dfrac{\partial}{\partial y}(f_y) = f_{yy}$ is positive at P.

71. $u = e^{-\alpha^2 k^2 t} \sin kx \quad \Rightarrow \quad u_x = k e^{-\alpha^2 k^2 t} \cos kx$, $u_{xx} = -k^2 e^{-\alpha^2 k^2 t} \sin kx$, and $u_t = -\alpha^2 k^2 e^{-\alpha^2 k^2 t} \sin kx$.
Thus $\alpha^2 u_{xx} = u_t$.

72. (a) $u = x^2 + y^2 \quad \Rightarrow \quad u_x = 2x,\ u_{xx} = 2;\quad u_y = 2y,\ u_{yy} = 2$. Thus $u_{xx} + u_{yy} \neq 0$ and $u = x^2 + y^2$ does not satisfy Laplace's Equation.

(b) $u = x^2 - y^2$ is a solution: $u_{xx} = 2,\ u_{yy} = -2$ so $u_{xx} + u_{yy} = 0$.

(c) $u = x^3 + 3xy^2$ is not a solution: $u_x = 3x^2 + 3y^2,\ u_{xx} = 6x;\quad u_y = 6xy,\ u_{yy} = 6x$.

(d) $u = \ln \sqrt{x^2 + y^2}$ is a solution: $u_x = \dfrac{1}{\sqrt{x^2 + y^2}} \left(\dfrac{1}{2}\right)(x^2 + y^2)^{-1/2}(2x) = \dfrac{x}{x^2 + y^2}$,

$u_{xx} = \dfrac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$. By symmetry, $u_{yy} = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$, so $u_{xx} + u_{yy} = 0$.

(e) $u = \sin x \cosh y + \cos x \sinh y$ is a solution: $u_x = \cos x \cosh y - \sin x \sinh y$, $u_{xx} = -\sin x \cosh y - \cos x \sinh y$, and $u_y = \sin x \sinh y + \cos x \cosh y$, $u_{yy} = \sin x \cosh y + \cos x \sinh y$.

(f) $u = e^{-x} \cos y - e^{-y} \cos x$ is a solution: $u_x = -e^{-x} \cos y + e^{-y} \sin x,\quad u_{xx} = e^{-x} \cos y + e^{-y} \cos x$, and $u_y = -e^{-x} \sin y + e^{-y} \cos x,\quad u_{yy} = -e^{-x} \cos y - e^{-y} \cos x$.

73. $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \quad\Rightarrow\quad u_x = \left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$ and

$$u_{xx} = -(x^2+y^2+z^2)^{-3/2} - x\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}.$$

By symmetry, $u_{yy} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $u_{zz} = \dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$.

Thus $u_{xx}+u_{yy}+u_{zz} = \dfrac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}} = 0$.

74. (a) $u = \sin(kx)\,\sin(akt) \quad\Rightarrow\quad u_t = ak\sin(kx)\,\cos(akt)$, $u_{tt} = -a^2k^2\sin(kx)\,\sin(akt)$, $u_x = k\cos(kx)\,\sin(akt)$,

$u_{xx} = -k^2\sin(kx)\,\sin(akt)$. Thus $u_{tt} = a^2u_{xx}$.

(b) $u = \dfrac{t}{a^2t^2-x^2} \quad\Rightarrow\quad u_t = \dfrac{(a^2t^2-x^2)-t(2a^2t)}{(a^2t^2-x^2)^2} = -\dfrac{a^2t^2+x^2}{(a^2t^2-x^2)^2}$,

$$u_{tt} = \frac{-2a^2t(a^2t^2-x^2)^2+(a^2t^2-x^2)(2)(a^2t^2-x^2)(2a^2t)}{(a^2t^2-x^2)^4} = \frac{2a^4t^3+6a^2tx^2}{(a^2t^2-x^2)^3},$$

$$u_x = t(-1)(a^2t^2-x^2)^{-2}(2x) = \frac{2tx}{(a^2t^2-x^2)^2},$$

$$u_{xx} = \frac{2t(a^2t^2-x^2)^2-2tx\,(2)(a^2t^2-x^2)(-2x)}{(a^2t^2-x^2)^4} = \frac{2a^2t^3-2tx^2+8tx^2}{(a^2t^2-x^2)^3} = \frac{2a^2t^3+6tx^2}{(a^2t^2-x^2)^3}.$$

Thus $u_{tt} = a^2u_{xx}$.

