1. $z = f(x, y) = 4x^2 - y^2 + 2y \quad \Rightarrow \quad f_x(x, y) = 8x$, $f_y(x, y) = -2y + 2$, so $f_x(-1, 2) = -8$, $f_y(-1, 2) = -2$. By Equation 2, an equation of the tangent plane is $z - 4 = f_x(-1, 2)[x - (-1)] + f_y(-1, 2)(y - 2) \quad \Rightarrow$ $z - 4 = -8(x + 1) - 2(y - 2)$ or $z = -8x - 2y$.

2. $z = f(x, y) = 3(x - 1)^2 + 2(y + 3)^2 + 7 \quad \Rightarrow \quad f_x(x, y) = 6(x - 1)$, $f_y(x, y) = 4(y + 3)$, so $f_x(2, -2) = 6$ and $f_y(2, -2) = 4$. By Equation 2, an equation of the tangent plane is $z - 12 = f_x(2, -2)(x - 2) + f_y(2, -2)\,[y - (-2)] \quad \Rightarrow$ $z - 12 = 6(x - 2) + 4(y + 2)$ or $z = 6x + 4y + 8$.

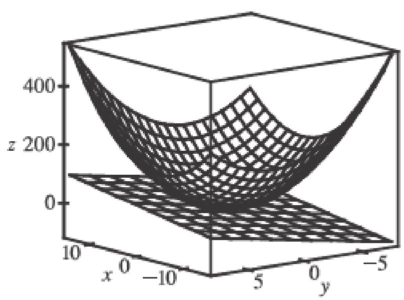
3. $z = f(x, y) = \sqrt{xy} \quad \Rightarrow \quad f_x(x, y) = \frac{1}{2}(xy)^{-1/2} \cdot y = \frac{1}{2}\sqrt{y/x}$, $f_y(x, y) = \frac{1}{2}(xy)^{-1/2} \cdot x = \frac{1}{2}\sqrt{x/y}$, so $f_x(1, 1) = \frac{1}{2}$ and $f_y(1, 1) = \frac{1}{2}$. Thus an equation of the tangent plane is $z - 1 = f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) \quad \Rightarrow$ $z - 1 = \frac{1}{2}(x - 1) + \frac{1}{2}(y - 1)$ or $x + y - 2z = 0$.

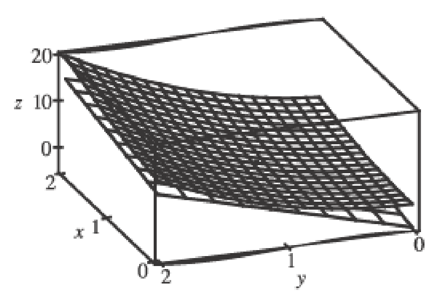
4. $z = f(x, y) = y \ln x \quad \Rightarrow \quad f_x(x, y) = y/x$, $f_y(x, y) = \ln x$, so $f_x(1, 4) = 4$, $f_y(1, 4) = 0$, and an equation of the tangent plane is $z - 0 = f_x(1, 4)(x - 1) + f_y(1, 4)(y - 4) \quad \Rightarrow \quad z = 4(x - 1) + 0(y - 4)$ or $z = 4x - 4$.

5. $z = f(x, y) = y\cos(x - y) \quad \Rightarrow \quad f_x = y(-\sin(x - y)(1)) = -y\sin(x - y)$, $f_y = y(-\sin(x - y)(-1)) + \cos(x - y) = y\sin(x - y) + \cos(x - y)$, so $f_x(2, 2) = -2\sin(0) = 0$, $f_y(2, 2) = 2\sin(0) + \cos(0) = 1$ and an equation of the tangent plane is $z - 2 = 0(x - 2) + 1(y - 2)$ or $z = y$.

6. $z = f(x, y) = e^{x^2 - y^2} \quad \Rightarrow \quad f_x(x, y) = 2xe^{x^2 - y^2}$, $f_y(x, y) = -2ye^{x^2 - y^2}$, so $f_x(1, -1) = 2$, $f_y(1, -1) = 2$. By Equation 2, an equation of the tangent plane is $z - 1 = f_x(1, -1)(x - 1) + f_y(1, -1)[y - (-1)] \quad \Rightarrow$ $z - 1 = 2(x - 1) + 2(y + 1)$ or $z = 2x + 2y + 1$.

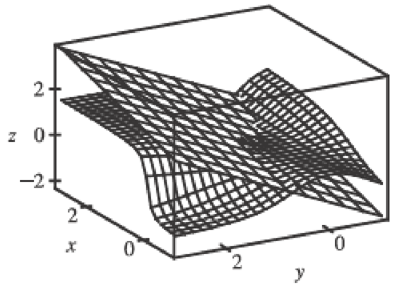
7. $z = f(x, y) = x^2 + xy + 3y^2$, so $f_x(x, y) = 2x + y \quad \Rightarrow \quad f_x(1, 1) = 3$, $f_y(x, y) = x + 6y \quad \Rightarrow \quad f_y(1, 1) = 7$ and an equation of the tangent plane is $z - 5 = 3(x - 1) + 7(y - 1)$ or $z = 3x + 7y - 5$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is below the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.

