1. $z = x^2 + y^2 + xy,\ x = \sin t,\ y = e^t \quad \Rightarrow \quad \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (2x + y)\cos t + (2y + x)e^t$

2. $z = \cos(x + 4y),\ x = 5t^4,\ y = 1/t \quad \Rightarrow$

$$\begin{aligned}\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} &= -\sin(x + 4y)(1)(20t^3) + [-\sin(x + 4y)(4)](-t^{-2}) \\ &= -20t^3\sin(x + 4y) + \frac{4}{t^2}\sin(x + 4y) = \left(\frac{4}{t^2} - 20t^3\right)\sin(x + 4y)\end{aligned}$$

3. $z = \sqrt{1 + x^2 + y^2},\quad x = \ln t,\quad y = \cos t \quad \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \tfrac{1}{2}(1 + x^2 + y^2)^{-1/2}(2x)\cdot\frac{1}{t} + \tfrac{1}{2}(1 + x^2 + y^2)^{-1/2}(2y)(-\sin t) = \frac{1}{\sqrt{1 + x^2 + y^2}}\left(\frac{x}{t} - y\sin t\right)$$

4. $z = \tan^{-1}(y/x),\ x = e^t,\ y = 1 - e^{-t} \quad \Rightarrow$

$$\begin{aligned}\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} &= \frac{1}{1 + (y/x)^2}(-yx^{-2})\cdot e^t + \frac{1}{1 + (y/x)^2}(1/x)\cdot(-e^{-t})(-1) \\ &= -\frac{y}{x^2 + y^2}\cdot e^t + \frac{1}{x + y^2/x}\cdot e^{-t} = \frac{xe^{-t} - ye^t}{x^2 + y^2}\end{aligned}$$

5. $w = xe^{y/z},\ x = t^2,\ y = 1 - t,\ z = 1 + 2t \quad \Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = e^{y/z}\cdot 2t + xe^{y/z}\left(\frac{1}{z}\right)\cdot(-1) + xe^{y/z}\left(-\frac{y}{z^2}\right)\cdot 2 = e^{y/z}\left(2t - \frac{x}{z} - \frac{2xy}{z^2}\right)$$

6. $w = \ln\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\ln(x^2 + y^2 + z^2),\ x = \sin t,\ y = \cos t,\ z = \tan t \quad \Rightarrow$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \frac{1}{2}\cdot\frac{2x}{x^2 + y^2 + z^2}\cdot\cos t + \frac{1}{2}\cdot\frac{2y}{x^2 + y^2 + z^2}\cdot(-\sin t) + \frac{1}{2}\cdot\frac{2z}{x^2 + y^2 + z^2}\cdot\sec^2 t \\ &= \frac{x\cos t - y\sin t + z\sec^2 t}{x^2 + y^2 + z^2}\end{aligned}$$

7. $z = x^2y^3,\ x = s\cos t,\ y = s\sin t \quad \Rightarrow$

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2xy^3\cos t + 3x^2y^2\sin t \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2xy^3)(-s\sin t) + (3x^2y^2)(s\cos t) = -2sxy^3\sin t + 3sx^2y^2\cos t\end{aligned}$$

8. $z = \arcsin(x - y),\quad x = s^2 + t^2,\quad y = 1 - 2st \quad \Rightarrow$

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{1}{\sqrt{1 - (x - y)^2}}(1)\cdot 2s + \frac{1}{\sqrt{1 - (x - y)^2}}(-1)\cdot(-2t) = \frac{2s + 2t}{\sqrt{1 - (x - y)^2}} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{1}{\sqrt{1 - (x - y)^2}}(1)\cdot 2t + \frac{1}{\sqrt{1 - (x - y)^2}}(-1)\cdot(-2s) = \frac{2s + 2t}{\sqrt{1 - (x - y)^2}}\end{aligned}$$

9. $z = \sin\theta\cos\phi,\ \ \theta = st^2,\ \ \phi = s^2 t \ \ \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial s} = (\cos\theta\ \cos\phi)(t^2) + (-\sin\theta\ \sin\phi)(2st) = t^2\cos\theta\ \cos\phi - 2st\sin\theta\ \sin\phi$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial t} = (\cos\theta\ \cos\phi)(2st) + (-\sin\theta\ \sin\phi)(s^2) = 2st\cos\theta\ \cos\phi - s^2\sin\theta\ \sin\phi$$