(c) $u = (x-at)^6+(x+at)^6 \quad\Rightarrow\quad u_t = -6a(x-at)^5+6a(x+at)^5,\; u_{tt} = 30a^2(x-at)^4+30a^2(x+at)^4$,

$u_x = 6(x-at)^5+6(x+at)^5,\; u_{xx} = 30(x-at)^4+30(x+at)^4$. Thus $u_{tt} = a^2u_{xx}$.

(d) $u = \sin(x-at)+\ln(x+at) \quad\Rightarrow\quad u_t = -a\cos(x-at)+\dfrac{a}{x+at},\quad u_{tt} = -a^2\sin(x-at)-\dfrac{a^2}{(x+at)^2}$,

$u_x = \cos(x-at)+\dfrac{1}{x+at},\quad u_{xx} = -\sin(x-at)-\dfrac{1}{(x+at)^2}$. Thus $u_{tt} = a^2u_{xx}$.

75. Let $v = x+at,\;\; w = x-at$. Then $u_t = \dfrac{\partial[f(v)+g(w)]}{\partial t} = \dfrac{df(v)}{dv}\dfrac{\partial v}{\partial t}+\dfrac{dg(w)}{dw}\dfrac{\partial w}{\partial t} = af'(v)-ag'(w)$ and

$u_{tt} = \dfrac{\partial[af'(v)-ag'(w)]}{\partial t} = a[af''(v)+ag''(w)] = a^2[f''(v)+g''(w)]$. Similarly, by using the Chain Rule we have

$u_x = f'(v)+g'(w)$ and $u_{xx} = f''(v)+g''(w)$. Thus $u_{tt} = a^2u_{xx}$.

76. For each i, $i = 1,\ldots,n$, $\partial u/\partial x_i = a_ie^{a_1x_1+a_2x_2+\cdots+a_nx_n}$ and $\partial^2u/\partial x_i^2 = a_i^2e^{a_1x_1+a_2x_2+\cdots+a_nx_n}$.

Then $\dfrac{\partial^2u}{\partial x_1^2}+\dfrac{\partial^2u}{\partial x_2^2}+\cdots+\dfrac{\partial^2u}{\partial x_n^2} = (a_1^2+a_2^2+\cdots+a_n^2)e^{a_1x_1+a_2x_2+\cdots+a_nx_n} = e^{a_1x_1+a_2x_2+\cdots+a_nx_n} = u$

since $a_1^2+a_2^2+\cdots+a_n^2 = 1$.

77. $z = \ln(e^x + e^y) \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = \dfrac{e^x}{e^x + e^y}$ and $\dfrac{\partial z}{\partial y} = \dfrac{e^y}{e^x + e^y}$, so $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = \dfrac{e^x}{e^x + e^y} + \dfrac{e^y}{e^x + e^y} = \dfrac{e^x + e^y}{e^x + e^y} = 1$.

$\dfrac{\partial^2 z}{\partial x^2} = \dfrac{e^x(e^x + e^y) - e^x(e^x)}{(e^x + e^y)^2} = \dfrac{e^{x+y}}{(e^x + e^y)^2}$, $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{0 - e^y(e^x)}{(e^x + e^y)^2} = -\dfrac{e^{x+y}}{(e^x + e^y)^2}$, and

$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{e^y(e^x + e^y) - e^y(e^y)}{(e^x + e^y)^2} = \dfrac{e^{x+y}}{(e^x + e^y)^2}$. Thus

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2 = \frac{e^{x+y}}{(e^x + e^y)^2} \cdot \frac{e^{x+y}}{(e^x + e^y)^2} - \left(-\frac{e^{x+y}}{(e^x + e^y)^2}\right)^2 = \frac{(e^{x+y})^2}{(e^x + e^y)^4} - \frac{(e^{x+y})^2}{(e^x + e^y)^4} = 0$$

78. $P = bL^\alpha K^\beta$, so $\dfrac{\partial P}{\partial L} = \alpha b L^{\alpha-1} K^\beta$ and $\dfrac{\partial P}{\partial K} = \beta b L^\alpha K^{\beta-1}$. Then

