

51.  $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} 2s + \frac{\partial z}{\partial y} 2r$ . Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r \partial s} &= \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial x} 2s \right) + \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial y} 2r \right) \\ &= \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial r} 2s + \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) \frac{\partial y}{\partial r} 2s + \frac{\partial z}{\partial x} \frac{\partial}{\partial r} 2s + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial r} 2r + \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) \frac{\partial x}{\partial r} 2r + \frac{\partial z}{\partial y} 2 \\ &= 4rs \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y \partial x} 4s^2 + 0 + 4rs \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} 4r^2 + 2 \frac{\partial z}{\partial y}\end{aligned}$$

By the continuity of the partials,  $\frac{\partial^2 z}{\partial r \partial s} = 4rs \frac{\partial^2 z}{\partial x^2} + 4rs \frac{\partial^2 z}{\partial y^2} + (4r^2 + 4s^2) \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial z}{\partial y}$ .

52. By the Chain Rule,

(a)  $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$

(b)  $\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} r \sin \theta + \frac{\partial z}{\partial y} r \cos \theta$

(c) 
$$\begin{aligned}\frac{\partial^2 z}{\partial r \partial \theta} &= \frac{\partial^2 z}{\partial \theta \partial r} = \frac{\partial}{\partial \theta} \left( \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta \right) = -\sin \theta \frac{\partial z}{\partial x} + \cos \theta \frac{\partial}{\partial \theta} \left( \frac{\partial z}{\partial x} \right) + \cos \theta \frac{\partial z}{\partial y} + \sin \theta \frac{\partial}{\partial \theta} \left( \frac{\partial z}{\partial y} \right) \\ &= -\sin \theta \frac{\partial z}{\partial x} + \cos \theta \left( \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial \theta} \right) + \cos \theta \frac{\partial z}{\partial y} + \sin \theta \left( \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial \theta} + \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial \theta} \right) \\ &= -\sin \theta \frac{\partial z}{\partial x} + \cos \theta \left( -r \sin \theta \frac{\partial^2 z}{\partial x^2} + r \cos \theta \frac{\partial^2 z}{\partial y \partial x} \right) + \cos \theta \frac{\partial z}{\partial y} + \sin \theta \left( r \cos \theta \frac{\partial^2 z}{\partial y^2} - r \sin \theta \frac{\partial^2 z}{\partial x \partial y} \right) \\ &= -\sin \theta \frac{\partial z}{\partial x} - r \cos \theta \sin \theta \frac{\partial^2 z}{\partial x^2} + r \cos^2 \theta \frac{\partial^2 z}{\partial y \partial x} + \cos \theta \frac{\partial z}{\partial y} + r \cos \theta \sin \theta \frac{\partial^2 z}{\partial y^2} - r \sin^2 \theta \frac{\partial^2 z}{\partial x \partial y} \\ &= \cos \theta \frac{\partial z}{\partial y} - \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \sin \theta \left( \frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2} \right) + r(\cos^2 \theta - \sin^2 \theta) \frac{\partial^2 z}{\partial y \partial x}\end{aligned}$$