10. $z = e^{x+2y},\ \ x = s/t,\ \ y = t/s \ \ \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^{x+2y})(1/t) + (2e^{x+2y})(-ts^{-2}) = e^{x+2y}\left(\frac{1}{t} - \frac{2t}{s^2}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^{x+2y})(-st^{-2}) + (2e^{x+2y})(1/s) = e^{x+2y}\left(\frac{2}{s} - \frac{s}{t^2}\right)$$

11. $z = e^r\cos\theta,\ r = st,\ \theta = \sqrt{s^2+t^2} \ \ \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} = e^r\cos\theta\cdot t + e^r(-\sin\theta)\cdot\tfrac{1}{2}(s^2+t^2)^{-1/2}(2s) = te^r\cos\theta - e^r\sin\theta\cdot\frac{s}{\sqrt{s^2+t^2}}$$

$$= e^r\left(t\cos\theta - \frac{s}{\sqrt{s^2+t^2}}\sin\theta\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} = e^r\cos\theta\cdot s + e^r(-\sin\theta)\cdot\tfrac{1}{2}(s^2+t^2)^{-1/2}(2t) = se^r\cos\theta - e^r\sin\theta\cdot\frac{t}{\sqrt{s^2+t^2}}$$

$$= e^r\left(s\cos\theta - \frac{t}{\sqrt{s^2+t^2}}\sin\theta\right)$$

12. $z = \tan(u/v),\ u = 2s + 3t,\ \ v = 3s - 2t \ \ \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s} = \sec^2(u/v)(1/v)\cdot 2 + \sec^2(u/v)(-uv^{-2})\cdot 3$$

$$= \frac{2}{v}\sec^2\left(\frac{u}{v}\right) - \frac{3u}{v^2}\sec^2\left(\frac{u}{v}\right) = \frac{2v-3u}{v^2}\sec^2\left(\frac{u}{v}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} = \sec^2(u/v)(1/v)\cdot 3 + \sec^2(u/v)(-uv^{-2})\cdot(-2)$$

$$= \frac{3}{v}\sec^2\left(\frac{u}{v}\right) + \frac{2u}{v^2}\sec^2\left(\frac{u}{v}\right) = \frac{2u+3v}{v^2}\sec^2\left(\frac{u}{v}\right)$$

13. When $t = 3$, $x = g(3) = 2$ and $y = h(3) = 7$. By the Chain Rule (2),

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(2,7)g'(3) + f_y(2,7)\,h'(3) = (6)(5) + (-8)(-4) = 62.$$

14. By the Chain Rule (3), $\frac{\partial W}{\partial s} = \frac{\partial W}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial W}{\partial v}\frac{\partial v}{\partial s} \Rightarrow$

$$W_s(1,0) = F_u(u(1,0), v(1,0))\, u_s(1,0) + F_v(u(1,0), v(1,0))\, v_s(1,0) = F_u(2,3)u_s(1,0) + F_v(2,3)v_s(1,0)$$
$$= (-1)(-2) + (10)(5) = 52$$

Similarly, $\frac{\partial W}{\partial t} = \frac{\partial W}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial W}{\partial v}\frac{\partial v}{\partial t} \Rightarrow$

$$W_t(1,0) = F_u(u(1,0), v(1,0))\, u_t(1,0) + F_v(u(1,0), v(1,0))\, v_t(1,0) = F_u(2,3)u_t(1,0) + F_v(2,3)v_t(1,0)$$
$$= (-1)(6) + (10)(4) = 34$$

15. $g(u,v) = f(x(u,v), y(u,v))$ where $x = e^u + \sin v,\ y = e^u + \cos v \Rightarrow$

$\frac{\partial x}{\partial u} = e^u,\ \frac{\partial x}{\partial v} = \cos v,\ \frac{\partial y}{\partial u} = e^u,\ \frac{\partial y}{\partial v} = -\sin v$. By the Chain Rule (3), $\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$. Then

$$g_u(0,0) = f_x(x(0,0), y(0,0))\, x_u(0,0) + f_y(x(0,0), y(0,0))\, y_u(0,0) = f_x(1,2)(e^0) + f_y(1,2)(e^0) = 2(1) + 5(1) = 7.$$

Similarly, $\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$. Then

$$g_v(0,0) = f_x(x(0,0), y(0,0))\, x_v(0,0) + f_y(x(0,0), y(0,0))\, y_v(0,0) = f_x(1,2)(\cos 0) + f_y(1,2)(-\sin 0)$$
$$= 2(1) + 5(0) = 2$$

16. $g(r,s) = f(x(r,s), y(r,s))$ where $x = 2r - s,\ y = s^2 - 4r \Rightarrow \frac{\partial x}{\partial r} = 2,\ \frac{\partial x}{\partial s} = -1,\ \frac{\partial y}{\partial r} = -4,\ \frac{\partial y}{\partial s} = 2s.$