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = L(\alpha b L^{\alpha-1} K^\beta) + K(\beta b L^\alpha K^{\beta-1}) = \alpha b L^{1+\alpha-1} K^\beta + \beta b L^\alpha K^{1+\beta-1} = (\alpha+\beta) b L^\alpha K^\beta = (\alpha+\beta)P$$

79. If we fix $K = K_0$, $P(L, K_0)$ is a function of a single variable L, and $\dfrac{dP}{dL} = \alpha\dfrac{P}{L}$ is a separable differential equation. Then

$\dfrac{dP}{P} = \alpha\dfrac{dL}{L} \quad \Rightarrow \quad \displaystyle\int \frac{dP}{P} = \int \alpha\frac{dL}{L} \quad \Rightarrow \quad \ln|P| = \alpha\ln|L| + C(K_0)$, where $C(K_0)$ can depend on K_0. Then

$|P| = e^{\alpha\ln|L| + C(K_0)}$, and since $P > 0$ and $L > 0$, we have $P = e^{\alpha\ln L}e^{C(K_0)} = e^{C(K_0)}e^{\ln L^\alpha} = C_1(K_0)L^\alpha$ where

$C_1(K_0) = e^{C(K_0)}$.

80. (a) $\partial T/\partial x = -60(2x)/(1 + x^2 + y^2)^2$, so at $(2, 1)$, $T_x = -240/(1 + 4 + 1)^2 = -\frac{20}{3}$.

(b) $\partial T/\partial y = -60(2y)/(1 + x^2 + y^2)^2$, so at $(2, 1)$, $T_y = -120/36 = -\frac{10}{3}$. Thus from the point $(2, 1)$ the temperature is decreasing at a rate of $\frac{20}{3}\,^\circ\mathrm{C/m}$ in the x-direction and is decreasing at a rate of $\frac{10}{3}\,^\circ\mathrm{C/m}$ in the y-direction.

81. By the Chain Rule, taking the partial derivative of both sides with respect to R_1 gives

$\dfrac{\partial R^{-1}}{\partial R}\dfrac{\partial R}{\partial R_1} = \dfrac{\partial\left[(1/R_1) + (1/R_2) + (1/R_3)\right]}{\partial R_1}$ or $-R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2}$. Thus $\dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$.

82. $P = \dfrac{mRT}{V}$ so $\dfrac{\partial P}{\partial V} = \dfrac{-mRT}{V^2}$; $\quad V = \dfrac{mRT}{P}$, so $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$; $\quad T = \dfrac{PV}{mR}$, so $\dfrac{\partial T}{\partial P} = \dfrac{V}{mR}$.

Thus $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P} = \dfrac{-mRT}{V^2}\dfrac{mR}{P}\dfrac{V}{mR} = \dfrac{-mRT}{PV} = -1$, since $PV = mRT$.

83. By Exercise 82, $PV = mRT \quad \Rightarrow \quad P = \dfrac{mRT}{V}$, so $\dfrac{\partial P}{\partial T} = \dfrac{mR}{V}$. Also, $PV = mRT \quad \Rightarrow \quad V = \dfrac{mRT}{P}$ and $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$.

Since $T = \dfrac{PV}{mR}$, we have $T\dfrac{\partial P}{\partial T}\dfrac{\partial V}{\partial T} = \dfrac{PV}{mR} \cdot \dfrac{mR}{V} \cdot \dfrac{mR}{P} = mR$.

84. $\dfrac{\partial W}{\partial T} = 0.6215 + 0.3965v^{0.16}$. When $T = -15°\text{C}$ and $v = 30$ km/h, $\dfrac{\partial W}{\partial T} = 0.6215 + 0.3965(30)^{0.16} \approx 1.3048$, so we would expect the apparent temperature to drop by approximately $1.3°\text{C}$ if the actual temperature decreases by $1°\text{C}$.

$\dfrac{\partial W}{\partial v} = -11.37(0.16)v^{-0.84} + 0.3965T(0.16)v^{-0.84}$ and when $T = -15°\text{C}$ and $v = 30$ km/h,

$\dfrac{\partial W}{\partial v} = -11.37(0.16)(30)^{-0.84} + 0.3965(-15)(0.16)(30)^{-0.84} \approx -0.1592$, so we would expect the apparent temperature to drop by approximately $0.16°\text{C}$ if the wind speed increases by 1 km/h.