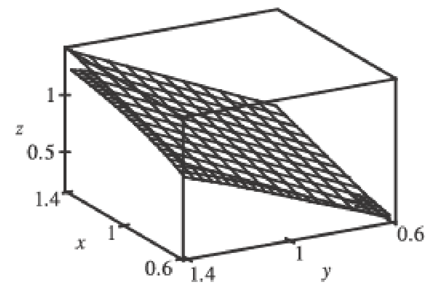




8. $z = f(x,y) = \arctan(xy^2) \quad \Rightarrow \quad f_x = \dfrac{1}{1+(xy^2)^2}\,(y^2) = \dfrac{y^2}{1+x^2y^4},\; f_y = \dfrac{1}{1+(xy^2)^2}\,(2xy) = \dfrac{2xy}{1+x^2y^4},$

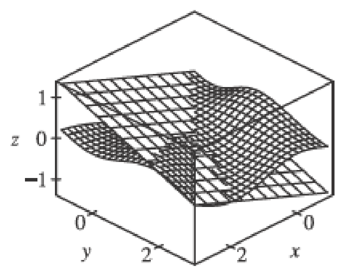
$f_x(1,1) = \frac{1}{1+1} = \frac{1}{2},\; f_y(1,1) = \frac{2}{1+1} = 1$, so an equation of the tangent plane is $z - \frac{\pi}{4} = \frac{1}{2}(x-1) + 1(y-1)$ or $z = \frac{1}{2}x + y - \frac{3}{2} + \frac{\pi}{4}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.

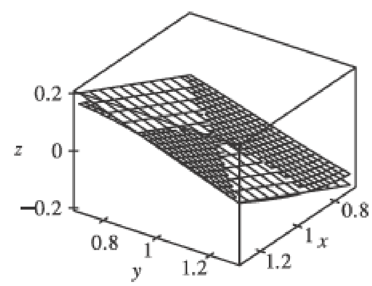




9. $f(x,y) = \dfrac{xy\sin(x-y)}{1+x^2+y^2}$. A CAS gives $f_x(x,y) = \dfrac{y\sin(x-y) + xy\cos(x-y)}{1+x^2+y^2} - \dfrac{2x^2y\sin(x-y)}{(1+x^2+y^2)^2}$ and

$f_y(x,y) = \dfrac{x\sin(x-y) - xy\cos(x-y)}{1+x^2+y^2} - \dfrac{2xy^2\sin(x-y)}{(1+x^2+y^2)^2}$. We use the CAS to evaluate these at $(1,1)$, and then substitute the results into Equation 2 to compute an equation of the tangent plane: $z = \frac{1}{3}x - \frac{1}{3}y$. The surface and tangent plane are shown in the first graph below. After zooming in, the surface and the tangent plane become almost indistinguishable, as shown in the second graph. (Here, the tangent plane is shown with fewer traces than the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.

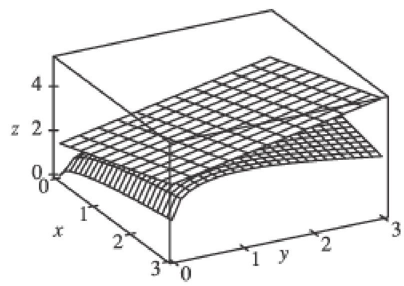


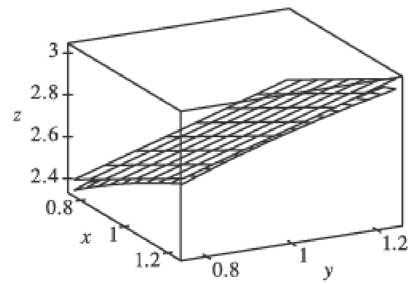


10. $f(x,y) = e^{-xy/10}\left(\sqrt{x}+\sqrt{y}+\sqrt{xy}\right)$. A CAS gives

$f_x(x,y) = -\frac{1}{10}ye^{-xy/10}\left(\sqrt{x}+\sqrt{y}+\sqrt{xy}\right) + e^{-xy/10}\left(\frac{1}{2\sqrt{x}}+\frac{y}{2\sqrt{xy}}\right)$ and

$f_y = -\frac{1}{10}xe^{-xy/10}\left(\sqrt{x}+\sqrt{y}+\sqrt{xy}\right) + e^{-xy/10}\left(\frac{1}{2\sqrt{y}}+\frac{x}{2\sqrt{xy}}\right)$. We use the CAS to evaluate these at $(1,1)$, and then substitute the results into Equation 2 to get an equation of the tangent plane: $z = 0.7e^{-0.1}x + 0.7e^{-0.1}y + 1.6e^{-0.1}$. The surface and tangent plane are shown in the first graph below. After zooming in, the surface and the tangent plane become almost indistinguishable, as shown in the second graph. (Here, the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.