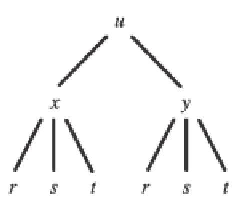
By the Chain Rule (3), $\frac{\partial g}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$. Then

$$g_r(1,2) = f_x(x(1,2), y(1,2))\, x_r(1,2) + f_y(x(1,2), y(1,2))\, y_r(1,2) = f_x(0,0)(2) + f_y(0,0)(-4)$$
$$= 4(2) + 8(-4) = -24$$

Similarly, $\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$. Then

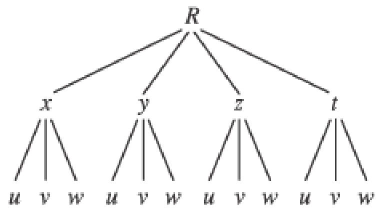
$$g_s(1,2) = f_x(x(1,2), y(1,2))\, x_s(1,2) + f_y(x(1,2), y(1,2))\, y_s(1,2) = f_x(0,0)(-1) + f_y(0,0)(4)$$
$$= 4(-1) + 8(4) = 28$$

17.



$u = f(x,y),\ x = x(r,s,t),\ y = y(r,s,t) \Rightarrow$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r},\ \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},\ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

18. 

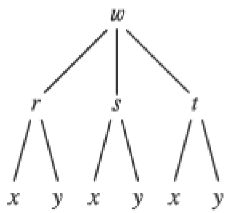
$R = f(x, y, z, t)$, $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$,

$t = t(u, v, w) \quad \Rightarrow$

$$\frac{\partial R}{\partial u} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial R}{\partial t}\frac{\partial t}{\partial u},$$

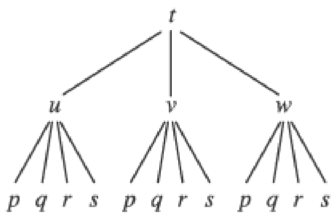
$$\frac{\partial R}{\partial v} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial R}{\partial t}\frac{\partial t}{\partial v},$$

$$\frac{\partial R}{\partial w} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial w} + \frac{\partial R}{\partial t}\frac{\partial t}{\partial w}$$

19. 

$w = f(r, s, t), \;\; r = r(x, y), \;\; s = s(x, y), \;\; t = t(x, y) \quad \Rightarrow$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}, \quad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$$

20. 

$t = f(u, v, w), \;\; u = u(p, q, r, s), \;\; v = v(p, q, r, s), \;\; w = w(p, q, r, s) \quad \Rightarrow$

$$\frac{\partial t}{\partial p} = \frac{\partial t}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial t}{\partial v}\frac{\partial v}{\partial p} + \frac{\partial t}{\partial w}\frac{\partial w}{\partial p}, \quad \frac{\partial t}{\partial q} = \frac{\partial t}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial t}{\partial v}\frac{\partial v}{\partial q} + \frac{\partial t}{\partial w}\frac{\partial w}{\partial q},$$

$$\frac{\partial t}{\partial r} = \frac{\partial t}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial t}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial t}{\partial w}\frac{\partial w}{\partial r}, \quad \frac{\partial t}{\partial s} = \frac{\partial t}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial t}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial w}\frac{\partial w}{\partial s}$$

21. $z = x^2 + xy^3, \;\; x = uv^2 + w^3, \;\; y = u + ve^w \quad \Rightarrow$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (2x + y^3)(v^2) + (3xy^2)(1),$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (2x + y^3)(2uv) + (3xy^2)(e^w),$$

$$\frac{\partial z}{\partial w} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial w} = (2x + y^3)(3w^2) + (3xy^2)(ve^w).$$

When $u = 2$, $v = 1$, and $w = 0$, we have $x = 2$, $y = 3$,

so $\dfrac{\partial z}{\partial u} = (31)(1) + (54)(1) = 85$, $\dfrac{\partial z}{\partial v} = (31)\,(4) + (54)(1) = 178$, $\dfrac{\partial z}{\partial w} = (31)(0) + (54)(1) = 54$.