53.  $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$  and  $\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} r \sin \theta + \frac{\partial z}{\partial y} r \cos \theta$ . Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \cos \theta \left( \frac{\partial^2 z}{\partial x^2} \cos \theta + \frac{\partial^2 z}{\partial y \partial x} \sin \theta \right) + \sin \theta \left( \frac{\partial^2 z}{\partial y^2} \sin \theta + \frac{\partial^2 z}{\partial x \partial y} \cos \theta \right) \\ &= \cos^2 \theta \frac{\partial^2 z}{\partial x^2} + 2 \cos \theta \sin \theta \frac{\partial^2 z}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 z}{\partial y^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 z}{\partial \theta^2} &= -r \cos \theta \frac{\partial z}{\partial x} + (-r \sin \theta) \left( \frac{\partial^2 z}{\partial x^2} (-r \sin \theta) + \frac{\partial^2 z}{\partial y \partial x} r \cos \theta \right) \\ &\quad -r \sin \theta \frac{\partial z}{\partial y} + r \cos \theta \left( \frac{\partial^2 z}{\partial y^2} r \cos \theta + \frac{\partial^2 z}{\partial x \partial y} (-r \sin \theta) \right) \\ &= -r \cos \theta \frac{\partial z}{\partial x} - r \sin \theta \frac{\partial z}{\partial y} + r^2 \sin^2 \theta \frac{\partial^2 z}{\partial x^2} - 2r^2 \cos \theta \sin \theta \frac{\partial^2 z}{\partial x \partial y} + r^2 \cos^2 \theta \frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r} \frac{\partial z}{\partial r} &= (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2 z}{\partial x^2} + (\sin^2 \theta + \cos^2 \theta) \frac{\partial^2 z}{\partial y^2} \\ &\quad - \frac{1}{r} \cos \theta \frac{\partial z}{\partial x} - \frac{1}{r} \sin \theta \frac{\partial z}{\partial y} + \frac{1}{r} \left( \cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} \right) \\ &= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \text{ as desired.}\end{aligned}$$

54. (a)  $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$ . Then

$$\begin{aligned}\frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t} \left( \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} \right) + \frac{\partial}{\partial t} \left( \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \right) = \frac{\partial}{\partial t} \left( \frac{\partial z}{\partial x} \right) \frac{\partial x}{\partial t} + \frac{\partial^2 x}{\partial t^2} \frac{\partial z}{\partial x} + \frac{\partial}{\partial t} \left( \frac{\partial z}{\partial y} \right) \frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2} \frac{\partial z}{\partial y} \\ &= \frac{\partial^2 z}{\partial x^2} \left( \frac{\partial x}{\partial t} \right)^2 + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 x}{\partial t^2} \frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} \left( \frac{\partial y}{\partial t} \right)^2 + \frac{\partial^2 z}{\partial x \partial y} \frac{\partial y}{\partial t} \frac{\partial x}{\partial t} + \frac{\partial^2 y}{\partial t^2} \frac{\partial z}{\partial y} \\ &= \frac{\partial^2 z}{\partial x^2} \left( \frac{\partial x}{\partial t} \right)^2 + 2 \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2} \left( \frac{\partial y}{\partial t} \right)^2 + \frac{\partial^2 x}{\partial t^2} \frac{\partial z}{\partial x} + \frac{\partial^2 y}{\partial t^2} \frac{\partial z}{\partial y}\end{aligned}$$

(b)  $\frac{\partial^2 z}{\partial s \partial t} = \frac{\partial}{\partial s} \left( \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \right)$

$$\begin{aligned}&= \left( \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial x} \frac{\partial x}{\partial s} \right) \frac{\partial x}{\partial t} + \frac{\partial z}{\partial x} \frac{\partial^2 x}{\partial s \partial t} + \left( \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial s} + \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial y} \frac{\partial y}{\partial s} \right) \frac{\partial y}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial^2 y}{\partial s \partial t} \\ &= \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial s} \frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x \partial y} \left( \frac{\partial y}{\partial s} \frac{\partial x}{\partial t} + \frac{\partial y}{\partial t} \frac{\partial x}{\partial s} \right) + \frac{\partial z}{\partial x} \frac{\partial^2 x}{\partial s \partial t} + \frac{\partial z}{\partial y} \frac{\partial^2 y}{\partial s \partial t} + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial s} \frac{\partial y}{\partial t}\end{aligned}$$

55. (a) Since  $f$  is a polynomial, it has continuous second-order partial derivatives, and

$$f(tx, ty) = (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3 = t^3x^2y + 2t^3xy^2 + 5t^3y^3 = t^3(x^2y + 2xy^2 + 5y^3) = t^3f(x, y).$$

Thus,  $f$  is homogeneous of degree 3.