11. $f(x,y) = x\sqrt{y}$. The partial derivatives are $f_x(x,y) = \sqrt{y}$ and $f_y(x,y) = \dfrac{x}{2\sqrt{y}}$, so $f_x(1,4) = 2$ and $f_y(1,4) = \frac{1}{4}$. Both f_x and f_y are continuous functions for $y > 0$, so by Theorem 8, f is differentiable at $(1,4)$. By Equation 3, the linearization of f at $(1,4)$ is given by $L(x,y) = f(1,4) + f_x(1,4)(x-1) + f_y(1,4)(y-4) = 2 + 2(x-1) + \frac{1}{4}(y-4) = 2x + \frac{1}{4}y - 1$.

12. $f(x,y) = x^3y^4$. The partial derivatives are $f_x(x,y) = 3x^2y^4$ and $f_y(x,y) = 4x^3y^3$, so $f_x(1,1) = 3$ and $f_y(1,1) = 4$. Both f_x and f_y are continuous functions, so f is differentiable at $(1,1)$ by Theorem 8. The linearization of f at $(1,1)$ is given by $L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) = 1 + 3(x-1) + 4(y-1) = 3x + 4y - 6$.

13. $f(x,y) = \dfrac{x}{x+y}$. The partial derivatives are $f_x(x,y) = \dfrac{1(x+y) - x(1)}{(x+y)^2} = y/(x+y)^2$ and $f_y(x,y) = x(-1)(x+y)^{-2}\cdot 1 = -x/(x+y)^2$, so $f_x(2,1) = \frac{1}{9}$ and $f_y(2,1) = -\frac{2}{9}$. Both f_x and f_y are continuous functions for $y \neq -x$, so f is differentiable at $(2,1)$ by Theorem 8. The linearization of f at $(2,1)$ is given by $L(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = \frac{2}{3} + \frac{1}{9}(x-2) - \frac{2}{9}(y-1) = \frac{1}{9}x - \frac{2}{9}y + \frac{2}{3}$.

14. $f(x,y) = \sqrt{x + e^{4y}} = (x + e^{4y})^{1/2}$. The partial derivatives are $f_x(x,y) = \frac{1}{2}(x+e^{4y})^{-1/2}$ and $f_y(x,y) = \frac{1}{2}(x+e^{4y})^{-1/2}(4e^{4y}) = 2e^{4y}(x+e^{4y})^{-1/2}$, so $f_x(3,0) = \frac{1}{2}(3+e^0)^{-1/2} = \frac{1}{4}$ and $f_y(3,0) = 2e^0(3+e^0)^{-1/2} = 1$. Both f_x and f_y are continuous functions near $(3,0)$, so f is differentiable at $(3,0)$ by Theorem 8. The linearization of f at $(3,0)$ is $L(x,y) = f(3,0) + f_x(3,0)(x-3) + f_y(3,0)(y-0) = 2 + \frac{1}{4}(x-3) + 1(y-0) = \frac{1}{4}x + y + \frac{5}{4}$.

15. $f(x,y)=e^{-xy}\cos y$. The partial derivatives are $f_x(x,y)=e^{-xy}(-y)\cos y=-ye^{-xy}\cos y$ and $f_y(x,y)=e^{-xy}(-\sin y)+(\cos y)e^{-xy}(-x)=-e^{-xy}(\sin y+x\cos y)$, so $f_x(\pi,0)=0$ and $f_y(\pi,0)=-\pi$. Both f_x and f_y are continuous functions, so f is differentiable at $(\pi,0)$, and the linearization of f at $(\pi,0)$ is

$L(x,y)=f(\pi,0)+f_x(\pi,0)(x-\pi)+f_y(\pi,0)(y-0)=1+0(x-\pi)-\pi(y-0)=1-\pi y.$

16. $f(x,y)=\sin(2x+3y)$. The partial derivatives are $f_x(x,y)=2\cos(2x+3y)$ and $f_y(x,y)=3\cos(2x+3y)$, so $f_x(-3,2)=2$ and $f_y(-3,2)=3$. Both f_x and f_y are continuous functions, so f is differentiable at $(-3,2)$, and the linearization of f at $(-3,2)$ is

$L(x,y)=f(-3,2)+f_x(-3,2)(x+3)+f_y(-3,2)(y-2)=0+2(x+3)+3(y-2)=2x+3y.$

17. Let $f(x,y)=\dfrac{2x+3}{4y+1}$. Then $f_x(x,y)=\dfrac{2}{4y+1}$ and $f_y(x,y)=(2x+3)(-1)(4y+1)^{-2}(4)=\dfrac{-8x-12}{(4y+1)^2}$. Both f_x and f_y are continuous functions for $y\neq-\frac{1}{4}$, so by Theorem 8, f is differentiable at $(0,0)$. We have $f_x(0,0)=2$, $f_y(0,0)=-12$ and the linear approximation of f at $(0,0)$ is $f(x,y)\approx f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)=3+2x-12y$.