22. $u = (r^2 + s^2)^{1/2}, \;\; r = y + x\cos t, \;\; s = x + y\sin t \quad \Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2r)(\cos t) + \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2s)(1) = (r\cos t + s)/\sqrt{r^2 + s^2},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2r)(1) + \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2s)(\sin t) = (r + s\sin t)/\sqrt{r^2 + s^2},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2r)(-x\sin t) + \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2s)(y\cos t) = \frac{-rx\sin t + sy\cos t}{\sqrt{r^2 + s^2}}.$$

When $x = 1$, $y = 2$, and $t = 0$ we have $r = 3$ and $s = 1$, so $\dfrac{\partial u}{\partial x} = \dfrac{4}{\sqrt{10}}$, $\dfrac{\partial u}{\partial y} = \dfrac{3}{\sqrt{10}}$, and $\dfrac{\partial u}{\partial t} = \dfrac{2}{\sqrt{10}}$.

23. $R = \ln(u^2+v^2+w^2)$, $u = x+2y$, $v = 2x-y$, $w = 2xy \quad \Rightarrow$

$$\frac{\partial R}{\partial x} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial x} = \frac{2u}{u^2+v^2+w^2}(1) + \frac{2v}{u^2+v^2+w^2}(2) + \frac{2w}{u^2+v^2+w^2}(2y)$$
$$= \frac{2u+4v+4wy}{u^2+v^2+w^2},$$

$$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial y} = \frac{2u}{u^2+v^2+w^2}(2) + \frac{2v}{u^2+v^2+w^2}(-1) + \frac{2w}{u^2+v^2+w^2}(2x)$$
$$= \frac{4u-2v+4wx}{u^2+v^2+w^2}.$$

When $x = y = 1$ we have $u = 3$, $v = 1$, and $w = 2$, so $\dfrac{\partial R}{\partial x} = \dfrac{9}{7}$ and $\dfrac{\partial R}{\partial y} = \dfrac{9}{7}$.

24. $M = xe^{y-z^2}$, $x = 2uv$, $y = u-v$, $z = u+v \quad \Rightarrow$

$$\frac{\partial M}{\partial u} = \frac{\partial M}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial u} = e^{y-z^2}(2v) + xe^{y-z^2}(1) + x(-2z)e^{y-z^2}(1) = e^{y-z^2}(2v+x-2xz),$$

$$\frac{\partial M}{\partial v} = \frac{\partial M}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial v} = e^{y-z^2}(2u) + xe^{y-z^2}(-1) + x(-2z)e^{y-z^2}(1) = e^{y-z^2}(2u-x-2xz).$$

When $u = 3$, $v = -1$ we have $x = -6$, $y = 4$, and $z = 2$, so $\dfrac{\partial M}{\partial u} = 16$ and $\dfrac{\partial M}{\partial v} = 36$.

25. $u = x^2 + yz$, $x = pr\cos\theta$, $y = pr\sin\theta$, $z = p+r \quad \Rightarrow$

$$\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial p} = (2x)(r\cos\theta) + (z)(r\sin\theta) + (y)(1) = 2xr\cos\theta + zr\sin\theta + y,$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} = (2x)(p\cos\theta) + (z)(p\sin\theta) + (y)(1) = 2xp\cos\theta + zp\sin\theta + y,$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} = (2x)(-pr\sin\theta) + (z)(pr\cos\theta) + (y)(0) = -2xpr\sin\theta + zpr\cos\theta.$$

When $p = 2$, $r = 3$, and $\theta = 0$ we have $x = 6$, $y = 0$, and $z = 5$, so $\dfrac{\partial u}{\partial p} = 36$, $\dfrac{\partial u}{\partial r} = 24$, and $\dfrac{\partial u}{\partial \theta} = 30$.

26. $Y = w\tan^{-1}(uv)$, $u = r+s$, $v = s+t$, $w = t+r \quad \Rightarrow$

$$\frac{\partial Y}{\partial r} = \frac{\partial Y}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial Y}{\partial w}\frac{\partial w}{\partial r} = \frac{w}{1+(uv)^2}(v)(1) + \frac{w}{1+(uv)^2}(u)(0) + \tan^{-1}(uv)(1) = \frac{vw}{1+u^2v^2} + \tan^{-1}(uv),$$

$$\frac{\partial Y}{\partial s} = \frac{\partial Y}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial Y}{\partial w}\frac{\partial w}{\partial s} = \frac{wv}{1+u^2v^2}(1) + \frac{wu}{1+u^2v^2}(1) + \tan^{-1}(uv)(0) = \frac{w(v+u)}{1+u^2v^2},$$

$$\frac{\partial Y}{\partial t} = \frac{\partial Y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial Y}{\partial w}\frac{\partial w}{\partial t} = \frac{wv}{1+u^2v^2}(0) + \frac{wu}{1+u^2v^2}(1) + \tan^{-1}(uv)(1) = \frac{wu}{1+u^2v^2} + \tan^{-1}(uv).$$

When $r = 1$, $s = 0$, and $t = 1$, we have $u = 1$, $v = 1$, and $w = 2$, so $\dfrac{\partial Y}{\partial r} = 1 + \dfrac{\pi}{4}$, $\dfrac{\partial Y}{\partial s} = 2$, and $\dfrac{\partial Y}{\partial t} = 1 + \dfrac{\pi}{4}$.