(b) Differentiating both sides of  $f(tx, ty) = t^n f(x, y)$  with respect to  $t$  using the Chain Rule, we get

$$\frac{\partial}{\partial t} f(tx, ty) = \frac{\partial}{\partial t} [t^n f(x, y)] \Leftrightarrow$$

$$\frac{\partial}{\partial(tx)} f(tx, ty) \cdot \frac{\partial(tx)}{\partial t} + \frac{\partial}{\partial(ty)} f(tx, ty) \cdot \frac{\partial(ty)}{\partial t} = x \frac{\partial}{\partial(tx)} f(tx, ty) + y \frac{\partial}{\partial(ty)} f(tx, ty) = nt^{n-1} f(x, y).$$

Setting  $t = 1$ :  $x \frac{\partial}{\partial x} f(x, y) + y \frac{\partial}{\partial y} f(x, y) = nf(x, y)$ .

56. Differentiating both sides of  $f(tx, ty) = t^n f(x, y)$  with respect to  $t$  using the Chain Rule, we get

$$\frac{\partial}{\partial(tx)} f(tx, ty) \cdot \frac{\partial(tx)}{\partial t} + \frac{\partial}{\partial(ty)} f(tx, ty) \cdot \frac{\partial(ty)}{\partial t} = x \frac{\partial}{\partial(tx)} f(tx, ty) + y \frac{\partial}{\partial(ty)} f(tx, ty) = nt^{n-1} f(x, y) \text{ and}$$

differentiating again with respect to  $t$  gives

$$x \left[ \frac{\partial^2}{\partial(tx)^2} f(tx, ty) \cdot \frac{\partial(tx)}{\partial t} + \frac{\partial^2}{\partial(tx)\partial(ty)} f(tx, ty) \cdot \frac{\partial(ty)}{\partial t} \right] + y \left[ \frac{\partial^2}{\partial(tx)\partial(ty)} f(tx, ty) \cdot \frac{\partial(tx)}{\partial t} + \frac{\partial^2}{\partial(ty)^2} f(tx, ty) \cdot \frac{\partial(ty)}{\partial t} \right] = n(n-1)t^{n-1} f(x, y).$$

Setting  $t = 1$  and using the fact that  $f_{yx} = f_{xy}$ , we have  $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1)f(x, y)$ .

57. Differentiating both sides of  $f(tx, ty) = t^n f(x, y)$  with respect to  $x$  using the Chain Rule, we get

$$\frac{\partial}{\partial x} f(tx, ty) = \frac{\partial}{\partial x} [t^n f(x, y)] \Leftrightarrow$$

$$\frac{\partial}{\partial(tx)} f(tx, ty) \cdot \frac{\partial(tx)}{\partial x} + \frac{\partial}{\partial(ty)} f(tx, ty) \cdot \frac{\partial(ty)}{\partial x} = t^n \frac{\partial}{\partial x} f(x, y) \Leftrightarrow t f_x(tx, ty) = t^n f_x(x, y).$$

Thus  $f_x(tx, ty) = t^{n-1} f_x(x, y)$ .

58.  $F(x, y, z) = 0$  is assumed to define  $z$  as a function of  $x$  and  $y$ , that is,  $z = f(x, y)$ . So by (7),  $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$  since  $F_z \neq 0$ .

Similarly, it is assumed that  $F(x, y, z) = 0$  defines  $x$  as a function of  $y$  and  $z$ , that is  $x = h(y, z)$ . Then  $F(h(y, z), y, z) = 0$

and by the Chain Rule,  $F_x \frac{\partial x}{\partial y} + F_y \frac{\partial y}{\partial y} + F_z \frac{\partial z}{\partial y} = 0$ . But  $\frac{\partial z}{\partial y} = 0$  and  $\frac{\partial y}{\partial y} = 1$ , so  $F_x \frac{\partial x}{\partial y} + F_y = 0 \Rightarrow \frac{\partial x}{\partial y} = -\frac{F_y}{F_x}$ .

A similar calculation shows that  $\frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$ . Thus  $\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = \left(-\frac{F_x}{F_z}\right) \left(-\frac{F_y}{F_x}\right) \left(-\frac{F_z}{F_y}\right) = -1$ .