18. Let $f(x,y)=\sqrt{y+\cos^2 x}$. Then $f_x(x,y)=\frac{1}{2}(y+\cos^2 x)^{-1/2}(2\cos x)(-\sin x)=-\cos x\sin x/\sqrt{y+\cos^2 x}$ and $f_y(x,y)=\frac{1}{2}(y+\cos^2 x)^{-1/2}(1)=1/\left(2\sqrt{y+\cos^2 x}\right)$. Both f_x and f_y are continuous functions for $y>-\cos^2 x$, so f is differentiable at $(0,0)$ by Theorem 8. We have $f_x(0,0)=0$ and $f_y(0,0)=\frac{1}{2}$, so the linear approximation of f at $(0,0)$ is $f(x,y)\approx f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)=1+0x+\frac{1}{2}y=1+\frac{1}{2}y$.

19. $f(x,y)=\sqrt{20-x^2-7y^2}\quad\Rightarrow\quad f_x(x,y)=-\dfrac{x}{\sqrt{20-x^2-7y^2}}$ and $f_y(x,y)=-\dfrac{7y}{\sqrt{20-x^2-7y^2}}$, so $f_x(2,1)=-\frac{2}{3}$ and $f_y(2,1)=-\frac{7}{3}$. Then the linear approximation of f at $(2,1)$ is given by

$f(x,y)\approx f(2,1)+f_x(2,1)(x-2)+f_y(2,1)(y-1)=3-\frac{2}{3}(x-2)-\frac{7}{3}(y-1)=-\frac{2}{3}x-\frac{7}{3}y+\frac{20}{3}.$

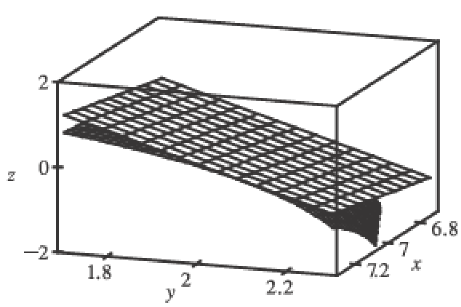
Thus $f(1.95,1.08)\approx-\frac{2}{3}(1.95)-\frac{7}{3}(1.08)+\frac{20}{3}=2.84\overline{6}$.

20. $f(x,y)=\ln(x-3y)\quad\Rightarrow\quad f_x(x,y)=\dfrac{1}{x-3y}$ and $f_y(x,y)=-\dfrac{3}{x-3y}$, so $f_x(7,2)=1$ and $f_y(7,2)=-3$.

Then the linear approximation of f at $(7,2)$ is given by

$$\begin{aligned}f(x,y)&\approx f(7,2)+f_x(7,2)(x-7)+f_y(7,2)(y-2)\\&=0+1(x-7)-3(y-2)=x-3y-1\end{aligned}$$

Thus $f(6.9,2.06)\approx6.9-3(2.06)-1=-0.28$. The graph shows that our approximated value is slightly greater than the actual value.



21. $f(x,y,z) = \sqrt{x^2+y^2+z^2} \quad\Rightarrow\quad f_x(x,y,z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}},\ f_y(x,y,z) = \dfrac{y}{\sqrt{x^2+y^2+z^2}}$, and

$f_z(x,y,z) = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$, so $f_x(3,2,6) = \frac{3}{7}$, $f_y(3,2,6) = \frac{2}{7}$, $f_z(3,2,6) = \frac{6}{7}$. Then the linear approximation of f at $(3,2,6)$ is given by

$$\begin{aligned} f(x,y,z) &\approx f(3,2,6) + f_x(3,2,6)(x-3) + f_y(3,2,6)(y-2) + f_z(3,2,6)(z-6) \\ &= 7 + \tfrac{3}{7}(x-3) + \tfrac{2}{7}(y-2) + \tfrac{6}{7}(z-6) = \tfrac{3}{7}x + \tfrac{2}{7}y + \tfrac{6}{7}z \end{aligned}$$

Thus $\sqrt{(3.02)^2+(1.97)^2+(5.99)^2} = f(3.02, 1.97, 5.99) \approx \frac{3}{7}(3.02) + \frac{2}{7}(1.97) + \frac{6}{7}(5.99) \approx 6.9914$.