27. $\sqrt{xy} = 1 + x^2 y$, so let $F(x,y) = (xy)^{1/2} - 1 - x^2 y = 0$. Then by Equation 6

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{1}{2}(xy)^{-1/2}(y) - 2xy}{\frac{1}{2}(xy)^{-1/2}(x) - x^2} = -\frac{y - 4xy\sqrt{xy}}{x - 2x^2\sqrt{xy}} = \frac{4(xy)^{3/2} - y}{x - 2x^2\sqrt{xy}}.$$

28. $y^5 + x^2 y^3 = 1 + ye^{x^2}$, so let $F(x,y) = y^5 + x^2 y^3 - 1 - ye^{x^2} = 0$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2xy^3 - 2xye^{x^2}}{5y^4 + 3x^2y^2 - e^{x^2}} = \frac{2xye^{x^2} - 2xy^3}{5y^4 + 3x^2y^2 - e^{x^2}}.$$

29. $\cos(x - y) = xe^y$, so let $F(x,y) = \cos(x-y) - xe^y = 0$.

Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{-\sin(x-y) - e^y}{-\sin(x-y)(-1) - xe^y} = \dfrac{\sin(x-y) + e^y}{\sin(x-y) - xe^y}$.

30. $\sin x + \cos y = \sin x\, \cos y$, so let $F(x,y) = \sin x + \cos y - \sin x\, \cos y = 0$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\cos x - \cos x\, \cos y}{-\sin y + \sin x\, \sin y} = \frac{\cos x(\cos y - 1)}{\sin y(\sin x - 1)}.$$

31. $x^2 + y^2 + z^2 = 3xyz$, so let $F(x,y,z) = x^2 + y^2 + z^2 - 3xyz = 0$. Then by Equations 7

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x - 3yz}{2z - 3xy} = \frac{3yz - 2x}{2z - 3xy} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y - 3xz}{2z - 3xy} = \frac{3xz - 2y}{2z - 3xy}.$$

32. $xyz = \cos(x + y + z)$. Let $F(x,y,z) = xyz - \cos(x+y+z) = 0$, so

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}.$$

33. $x - z = \arctan(yz)$, so let $F(x,y,z) = x - z - \arctan(yz) = 0$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1}{-1 - \dfrac{1}{1 + (yz)^2}(y)} = \frac{1 + y^2z^2}{1 + y + y^2z^2}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-\dfrac{1}{1+(yz)^2}(z)}{-1 - \dfrac{1}{1+(yz)^2}(y)} = -\frac{\dfrac{z}{1 + y^2z^2}}{\dfrac{1 + y^2z^2 + y}{1 + y^2z^2}} = -\frac{z}{1 + y + y^2z^2}$$

34. $yz = \ln(x + z)$, so let $F(x,y,z) = yz - \ln(x+z) = 0$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-\dfrac{1}{x+z}(1)}{y - \dfrac{1}{x+z}(1)} = \frac{1}{y(x+z) - 1} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z}{y - \dfrac{1}{x+z}} = -\frac{z(x+z)}{y(x+z) - 1}$$

35. Since x and y are each functions of t, $T(x,y)$ is a function of t, so by the Chain Rule, $\dfrac{dT}{dt} = \dfrac{\partial T}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\dfrac{dy}{dt}$. After 3 seconds, $x = \sqrt{1+t} = \sqrt{1+3} = 2$, $y = 2 + \frac{1}{3}t = 2 + \frac{1}{3}(3) = 3$, $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{1+t}} = \dfrac{1}{2\sqrt{1+3}} = \dfrac{1}{4}$, and $\dfrac{dy}{dt} = \dfrac{1}{3}$.

Then $\dfrac{dT}{dt} = T_x(2,3)\,\dfrac{dx}{dt} + T_y(2,3)\,\dfrac{dy}{dt} = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) = 2$. Thus the temperature is rising at a rate of $2°\text{C/s}$.

36. (a) Since $\partial W/\partial T$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since $\partial W/\partial R$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.