85. $\dfrac{\partial K}{\partial m} = \frac{1}{2}v^2$, $\dfrac{\partial K}{\partial v} = mv$, $\dfrac{\partial^2 K}{\partial v^2} = m$. Thus $\dfrac{\partial K}{\partial m} \cdot \dfrac{\partial^2 K}{\partial v^2} = \frac{1}{2}v^2 m = K$.

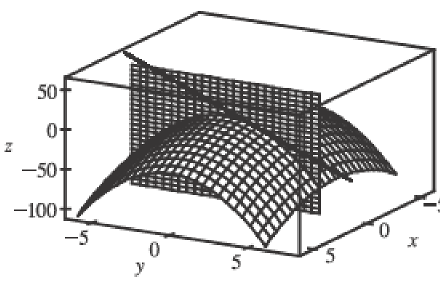
86. The Law of Cosines says that $a^2 = b^2 + c^2 - 2bc\cos A$. Thus $\dfrac{\partial(a^2)}{\partial a} = \dfrac{\partial(b^2 + c^2 - 2ab\cos A)}{\partial a}$ or

$2a = -2bc(-\sin A)\dfrac{\partial A}{\partial a}$, implying that $\dfrac{\partial A}{\partial a} = \dfrac{a}{bc\sin A}$. Taking the partial derivative of both sides with respect to b gives

$0 = 2b - 2c(\cos A) - 2bc(-\sin A)\dfrac{\partial A}{\partial b}$. Thus $\dfrac{\partial A}{\partial b} = \dfrac{c\cos A - b}{bc\sin A}$. By symmetry, $\dfrac{\partial A}{\partial c} = \dfrac{b\cos A - c}{bc\sin A}$.

87. $f_x(x, y) = x + 4y \;\Rightarrow\; f_{xy}(x, y) = 4$ and $f_y(x, y) = 3x - y \;\Rightarrow\; f_{yx}(x, y) = 3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x, y) \neq f_{yx}(x, y)$, Clairaut's Theorem implies that such a function $f(x, y)$ does not exist.

88. Setting $x = 1$, the equation of the parabola of intersection is $z = 6 - 1 - 1 - 2y^2 = 4 - 2y^2$. The slope of the tangent is $\partial z/\partial y = -4y$, so at $(1, 2, -4)$ the slope is -8. Parametric equations for the line are therefore $x = 1$, $y = 2 + t$, $z = -4 - 8t$.



89. By the geometry of partial derivatives, the slope of the tangent line is $f_x(1, 2)$. By implicit differentiation of $4x^2 + 2y^2 + z^2 = 16$, we get $8x + 2z(\partial z/\partial x) = 0 \;\Rightarrow\; \partial z/\partial x = -4x/z$, so when $x = 1$ and $z = 2$ we have $\partial z/\partial x = -2$. So the slope is $f_x(1, 2) = -2$. Thus the tangent line is given by $z - 2 = -2(x - 1)$, $y = 2$. Taking the parameter to be $t = x - 1$, we can write parametric equations for this line: $x = 1 + t$, $y = 2$, $z = 2 - 2t$.

90. $T(x,t) = T_0 + T_1 e^{-\lambda x} \sin(\omega t - \lambda x)$

(a) $\partial T/\partial x = T_1 e^{-\lambda x}\left[\cos(\omega t - \lambda x)(-\lambda)\right] + T_1(-\lambda e^{-\lambda x})\sin(\omega t - \lambda x) = -\lambda T_1 e^{-\lambda x}\left[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)\right]$.
This quantity represents the rate of change of temperature with respect to depth below the surface, at a given time t.

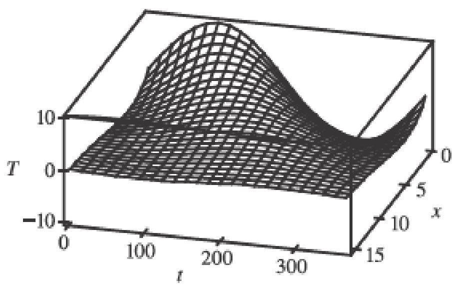
(b) $\partial T/\partial t = T_1 e^{-\lambda x}\left[\cos(\omega t - \lambda x)(\omega)\right] = \omega T_1 e^{-\lambda x}\cos(\omega t - \lambda x)$. This quantity represents the rate of change of temperature with respect to time at a fixed depth x.