22. From the table, $f(40,20) = 28$. To estimate $f_v(40,20)$ and $f_t(40,20)$ we follow the procedure used in Exercise 15.3.4 [ET 14.3.4]. Since $f_v(40,20) = \lim\limits_{h\to 0} \dfrac{f(40+h,20) - f(40,20)}{h}$, we approximate this quantity with $h = \pm 10$ and use the values given in the table:

$$f_v(40,20) \approx \frac{f(50,20) - f(40,20)}{10} = \frac{40-28}{10} = 1.2, \qquad f_v(40,20) \approx \frac{f(30,20) - f(40,20)}{-10} = \frac{17-28}{-10} = 1.1$$

Averaging these values gives $f_v(40,20) \approx 1.15$. Similarly, $f_t(40,20) = \lim\limits_{h\to 0} \dfrac{f(40,20+h) - f(40,20)}{h}$, so we use $h = 10$ and $h = -5$:

$$f_t(40,20) \approx \frac{f(40,30) - f(40,20)}{10} = \frac{31-28}{10} = 0.3, \qquad f_t(40,20) \approx \frac{f(40,15) - f(40,20)}{-5} = \frac{25-28}{-5} = 0.6$$

Averaging these values gives $f_t(40,15) \approx 0.45$. The linear approximation, then, is

$$f(v,t) \approx f(40,20) + f_v(40,20)(v-40) + f_t(40,20)(t-20) \approx 28 + 1.15(v-40) + 0.45(t-20)$$

When $v = 43$ and $t = 24$, we estimate $f(43,24) \approx 28 + 1.15(43-40) + 0.45(24-20) = 33.25$, so we would expect the wave heights to be approximately 33.25 ft.

23. From the table, $f(94, 80) = 127$. To estimate $f_T(94, 80)$ and $f_H(94, 80)$ we follow the procedure used in Section 15.3 [ET 14.3]. Since $f_T(94, 80) = \lim_{h \to 0} \dfrac{f(94 + h, 80) - f(94, 80)}{h}$, we approximate this quantity with $h = \pm 2$ and use the values given in the table:

$$f_T(94, 80) \approx \frac{f(96, 80) - f(94, 80)}{2} = \frac{135 - 127}{2} = 4, \quad f_T(94, 80) \approx \frac{f(92, 80) - f(94, 80)}{-2} = \frac{119 - 127}{-2} = 4$$

Averaging these values gives $f_T(94, 80) \approx 4$. Similarly, $f_H(94, 80) = \lim_{h \to 0} \dfrac{f(94, 80 + h) - f(94, 80)}{h}$, so we use $h = \pm 5$:

$$f_H(94, 80) \approx \frac{f(94, 85) - f(94, 80)}{5} = \frac{132 - 127}{5} = 1, \quad f_H(94, 80) \approx \frac{f(94, 75) - f(94, 80)}{-5} = \frac{122 - 127}{-5} = 1$$

Averaging these values gives $f_H(94, 80) \approx 1$. The linear approximation, then, is

$$\begin{aligned} f(T, H) &\approx f(94, 80) + f_T(94, 80)(T - 94) + f_H(94, 80)(H - 80) \\ &\approx 127 + 4(T - 94) + 1(H - 80) \qquad [\text{or } 4T + H - 329] \end{aligned}$$

Thus when $T = 95$ and $H = 78$, $f(95, 78) \approx 127 + 4(95 - 94) + 1(78 - 80) = 129$, so we estimate the heat index to be approximately $129°\text{F}$.

24. From the table, $f(-15, 50) = -29$. To estimate $f_T(-15, 50)$ and $f_v(-15, 50)$ we follow the procedure used in Section 15.3 [ET 14.3]. Since $f_T(-15, 50) = \lim_{h \to 0} \dfrac{f(-15 + h, 50) - f(-15, 50)}{h}$, we approximate this quantity with $h = \pm 5$ and use the values given in the table:

$$f_T(-15, 50) \approx \frac{f(-10, 50) - f(-15, 50)}{5} = \frac{-22 - (-29)}{5} = 1.4$$

$$f_T(-15, 50) \approx \frac{f(-20, 50) - f(-15, 50)}{-5} = \frac{-35 - (-29)}{-5} = 1.2$$

Averaging these values gives $f_T(-15, 50) \approx 1.3$. Similarly $f_v(-15, 50) = \lim_{h \to 0} \dfrac{f(-15, 50 + h) - f(-15, 50)}{h}$, so we use $h = \pm 10$:

$$f_v(-15, 50) \approx \frac{f(-15, 60) - f(-15, 50)}{10} = \frac{-30 - (-29)}{10} = -0.1$$

$$f_v(-15, 50) \approx \frac{f(-15, 40) - f(-15, 50)}{-10} = \frac{-27 - (-29)}{-10} = -0.2$$

Averaging these values gives $f_v(-15, 50) \approx -0.15$. The linear approximation to the wind-chill index function, then, is $f(T, v) \approx f(-15, 50) + f_T(-15, 50)(T - (-15)) + f_v(-15, 50)(v - 50) \approx -29 + (1.3)(T + 15) - (0.15)(v - 50)$. Thus when $T = -17°\text{C}$ and $v = 55$ km/h, $f(-17, 55) \approx -29 + (1.3)(-17 + 15) - (0.15)(55 - 50) = -32.35$, so we estimate the wind-chill index to be approximately $-32.35°\text{C}$.