(b) Since the average temperature is rising at a rate of $0.15°\text{C/year}$, we know that $dT/dt = 0.15$. Since rainfall is decreasing at a rate of 0.1 cm/year, we know $dR/dt = -0.1$. Then, by the Chain Rule,

$\dfrac{dW}{dt} = \dfrac{\partial W}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial W}{\partial R}\dfrac{dR}{dt} = (-2)(0.15) + (8)(-0.1) = -1.1$. Thus we estimate that wheat production will decrease at a rate of 1.1 units/year.

37. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11T + 0.00087T^2$ and $\dfrac{\partial C}{\partial D} = 0.016$.

According to the graph, the diver is experiencing a temperature of approximately $12.5°\text{C}$ at $t = 20$ minutes, so

$\dfrac{\partial C}{\partial T} = 4.6 - 0.11(12.5) + 0.00087(12.5)^2 \approx 3.36$. By sketching tangent lines at $t = 20$ to the graphs given, we estimate

$\dfrac{dD}{dt} \approx \dfrac{1}{2}$ and $\dfrac{dT}{dt} \approx -\dfrac{1}{10}$. Then, by the Chain Rule, $\dfrac{dC}{dt} = \dfrac{\partial C}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial C}{\partial D}\dfrac{dD}{dt} \approx (3.36)\left(-\frac{1}{10}\right) + (0.016)\left(\frac{1}{2}\right) \approx -0.33$.

Thus the speed of sound experienced by the diver is decreasing at a rate of approximately 0.33 m/s per minute.

38. $V = \pi r^2 h/3$, so $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt} = \dfrac{2\pi rh}{3}\,1.8 + \dfrac{\pi r^2}{3}(-2.5) = 20{,}160\pi - 12{,}000\pi = 8160\pi \text{ in}^3\text{/s}$.

39. (a) $V = \ell wh$, so by the Chain Rule,

$$\frac{dV}{dt} = \frac{\partial V}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = wh\,\frac{d\ell}{dt} + \ell h\,\frac{dw}{dt} + \ell w\,\frac{dh}{dt} = 2\cdot 2\cdot 2 + 1\cdot 2\cdot 2 + 1\cdot 2\cdot(-3) = 6 \text{ m}^3\text{/s}.$$

(b) $S = 2(\ell w + \ell h + wh)$, so by the Chain Rule,

$$\begin{aligned}\frac{dS}{dt} &= \frac{\partial S}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial S}{\partial w}\frac{dw}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt} = 2(w+h)\,\frac{d\ell}{dt} + 2(\ell+h)\,\frac{dw}{dt} + 2(\ell+w)\,\frac{dh}{dt}\\ &= 2(2+2)2 + 2(1+2)2 + 2(1+2)(-3) = 10 \text{ m}^2\text{/s}\end{aligned}$$

(c) $L^2 = \ell^2 + w^2 + h^2 \;\Rightarrow\; 2L\,\dfrac{dL}{dt} = 2\ell\,\dfrac{d\ell}{dt} + 2w\,\dfrac{dw}{dt} + 2h\,\dfrac{dh}{dt} = 2(1)(2) + 2(2)(2) + 2(2)(-3) = 0 \;\Rightarrow$ $dL/dt = 0$ m/s.

40. $I = \dfrac{V}{R} \;\Rightarrow$

$$\frac{dI}{dt} = \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{V}{R^2}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{I}{R}\frac{dR}{dt} = \frac{1}{400}(-0.01) - \frac{0.08}{400}(0.03) = -0.000031 \text{ A/s}$$

41. $\frac{dP}{dt} = 0.05$, $\frac{dT}{dt} = 0.15$, $V = 8.31\frac{T}{P}$ and $\frac{dV}{dt} = \frac{8.31}{P}\frac{dT}{dt} - 8.31\frac{T}{P^2}\frac{dP}{dt}$. Thus when $P = 20$ and $T = 320$,

$\frac{dV}{dt} = 8.31\left[\frac{0.15}{20} - \frac{(0.05)(320)}{400}\right] \approx -0.27$ L/s.

42. Let x and y be the respective distances of car A and car B from the intersection and let z be the distance between the two cars. Then $dx/dt = -90$, $dy/dt = -80$ and $z^2 = x^2 + y^2$. When $x = 0.3$ and $y = 0.4$, $z = \sqrt{0.25} = 0.5$ and $2z\,(dz/dt) = 2x\,(dx/dt) + 2y\,(dy/dt)$ or $dz/dt = 0.6(-90) + 0.8(-80) = -118$ km/h.

