1. We can approximate the directional derivative of the pressure function at K in the direction of S by the average rate of change of pressure between the points where the red line intersects the contour lines closest to K (extend the red line slightly at the left). In the direction of S, the pressure changes from 1000 millibars to 996 millibars and we estimate the distance between these two points to be approximately 50 km (using the fact that the distance from K to S is 300 km). Then the rate of change of pressure in the direction given is approximately $\frac{996-1000}{50} = -0.08$ millibar/km.

2. First we draw a line passing through Dubbo and Sydney. We approximate the directional derivative at Dubbo in the direction of Sydney by the average rate of change of temperature between the points where the line intersects the contour lines closest to Dubbo. In the direction of Sydney, the temperature changes from 30°C to 27°C. We estimate the distance between these two points to be approximately 120 km, so the rate of change of maximum temperature in the direction given is approximately $\frac{27-30}{120} = -0.025°\text{C/km}$.

3. $D_{\mathbf{u}}\, f(-20, 30) = \nabla f\,(-20, 30) \cdot \mathbf{u} = f_T(-20, 30)\left(\frac{1}{\sqrt{2}}\right) + f_v(-20, 30)\left(\frac{1}{\sqrt{2}}\right)$.

$f_T(-20, 30) = \lim\limits_{h \to 0} \dfrac{f(-20+h, 30) - f(-20, 30)}{h}$, so we can approximate $f_T(-20, 30)$ by considering $h = \pm 5$ and using the values given in the table: $f_T(-20, 30) \approx \dfrac{f(-15, 30) - f(-20, 30)}{5} = \dfrac{-26 - (-33)}{5} = 1.4$,

$f_T(-20, 30) \approx \dfrac{f(-25, 30) - f(-20, 30)}{-5} = \dfrac{-39 - (-33)}{-5} = 1.2$. Averaging these values gives $f_T(-20, 30) \approx 1.3$.

Similarly, $f_v(-20, 30) = \lim\limits_{h \to 0} \dfrac{f(-20, 30+h) - f(-20, 30)}{h}$, so we can approximate $f_v(-20, 30)$ with $h = \pm 10$:

$f_v(-20, 30) \approx \dfrac{f(-20, 40) - f(-20, 30)}{10} = \dfrac{-34 - (-33)}{10} = -0.1$,

$f_v(-20, 30) \approx \dfrac{f(-20, 20) - f(-20, 30)}{-10} = \dfrac{-30 - (-33)}{-10} = -0.3$. Averaging these values gives $f_v(-20, 30) \approx -0.2$.

Then $D_{\mathbf{u}} f(-20, 30) \approx 1.3\left(\frac{1}{\sqrt{2}}\right) + (-0.2)\left(\frac{1}{\sqrt{2}}\right) \approx 0.778$.

4. $f(x, y) = x^2y^3 - y^4 \quad \Rightarrow \quad f_x(x, y) = 2xy^3$ and $f_y(x, y) = 3x^2y^2 - 4y^3$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \frac{\pi}{4}$, then from Equation 6, $D_{\mathbf{u}} f(2, 1) = f_x(2, 1)\cos\left(\frac{\pi}{4}\right) + f_y(2, 1)\sin\left(\frac{\pi}{4}\right) = 4 \cdot \frac{\sqrt{2}}{2} + 8 \cdot \frac{\sqrt{2}}{2} = 6\sqrt{2}$.

5. $f(x, y) = ye^{-x} \quad \Rightarrow \quad f_x(x, y) = -ye^{-x}$ and $f_y(x, y) = e^{-x}$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = 2\pi/3$, then from Equation 6, $D_{\mathbf{u}}\, f(0, 4) = f_x(0, 4)\cos\left(\frac{2\pi}{3}\right) + f_y(0, 4)\sin\left(\frac{2\pi}{3}\right) = -4 \cdot \left(-\frac{1}{2}\right) + 1 \cdot \frac{\sqrt{3}}{2} = 2 + \frac{\sqrt{3}}{2}$.

6. $f(x, y) = x\sin(xy) \quad \Rightarrow \quad f_x(x, y) = x\cos(xy) \cdot y + \sin(xy) = xy\cos(xy) + \sin(xy)$ and $f_y(x, y) = x\cos(xy) \cdot x = x^2\cos(xy)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \frac{\pi}{3}$, then from Equation 6, $D_{\mathbf{u}}\, f(2, 0) = f_x(2, 0)\cos\frac{\pi}{3} + f_y(2, 0)\sin\frac{\pi}{3} = 0 + 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$.