25. $z = x^3 \ln(y^2) \quad \Rightarrow \quad dz = \dfrac{\partial z}{\partial x} dx + \dfrac{\partial z}{\partial y} dy = 3x^2 \ln(y^2)\, dx + x^3 \cdot \dfrac{1}{y^2}(2y)\, dy = 3x^2 \ln(y^2)\, dx + \dfrac{2x^3}{y} dy$

26. $v = y\cos xy \quad \Rightarrow$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = y(-\sin xy)y\,dx + [y(-\sin xy)x + \cos xy]\,dy = -y^2\sin xy\,dx + (\cos xy - xy\sin xy)\,dy$$

27. $m = p^5q^3 \quad \Rightarrow \quad dm = \dfrac{\partial m}{\partial p}\,dp + \dfrac{\partial m}{\partial q}\,dq = 5p^4q^3\,dp + 3p^5q^2\,dq$

28. $T = \dfrac{v}{1+uvw} \quad \Rightarrow$

$$\begin{aligned} dT &= \frac{\partial T}{\partial u}\,du + \frac{\partial T}{\partial v}\,dv + \frac{\partial T}{\partial w}\,dw \\ &= v(-1)(1+uvw)^{-2}(vw)\,du + \frac{1(1+uvw) - v(uw)}{(1+uvw)^2}\,dv + v(-1)(1+uvw)^{-2}(uv)\,dw \\ &= -\frac{v^2w}{(1+uvw)^2}\,du + \frac{1}{(1+uvw)^2}\,dv - \frac{uv^2}{(1+uvw)^2}\,dw \end{aligned}$$

29. $R = \alpha\beta^2\cos\gamma \quad \Rightarrow \quad dR = \dfrac{\partial R}{\partial \alpha}\,d\alpha + \dfrac{\partial R}{\partial \beta}\,d\beta + \dfrac{\partial R}{\partial \gamma}\,d\gamma = \beta^2\cos\gamma\,d\alpha + 2\alpha\beta\,\cos\gamma\,d\beta - \alpha\beta^2\sin\gamma\,d\gamma$

30. $w = xye^{xz} \quad \Rightarrow$

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz = (xyze^{xz} + ye^{xz})\,dx + xe^{xz}\,dy + x^2ye^{xz}\,dz = (xz+1)ye^{xz}\,dx + xe^{xz}\,dy + x^2ye^{xz}\,dz$$

31. $dx = \Delta x = 0.05$, $dy = \Delta y = 0.1$, $z = 5x^2 + y^2$, $z_x = 10x$, $z_y = 2y$. Thus when $x = 1$ and $y = 2$,

$dz = z_x(1,2)\,dx + z_y(1,2)\,dy = (10)(0.05) + (4)(0.1) = 0.9$ while

$\Delta z = f(1.05, 2.1) - f(1,2) = 5(1.05)^2 + (2.1)^2 - 5 - 4 = 0.9225$.

32. $dx = \Delta x = -0.04$, $dy = \Delta y = 0.05$, $z = x^2 - xy + 3y^2$, $z_x = 2x - y$, $z_y = 6y - x$. Thus when $x = 3$ and $y = -1$,

$dz = (7)(-0.04) + (-9)(0.05) = -0.73$ while $\Delta z = (2.96)^2 - (2.96)(-0.95) + 3(-0.95)^2 - (9+3+3) = -0.7189$.

33. $dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy = y\,dx + x\,dy$ and $|\Delta x| \le 0.1$, $|\Delta y| \le 0.1$. We use $dx = 0.1$, $dy = 0.1$ with $x = 30$, $y = 24$;

then the maximum error in the area is about $dA = 24(0.1) + 30(0.1) = 5.4\text{ cm}^2$.

34. Let S be surface area. Then $S = 2(xy + xz + yz)$ and $dS = 2(y+z)\,dx + 2(x+z)\,dy + 2(x+y)\,dz$. The maximum error occurs with $\Delta x = \Delta y = \Delta z = 0.2$. Using $dx = \Delta x$, $dy = \Delta y$, $dz = \Delta z$ we find the maximum error in calculated surface area to be about $dS = (220)(0.2) + (260)(0.2) + (280)(0.2) = 152\text{ cm}^2$.

35. The volume of a can is $V = \pi r^2 h$ and $\Delta V \approx dV$ is an estimate of the amount of tin. Here $dV = 2\pi rh\,dr + \pi r^2\,dh$, so put $dr = 0.04$, $dh = 0.08$ (0.04 on top, 0.04 on bottom) and then $\Delta V \approx dV = 2\pi(48)(0.04) + \pi(16)(0.08) \approx 16.08\text{ cm}^3$. Thus the amount of tin is about 16 cm^3.