43. Let x be the length of the first side of the triangle and y the length of the second side. The area A of the triangle is given by $A = \frac{1}{2}xy\sin\theta$ where θ is the angle between the two sides. Thus A is a function of x, y, and θ, and x, y, and θ are each in turn functions of time t. We are given that $\frac{dx}{dt} = 3$, $\frac{dy}{dt} = -2$, and because A is constant, $\frac{dA}{dt} = 0$. By the Chain Rule,

$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt} \quad\Rightarrow\quad \frac{dA}{dt} = \frac{1}{2}y\sin\theta\cdot\frac{dx}{dt} + \frac{1}{2}x\sin\theta\cdot\frac{dy}{dt} + \frac{1}{2}xy\cos\theta\cdot\frac{d\theta}{dt}$. When $x = 20$, $y = 30$, and $\theta = \pi/6$ we have

$$\begin{aligned} 0 &= \tfrac{1}{2}(30)\left(\sin\tfrac{\pi}{6}\right)(3) + \tfrac{1}{2}(20)\left(\sin\tfrac{\pi}{6}\right)(-2) + \tfrac{1}{2}(20)(30)\left(\cos\tfrac{\pi}{6}\right)\frac{d\theta}{dt} \\ &= 45\cdot\tfrac{1}{2} - 20\cdot\tfrac{1}{2} + 300\cdot\frac{\sqrt{3}}{2}\cdot\frac{d\theta}{dt} = \tfrac{25}{2} + 150\sqrt{3}\,\frac{d\theta}{dt} \end{aligned}$$

Solving for $\frac{d\theta}{dt}$ gives $\frac{d\theta}{dt} = \frac{-25/2}{150\sqrt{3}} = -\frac{1}{12\sqrt{3}}$, so the angle between the sides is decreasing at a rate of $1/(12\sqrt{3}) \approx 0.048$ rad/s.

44. $f_o = \left(\frac{c + v_o}{c - v_s}\right)f_s = \left(\frac{332 + 34}{332 - 40}\right)460 \approx 576.6$ Hz. v_o and v_s are functions of time t, so

$$\begin{aligned} \frac{df_o}{dt} &= \frac{\partial f_o}{\partial v_o}\frac{dv_o}{dt} + \frac{\partial f_o}{\partial v_s}\frac{dv_s}{dt} = \left(\frac{1}{c - v_s}\right)f_s\cdot\frac{dv_o}{dt} + \frac{c + v_o}{(c - v_s)^2}\,f_s\cdot\frac{dv_s}{dt} \\ &= \left(\tfrac{1}{332-40}\right)(460)(1.2) + \tfrac{332+34}{(332-40)^2}(460)(1.4) \approx 4.65 \text{ Hz/s} \end{aligned}$$

45. (a) By the Chain Rule, $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$, $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}r\cos\theta$.

(b) $\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta$,

$\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}r^2\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. Thus

$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right](\cos^2\theta + \sin^2\theta) = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$.

46. By the Chain Rule, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\, e^s \cos t + \dfrac{\partial u}{\partial y}\, e^s \sin t$, $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}\,(-e^s \sin t) + \dfrac{\partial u}{\partial y}\, e^s \cos t$. Then

$\left(\dfrac{\partial u}{\partial s}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 e^{2s} \cos^2 t + 2\,\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}\, e^{2s} \cos t\, \sin t + \left(\dfrac{\partial u}{\partial y}\right)^2 e^{2s} \sin^2 t$ and

$\left(\dfrac{\partial u}{\partial t}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 e^{2s} \sin^2 t - 2\,\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}\, e^{2s} \cos t\, \sin t + \left(\dfrac{\partial u}{\partial y}\right)^2 e^{2s} \sin^2 t$. Thus

$\left[\left(\dfrac{\partial u}{\partial s}\right)^2 + \left(\dfrac{\partial u}{\partial t}\right)^2\right] e^{-2s} = \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2.$

47. Let $u = x - y$. Then $\dfrac{\partial z}{\partial x} = \dfrac{dz}{du}\dfrac{\partial u}{\partial x} = \dfrac{dz}{du}$ and $\dfrac{\partial z}{\partial y} = \dfrac{dz}{du}\,(-1)$. Thus $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

48. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}$ and $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}$. Thus $\dfrac{\partial z}{\partial s}\dfrac{\partial z}{\partial t} = \left(\dfrac{\partial z}{\partial x}\right)^2 - \left(\dfrac{\partial z}{\partial y}\right)^2$.