7. $f(x, y) = \sin(2x + 3y)$

(a) $\nabla f(x, y) = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} = [\cos(2x + 3y) \cdot 2]\,\mathbf{i} + [\cos(2x + 3y) \cdot 3]\,\mathbf{j} = 2\cos(2x + 3y)\,\mathbf{i} + 3\cos(2x + 3y)\,\mathbf{j}$

(b) $\nabla f(-6, 4) = (2\cos 0)\mathbf{i} + (3\cos 0)\mathbf{j} = 2\,\mathbf{i} + 3\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\, f(-6, 4) = \nabla f(-6, 4) \cdot \mathbf{u} = (2\,\mathbf{i} + 3\,\mathbf{j}) \cdot \frac{1}{2}(\sqrt{3}\,\mathbf{i} - \mathbf{j}) = \frac{1}{2}(2\sqrt{3} - 3) = \sqrt{3} - \frac{3}{2}$.

8. $f(x, y) = y^2/x$

(a) $\nabla f(x, y) = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} = y^2(-x^{-2})\mathbf{i} + (2y/x)\,\mathbf{j} = -\dfrac{y^2}{x^2}\mathbf{i} + \dfrac{2y}{x}\mathbf{j}$

(b) $\nabla f(1, 2) = -4\,\mathbf{i} + 4\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\, f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = (-4\,\mathbf{i} + 4\,\mathbf{j}) \cdot \frac{1}{3}(2\,\mathbf{i} + \sqrt{5}\,\mathbf{j}) = \frac{1}{3}(-8 + 4\sqrt{5}) = \frac{4}{3}(\sqrt{5} - 2)$.

9. $f(x, y, z) = xe^{2yz}$

(a) $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle = \langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz}\rangle$

(b) $\nabla f(3, 0, 2) = \langle 1, 12, 0\rangle$

(c) By Equation 14, $D_{\mathbf{u}} f(3, 0, 2) = \nabla f(3, 0, 2) \cdot \mathbf{u} = \langle 1, 12, 0\rangle \cdot \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\rangle = \frac{2}{3} - \frac{24}{3} + 0 = -\frac{22}{3}$.

10. $f(x, y, z) = \sqrt{x + yz} = (x + yz)^{1/2}$

(a) $\nabla f(x, y, z) = \left\langle \frac{1}{2}(x + yz)^{-1/2}(1), \frac{1}{2}(x + yz)^{-1/2}(z), \frac{1}{2}(x + yz)^{-1/2}(y)\right\rangle$

$= \langle 1/(2\sqrt{x + yz}\,), z/(2\sqrt{x + yz}\,), y/(2\sqrt{x + yz}\,)\rangle$

(b) $\nabla f(1, 3, 1) = \langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4}\rangle$

(c) $D_{\mathbf{u}} f(1, 3, 1) = \nabla f(1, 3, 1) \cdot \mathbf{u} = \langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4}\rangle \cdot \langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7}\rangle = \frac{2}{28} + \frac{3}{28} + \frac{18}{28} = \frac{23}{28}$

11. $f(x, y) = 1 + 2x\sqrt{y} \quad \Rightarrow \quad \nabla f(x, y) = \left\langle 2\sqrt{y}, 2x \cdot \frac{1}{2}y^{-1/2}\right\rangle = \left\langle 2\sqrt{y}, x/\sqrt{y}\right\rangle$, $\nabla f(3, 4) = \langle 4, \frac{3}{2}\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \dfrac{1}{\sqrt{4^2 + (-3)^2}}\langle 4, -3\rangle = \langle \frac{4}{5}, -\frac{3}{5}\rangle$, so $D_{\mathbf{u}}\, f(3, 4) = \nabla f(3, 4) \cdot \mathbf{u} = \langle 4, \frac{3}{2}\rangle \cdot \langle \frac{4}{5}, -\frac{3}{5}\rangle = \frac{23}{10}$.

12. $f(x, y) = \ln(x^2 + y^2) \quad \Rightarrow \quad \nabla f(x, y) = \left\langle \dfrac{2x}{x^2 + y^2}, \dfrac{2y}{x^2 + y^2}\right\rangle$, $\nabla f(2, 1) = \langle \frac{4}{5}, \frac{2}{5}\rangle$, and a unit vector in the direction of $\mathbf{v} = \langle -1, 2\rangle$ is $\mathbf{u} = \dfrac{1}{\sqrt{1 + 4}}\langle -1, 2\rangle = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$, so

$D_{\mathbf{u}}\, f(2, 1) = \nabla f(2, 1) \cdot \mathbf{u} = \langle \frac{4}{5}, \frac{2}{5}\rangle \cdot \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle = -\frac{4}{5\sqrt{5}} + \frac{4}{5\sqrt{5}} = 0$.