36. Let V be the volume. Then $V = \pi r^2 h$ and $\Delta V \approx dV = 2\pi rh\,dr + \pi r^2\,dh$ is an estimate of the amount of metal. With $dr = 0.05$ and $dh = 0.2$ we get $dV = 2\pi(2)(10)(0.05) + \pi(2)^2(0.2) = 2.80\pi \approx 8.8\text{ cm}^3$.

37. The area of the rectangle is $A = xy$, and $\Delta A \approx dA$ is an estimate of the area of paint in the stripe. Here $dA = y\,dx + x\,dy$, so with $dx = dy = \frac{3+3}{12} = \frac{1}{2}$, $\Delta A \approx dA = (100)\left(\frac{1}{2}\right) + (200)\left(\frac{1}{2}\right) = 150\text{ ft}^2$. Thus there are approximately 150 ft^2 of paint in the stripe.

38. Here $dV = \Delta V = 0.3$, $dT = \Delta T = -5$, $P = 8.31\,\dfrac{T}{V}$, so

$$dP = \left(\frac{8.31}{V}\right) dT - \frac{8.31 \cdot T}{V^2}\,dV = 8.31\left[-\frac{5}{12} - \frac{310}{144}\cdot\frac{3}{10}\right] \approx -8.83.$$ Thus the pressure will drop by about $8.83\,\text{kPa}$.

39. First we find $\dfrac{\partial R}{\partial R_1}$ implicitly by taking partial derivatives of both sides with respect to R_1:

$$\frac{\partial}{\partial R_1}\left(\frac{1}{R}\right) = \frac{\partial\left[(1/R_1) + (1/R_2) + (1/R_3)\right]}{\partial R_1} \quad\Rightarrow\quad -R^{-2}\frac{\partial R}{\partial R_1} = -R_1^{-2} \quad\Rightarrow\quad \frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}.$$ Then by symmetry,

$\dfrac{\partial R}{\partial R_2} = \dfrac{R^2}{R_2^2}$, $\dfrac{\partial R}{\partial R_3} = \dfrac{R^2}{R_3^2}$. When $R_1 = 25$, $R_2 = 40$ and $R_3 = 50$, $\dfrac{1}{R} = \dfrac{17}{200} \quad\Leftrightarrow\quad R = \frac{200}{17}\,\Omega$.

Since the possible error for each R_i is 0.5%, the maximum error of R is attained by setting $\Delta R_i = 0.005 R_i$. So

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1}\Delta R_1 + \frac{\partial R}{\partial R_2}\Delta R_2 + \frac{\partial R}{\partial R_3}\Delta R_3 = (0.005)R^2\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = (0.005)R = \tfrac{1}{17} \approx 0.059\,\Omega.$$

40. Let x, y, z and w be the four numbers with $p(x, y, z, w) = xyzw$. Since the largest error due to rounding for each number is 0.05, the maximum error in the calculated product is approximated by $dp = (yzw)(0.05) + (xzw)(0.05) + (xyw)(0.05) + (xyz)(0.05)$. Furthermore, each of the numbers is positive but less than 50, so the product of any three is between 0 and $(50)^3$. Thus $dp \le 4(50)^3(0.05) = 25{,}000$.

41. The errors in measurement are at most 2%, so $\left|\dfrac{\Delta w}{w}\right| \le 0.02$ and $\left|\dfrac{\Delta h}{h}\right| \le 0.02$. The relative error in the calculated surface area is

$$\frac{\Delta S}{S} \approx \frac{dS}{S} = \frac{0.1091(0.425w^{0.425-1})h^{0.725}\,dw + 0.1091w^{0.425}(0.725h^{0.725-1})\,dh}{0.1091w^{0.425}h^{0.725}} = 0.425\frac{dw}{w} + 0.725\frac{dh}{h}$$

To estimate the maximum relative error, we use $\dfrac{dw}{w} = \left|\dfrac{\Delta w}{w}\right| = 0.02$ and $\dfrac{dh}{h} = \left|\dfrac{\Delta h}{h}\right| = 0.02 \quad\Rightarrow$

$\dfrac{dS}{S} = 0.425\,(0.02) + 0.725\,(0.02) = 0.023$. Thus the maximum percentage error is approximately 2.3%.