49. Let $u = x + at$, $v = x - at$. Then $z = f(u) + g(v)$, so $\partial z/\partial u = f'(u)$ and $\partial z/\partial v = g'(v)$.

Thus $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial t} = af'(u) - ag'(v)$ and

$\dfrac{\partial^2 z}{\partial t^2} = a\,\dfrac{\partial}{\partial t}\,[f'(u) - g'(v)] = a\left(\dfrac{df'(u)}{du}\dfrac{\partial u}{\partial t} - \dfrac{dg'(v)}{dv}\dfrac{\partial v}{\partial t}\right) = a^2 f''(u) + a^2 g''(v).$

Similarly $\dfrac{\partial z}{\partial x} = f'(u) + g'(v)$ and $\dfrac{\partial^2 z}{\partial x^2} = f''(u) + g''(v)$. Thus $\dfrac{\partial^2 z}{\partial t^2} = a^2\,\dfrac{\partial^2 z}{\partial x^2}$.

50. By the Chain Rule, $\dfrac{\partial u}{\partial s} = e^s \cos t \dfrac{\partial u}{\partial x} + e^s \sin t \dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial t} = -e^s \sin t \dfrac{\partial u}{\partial x} + e^s \cos t \dfrac{\partial u}{\partial y}$.

Then $\dfrac{\partial^2 u}{\partial s^2} = e^s \cos t \dfrac{\partial u}{\partial x} + e^s \cos t \dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial x}\right) + e^s \sin t \dfrac{\partial u}{\partial y} + e^s \sin t \dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial y}\right)$. But

$\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial x^2}\dfrac{\partial x}{\partial s} + \dfrac{\partial^2 u}{\partial y\,\partial x}\dfrac{\partial y}{\partial s} = e^s \cos t \dfrac{\partial^2 u}{\partial x^2} + e^s \sin t \dfrac{\partial^2 u}{\partial y\,\partial x}$ and

$\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial^2 u}{\partial y^2}\dfrac{\partial y}{\partial s} + \dfrac{\partial^2 u}{\partial x\,\partial y}\dfrac{\partial x}{\partial s} = e^s \sin t \dfrac{\partial^2 u}{\partial y^2} + e^s \cos t \dfrac{\partial^2 u}{\partial x\,\partial y}$.

Also, by continuity of the partials, $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$. Thus

$$\frac{\partial^2 u}{\partial s^2} = e^s \cos t \frac{\partial u}{\partial x} + e^s \cos t\left(e^s \cos t \frac{\partial^2 u}{\partial x^2} + e^s \sin t \frac{\partial^2 u}{\partial x\,\partial y}\right) + e^s \sin t \frac{\partial u}{\partial y} + e^s \sin t\left(e^s \sin t \frac{\partial^2 u}{\partial y^2} + e^s \cos t \frac{\partial^2 u}{\partial x\,\partial y}\right)$$

$$= e^s \cos t \frac{\partial u}{\partial x} + e^s \sin t \frac{\partial u}{\partial y} + e^{2s} \cos^2 t \frac{\partial^2 u}{\partial x^2} + 2e^{2s} \cos t \sin t \frac{\partial^2 u}{\partial x\,\partial y} + e^{2s} \sin^2 t \frac{\partial^2 u}{\partial y^2}$$

Similarly

$$\frac{\partial^2 u}{\partial t^2} = -e^s \cos t \frac{\partial u}{\partial x} - e^s \sin t \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - e^s \sin t \frac{\partial u}{\partial y} + e^s \cos t \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)$$

$$= -e^s \cos t \frac{\partial u}{\partial x} - e^s \sin t\left(-e^s \sin t \frac{\partial^2 u}{\partial x^2} + e^s \cos t \frac{\partial^2 u}{\partial x\,\partial y}\right)$$

$$-e^s \sin t \frac{\partial u}{\partial y} + e^s \cos t\left(e^s \cos t \frac{\partial^2 u}{\partial y^2} - e^s \sin t \frac{\partial^2 u}{\partial x\,\partial y}\right)$$

$$= -e^s \cos t \frac{\partial u}{\partial x} - e^s \sin t \frac{\partial u}{\partial y} + e^{2s} \sin^2 t \frac{\partial^2 u}{\partial x^2} - 2e^{2s} \cos t \sin t \frac{\partial^2 u}{\partial x\,\partial y} + e^{2s} \cos^2 t \frac{\partial^2 u}{\partial y^2}$$

Thus $e^{-2s}\left(\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2}\right) = (\cos^2 t + \sin^2 t)\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$, as desired.