13. $g(p, q) = p^4 - p^2q^3 \quad \Rightarrow \quad \nabla g(p, q) = (4p^3 - 2pq^3)\,\mathbf{i} + (-3p^2q^2)\,\mathbf{j}$, $\nabla g(2, 1) = 28\,\mathbf{i} - 12\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{1^2 + 3^2}}(\mathbf{i} + 3\,\mathbf{j}) = \frac{1}{\sqrt{10}}(\mathbf{i} + 3\,\mathbf{j})$, so

$D_{\mathbf{u}}\, g(2, 1) = \nabla g(2, 1) \cdot \mathbf{u} = (28\,\mathbf{i} - 12\,\mathbf{j}) \cdot \frac{1}{\sqrt{10}}(\mathbf{i} + 3\,\mathbf{j}) = \frac{1}{\sqrt{10}}(28 - 36) = -\frac{8}{\sqrt{10}}$ or $-\frac{4\sqrt{10}}{5}$.

14. $g(r,s)=\tan^{-1}(rs) \Rightarrow \nabla g(r,s)=\left(\dfrac{1}{1+(rs)^2}\cdot s\right)\mathbf{i}+\left(\dfrac{1}{1+(rs)^2}\cdot r\right)\mathbf{j}=\dfrac{s}{1+r^2s^2}\mathbf{i}+\dfrac{r}{1+r^2s^2}\mathbf{j}$,

$\nabla g(1,2)=\frac{2}{5}\,\mathbf{i}+\frac{1}{5}\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u}=\frac{1}{\sqrt{5^2+10^2}}(5\,\mathbf{i}+10\,\mathbf{j})=\frac{1}{5\sqrt{5}}(5\,\mathbf{i}+10\,\mathbf{j})=\frac{1}{\sqrt{5}}\,\mathbf{i}+\frac{2}{\sqrt{5}}\,\mathbf{j}$,

so $D_{\mathbf{u}}\,g(1,2)=\nabla g(1,2)\cdot\mathbf{u}=(\frac{2}{5}\,\mathbf{i}+\frac{1}{5}\,\mathbf{j})\cdot(\frac{1}{\sqrt{5}}\,\mathbf{i}+\frac{2}{\sqrt{5}}\,\mathbf{j})=\frac{2}{5\sqrt{5}}+\frac{2}{5\sqrt{5}}=\frac{4}{5\sqrt{5}}$ or $\frac{4\sqrt{5}}{25}$.

15. $f(x,y,z)=xe^y+ye^z+ze^x \Rightarrow \nabla f(x,y,z)=\langle e^y+ze^x, xe^y+e^z, ye^z+e^x\rangle$, $\nabla f(0,0,0)=\langle 1,1,1\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u}=\frac{1}{\sqrt{25+1+4}}\langle 5,1,-2\rangle=\frac{1}{\sqrt{30}}\langle 5,1,-2\rangle$, so

$D_{\mathbf{u}}\,f(0,0,0)=\nabla f(0,0,0)\cdot\mathbf{u}=\langle 1,1,1\rangle\cdot\frac{1}{\sqrt{30}}\langle 5,1,-2\rangle=\frac{4}{\sqrt{30}}$.

16. $f(x,y,z)=\sqrt{xyz} \Rightarrow$

$\nabla f(x,y,z)=\left\langle \frac{1}{2}(xyz)^{-1/2}\cdot yz, \frac{1}{2}(xyz)^{-1/2}\cdot xz, \frac{1}{2}(xyz)^{-1/2}\cdot xy\right\rangle=\left\langle \dfrac{yz}{2\sqrt{xyz}}, \dfrac{xz}{2\sqrt{xyz}}, \dfrac{xy}{2\sqrt{xyz}}\right\rangle$,

$\nabla f(3,2,6)=\left\langle \frac{12}{2\sqrt{36}}, \frac{18}{2\sqrt{36}}, \frac{6}{2\sqrt{36}}\right\rangle=\langle 1,\frac{3}{2},\frac{1}{2}\rangle$, and a unit vector in the

direction of $\mathbf{v}$ is $\mathbf{u}=\frac{1}{\sqrt{1+4+4}}\langle -1,-2,2\rangle=\langle -\frac{1}{3},-\frac{2}{3},\frac{2}{3}\rangle$, so

$D_{\mathbf{u}}\,f(3,2,6)=\nabla f(3,2,6)\cdot\mathbf{u}=\langle 1,\frac{3}{2},\frac{1}{2}\rangle\cdot\langle -\frac{1}{3},-\frac{2}{3},\frac{2}{3}\rangle=-\frac{1}{3}-1+\frac{1}{3}=-1$.