42. $\mathbf{r}_1(t) = \langle 2 + 3t, 1 - t^2, 3 - 4t + t^2\rangle \quad\Rightarrow\quad \mathbf{r}_1'(t) = \langle 3, -2t, -4 + 2t\rangle$, $\mathbf{r}_2(u) = \langle 1 + u^2, 2u^3 - 1, 2u + 1\rangle \quad\Rightarrow$ $\mathbf{r}_2'(u) = \langle 2u, 6u^2, 2\rangle$. Both curves pass through P since $\mathbf{r}_1(0) = \mathbf{r}_2(1) = \langle 2, 1, 3\rangle$, so the tangent vectors $\mathbf{r}_1'(0) = \langle 3, 0, -4\rangle$ and $\mathbf{r}_2'(1) = \langle 2, 6, 2\rangle$ are both parallel to the tangent plane to S at P. A normal vector for the tangent plane is $\mathbf{r}_1'(0) \times \mathbf{r}_2'(1) = \langle 3, 0, -4\rangle \times \langle 2, 6, 2\rangle = \langle 24, -14, 18\rangle$, so an equation of the tangent plane is $24(x - 2) - 14(y - 1) + 18(z - 3) = 0$ or $12x - 7y + 9z = 44$.

43. $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) = (a + \Delta x)^2 + (b + \Delta y)^2 - (a^2 + b^2)$

$= a^2 + 2a\,\Delta x + (\Delta x)^2 + b^2 + 2b\,\Delta y + (\Delta y)^2 - a^2 - b^2 = 2a\,\Delta x + (\Delta x)^2 + 2b\,\Delta y + (\Delta y)^2$

But $f_x(a, b) = 2a$ and $f_y(a, b) = 2b$ and so $\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \Delta x\,\Delta x + \Delta y\,\Delta y$, which is Definition 7 with $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = \Delta y$. Hence f is differentiable.

44. $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) = (a + \Delta x)(b + \Delta y) - 5(b + \Delta y)^2 - (ab - 5b^2)$

$= ab + a\,\Delta y + b\,\Delta x + \Delta x\,\Delta y - 5b^2 - 10b\,\Delta y - 5(\Delta y)^2 - ab + 5b^2$

$= (a - 10b)\,\Delta y + b\,\Delta x + \Delta x\,\Delta y - 5\,\Delta y\,\Delta y,$

but $f_x(a, b) = b$ and $f_y(a, b) = a - 10b$ and so $\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \Delta x\,\Delta y - 5\Delta y\,\Delta y$, which is Definition 7 with $\varepsilon_1 = \Delta y$ and $\varepsilon_2 = -5\,\Delta y$. Hence f is differentiable.

45. To show that f is continuous at (a, b) we need to show that $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$ or equivalently $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$. Since f is differentiable at (a, b), $f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$, where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Thus $f(a + \Delta x, b + \Delta y) = f(a, b) + f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$. Taking the limit of both sides as $(\Delta x, \Delta y) \to (0, 0)$ gives $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$. Thus f is continuous at (a, b).

46. (a) $\lim\limits_{h\to 0} \dfrac{f(h, 0) - f(0, 0)}{h} = \lim\limits_{h\to 0} \dfrac{0 - 0}{h} = 0$ and $\lim\limits_{h\to 0} \dfrac{f(0, h) - f(0, 0)}{h} = \lim\limits_{h\to 0} \dfrac{0 - 0}{h} = 0$. Thus $f_x(0, 0) = f_y(0, 0) = 0$.

To show that f isn't differentiable at $(0, 0)$ we need only show that f is not continuous at $(0, 0)$ and apply Exercise 45. As $(x, y) \to (0, 0)$ along the x-axis $f(x, y) = 0/x^2 = 0$ for $x \neq 0$ so $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along the x-axis. But as $(x, y) \to (0, 0)$ along the line $y = x$, $f(x, x) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$ so $f(x, y) \to \frac{1}{2}$ as $(x, y) \to (0, 0)$ along this line. Thus $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ doesn't exist, so f is discontinuous at $(0, 0)$ and thus not differentiable there.

(b) For $(x, y) \neq (0, 0)$, $f_x(x, y) = \dfrac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} = \dfrac{y(y^2 - x^2)}{(x^2 + y^2)^2}$. If we approach $(0, 0)$ along the y-axis, then $f_x(x, y) = f_x(0, y) = \dfrac{y^3}{y^4} = \dfrac{1}{y}$, so $f_x(x, y) \to \pm\infty$ as $(x, y) \to (0, 0)$. Thus $\lim\limits_{(x,y)\to(0,0)} f_x(x, y)$ does not exist and $f_x(x, y)$ is not continuous at $(0, 0)$. Similarly, $f_y(x, y) = \dfrac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} = \dfrac{x(x^2 - y^2)}{(x^2 + y^2)^2}$ for $(x, y) \neq (0, 0)$, and if we approach $(0, 0)$ along the x-axis, then $f_y(x, y) = f_x(x, 0) = \dfrac{x^3}{x^4} = \dfrac{1}{x}$. Thus $\lim\limits_{(x,y)\to(0,0)} f_y(x, y)$ does not exist and $f_y(x, y)$ is not continuous at $(0, 0)$.