(c) $T_{xx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial T}{\partial x}\right)$

$$= -\lambda T_1\left(e^{-\lambda x}\left[\cos(\omega t - \lambda x)(-\lambda) - \sin(\omega t - \lambda x)(-\lambda)\right] + e^{-\lambda x}(-\lambda)\left[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)\right]\right)$$
$$= 2\lambda^2 T_1 e^{-\lambda x}\cos(\omega t - \lambda x)$$

But from part (b), $T_t = \omega T_1 e^{-\lambda x}\cos(\omega t - \lambda x) = \dfrac{\omega}{2\lambda^2} T_{xx}$. So with $k = \dfrac{\omega}{2\lambda^2}$, the function T satisfies the heat equation.

(d)



Note that near the surface (that is, for small x) the temperature varies greatly as t changes, but deeper (for large x) the temperature is more stable.

(e) The term $-\lambda x$ is a phase shift: it represents the fact that since heat diffuses slowly through soil, it takes time for changes in the surface temperature to affect the temperature at deeper points. As x increases, the phase shift also increases. For example, at the surface the highest temperature is reached at $t \approx 100$, whereas at a depth of 5 feet the peak temperature is attained at $t \approx 150$, and at a depth of 10 feet, at $t \approx 220$.

91. By Clairaut's Theorem, $f_{xyy} = (f_{xy})_y = (f_{yx})_y = f_{yxy} = (f_y)_{xy} = (f_y)_{yx} = f_{yyx}$.

92. (a) Since we are differentiating n times, with two choices of variable at each differentiation, there are 2^n nth-order partial derivatives.

(b) If these partial derivatives are all continuous, then the order in which the partials are taken doesn't affect the value of the result, that is, all nth-order partial derivatives with p partials with respect to x and $n - p$ partials with respect to y are equal. Since the number of partials taken with respect to x for an nth-order partial derivative can range from 0 to n, a function of two variables has $n + 1$ distinct partial derivatives of order n if these partial derivatives are all continuous.

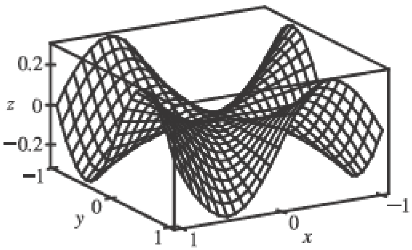
(c) Since n differentiations are to be performed with three choices of variable at each differentiation, there are 3^n nth-order partial derivatives of a function of three variables.

93. Let $g(x) = f(x, 0) = x(x^2)^{-3/2} e^0 = x\,|x|^{-3}$. But we are using the point $(1, 0)$, so near $(1, 0)$, $g(x) = x^{-2}$. Then $g'(x) = -2x^{-3}$ and $g'(1) = -2$, so using (1) we have $f_x(1, 0) = g'(1) = -2$.

94. $f_x(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \dfrac{(h^3+0)^{1/3} - 0}{h} = \lim_{h\to 0} \dfrac{h}{h} = 1.$

Or: Let $g(x) = f(x,0) = \sqrt[3]{x^3+0} = x$. Then $g'(x) = 1$ and $g'(0) = 1$ so, by (1), $f_x(0,0) = g'(0) = 1$.

95. (a)



(b) For $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2}$$
$$= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

and by symmetry $f_y(x,y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$.

(c) $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \dfrac{(0/h^2) - 0}{h} = 0$ and $f_y(0,0) = \lim_{h\to 0} \dfrac{f(0,h) - f(0,0)}{h} = 0$.

(d) By (3), $f_{xy}(0,0) = \dfrac{\partial f_x}{\partial y} = \lim_{h\to 0} \dfrac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0} \dfrac{(-h^5 - 0)/h^4}{h} = -1$ while by (2),

$$f_{yx}(0,0) = \frac{\partial f_y}{\partial x} = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h^5/h^4}{h} = 1.$$

(e) For $(x,y) \neq (0,0)$, we use a CAS to compute

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

Now as $(x,y) \to (0,0)$ along the x-axis, $f_{xy}(x,y) \to 1$ while as $(x,y) \to (0,0)$ along the y-axis, $f_{xy}(x,y) \to -1$. Thus f_{xy} isn't continuous at $(0,0)$ and Clairaut's Theorem doesn't apply, so there is no contradiction. The graphs of f_{xy} and f_{yx} are identical except at the origin, where we observe the discontinuity.