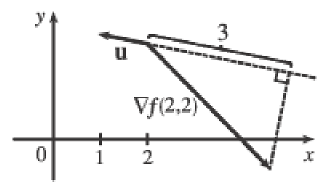
17. $g(x,y,z)=(x+2y+3z)^{3/2} \Rightarrow$

$$\nabla g(x,y,z)=\left\langle \tfrac{3}{2}(x+2y+3z)^{1/2}(1), \tfrac{3}{2}(x+2y+3z)^{1/2}(2), \tfrac{3}{2}(x+2y+3z)^{1/2}(3)\right\rangle$$
$$=\langle \tfrac{3}{2}\sqrt{x+2y+3z}, 3\sqrt{x+2y+3z}, \tfrac{9}{2}\sqrt{x+2y+3z}\,\rangle,\ \nabla g(1,1,2)=\langle \tfrac{9}{2},9,\tfrac{27}{2}\rangle,$$

and a unit vector in the direction of $\mathbf{v}=2\mathbf{j}-\mathbf{k}$ is $\mathbf{u}=\frac{2}{\sqrt{5}}\,\mathbf{j}-\frac{1}{\sqrt{5}}\,\mathbf{k}$, so

$D_{\mathbf{u}}\,g(1,1,2)=\langle \frac{9}{2},9,\frac{27}{2}\rangle\cdot\left\langle 0,\frac{2}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right\rangle=\frac{18}{\sqrt{5}}-\frac{27}{2\sqrt{5}}=\frac{9}{2\sqrt{5}}$.

18. $D_{\mathbf{u}}f(2,2)=\nabla f(2,2)\cdot\mathbf{u}$, the scalar projection of $\nabla f(2,2)$ onto $\mathbf{u}$, so we draw a perpendicular from the tip of $\nabla f(2,2)$ to the line containing $\mathbf{u}$. We can use the point $(2,2)$ to determine the scale of the axes, and we estimate the length of the projection to be approximately 3.0 units. Since the angle between $\nabla f(2,2)$ and $\mathbf{u}$ is greater than $90°$, the scalar projection is negative. Thus $D_{\mathbf{u}}\,f(2,2)\approx -3$.



19. $f(x,y)=\sqrt{xy} \Rightarrow \nabla f(x,y)=\left\langle \frac{1}{2}(xy)^{-1/2}(y), \frac{1}{2}(xy)^{-1/2}(x)\right\rangle=\left\langle \dfrac{y}{2\sqrt{xy}}, \dfrac{x}{2\sqrt{xy}}\right\rangle$, so $\nabla f(2,8)=\langle 1,\frac{1}{4}\rangle$.

The unit vector in the direction of $\overrightarrow{PQ}=\langle 5-2,4-8\rangle=\langle 3,-4\rangle$ is $\mathbf{u}=\langle \frac{3}{5},-\frac{4}{5}\rangle$, so

$D_{\mathbf{u}}\,f(2,8)=\nabla f(2,8)\cdot\mathbf{u}=\langle 1,\frac{1}{4}\rangle\cdot\langle \frac{3}{5},-\frac{4}{5}\rangle=\frac{2}{5}$.

20. $f(x,y,z) = xy + yz + zx \;\Rightarrow\; \nabla f(x,y,z) = \langle y+z, x+z, y+x\rangle$, so $\nabla f(1,-1,3) = \langle 2,4,0\rangle$. The unit vector in the direction of $\overrightarrow{PQ} = \langle 1,5,2\rangle$ is $\mathbf{u} = \frac{1}{\sqrt{30}}\langle 1,5,2\rangle$, so $D_{\mathbf{u}}\, f(1,-1,3) = \nabla f(1,-1,3)\cdot\mathbf{u} = \langle 2,4,0\rangle\cdot\frac{1}{\sqrt{30}}\langle 1,5,2\rangle = \frac{22}{\sqrt{30}}$.

21. $f(x,y) = y^2/x = y^2x^{-1} \;\Rightarrow\; \nabla f(x,y) = \langle -y^2x^{-2}, 2yx^{-1}\rangle = \langle -y^2/x^2, 2y/x\rangle$.
$\nabla f(2,4) = \langle -4,4\rangle$, or equivalently $\langle -1,1\rangle$, is the direction of maximum rate of change, and the maximum rate is $|\nabla f(2,4)| = \sqrt{16+16} = 4\sqrt{2}$.

22. $f(p,q) = qe^{-p} + pe^{-q} \;\Rightarrow\; \nabla f(p,q) = \langle -qe^{-p} + e^{-q}, e^{-p} - pe^{-q}\rangle$.
$\nabla f(0,0) = \langle 1,1\rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(0,0)| = \sqrt{2}$.

23. $f(x,y) = \sin(xy) \;\Rightarrow\; \nabla f(x,y) = \langle y\cos(xy), x\cos(xy)\rangle$, $\nabla f(1,0) = \langle 0,1\rangle$. Thus the maximum rate of change is $|\nabla f(1,0)| = 1$ in the direction $\langle 0,1\rangle$.

24. $f(x,y,z) = \dfrac{x+y}{z} \;\Rightarrow\; \nabla f(x,y,z) = \left\langle \dfrac{1}{z}, \dfrac{1}{z}, -\dfrac{x+y}{z^2}\right\rangle$, $\nabla f(1,1,-1) = \langle -1,-1,-2\rangle$. Thus the maximum rate of change is $|\nabla f(1,1,-1)| = \sqrt{1+1+4} = \sqrt{6}$ in the direction $\langle -1,-1,-2\rangle$.

25. $f(x,y,z) = \sqrt{x^2+y^2+z^2} \;\Rightarrow$

$$\nabla f(x,y,z) = \left\langle \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x, \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2y, \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2z\right\rangle$$
$$= \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right\rangle.$$

$\nabla f(3,6,-2) = \left\langle \frac{3}{\sqrt{49}}, \frac{6}{\sqrt{49}}, \frac{-2}{\sqrt{49}}\right\rangle = \left\langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7}\right\rangle$. Thus the maximum rate of change is

$|\nabla f(3,6,-2)| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2 + \left(-\frac{2}{7}\right)^2} = \sqrt{\frac{9+36+4}{49}} = 1$ in the direction $\left\langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7}\right\rangle$ or equivalently $\langle 3,6,-2\rangle$.

26. $f(x,y,z) = \tan(x+2y+3z) \;\Rightarrow$
$\nabla f(x,y,z) = \langle \sec^2(x+2y+3z)(1), \sec^2(x+2y+3z)(2), \sec^2(x+2y+3z)(3)\rangle$.
$\nabla f(-5,1,1) = \langle \sec^2(0), 2\sec^2(0), 3\sec^2(0)\rangle = \langle 1,2,3\rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(-5,1,1)| = \sqrt{14}$.

27. (a) As in the proof of Theorem 15, $D_{\mathbf{u}}\, f = |\nabla f|\cos\theta$. Since the minimum value of $\cos\theta$ is -1 occurring when $\theta = \pi$, the minimum value of $D_{\mathbf{u}}\, f$ is $-|\nabla f|$ occurring when $\theta = \pi$, that is when $\mathbf{u}$ is in the opposite direction of ∇f (assuming $\nabla f \neq \mathbf{0}$).

(b) $f(x,y) = x^4y - x^2y^3 \;\Rightarrow\; \nabla f(x,y) = \langle 4x^3y - 2xy^3, x^4 - 3x^2y^2\rangle$, so f decreases fastest at the point $(2,-3)$ in the direction $-\nabla f(2,-3) = -\langle 12,-92\rangle = \langle -12, 92\rangle$.

28. $f(x, y) = ye^{-xy} \;\Rightarrow\; f_x(x, y) = ye^{-xy}(-y) = -y^2e^{-xy},\;\; f_y(x, y) = ye^{-xy}(-x) + e^{-xy} = (1 - xy)e^{-xy}$ and $f_x(0, 2) = -4e^0 = -4,\;\; f_y(0, 2) = (1 - 0)e^0 = 1$. If $\mathbf{u}$ is a unit vector which makes an angle θ with the positive x-axis, then $D_\mathbf{u}f(0, 2) = f_x(0, 2)\cos\theta + f_y(0, 2)\sin\theta = -4\cos\theta + \sin\theta$. We want $D_\mathbf{u}f(0, 2) = 1$, so $-4\cos\theta + \sin\theta = 1 \;\Rightarrow\; \sin\theta = 1 + 4\cos\theta \;\Rightarrow\; \sin^2\theta = (1 + 4\cos\theta)^2 \;\Rightarrow\; 1 - \cos^2\theta = 1 + 8\cos\theta + 16\cos^2\theta \;\Rightarrow\; 17\cos^2\theta + 8\cos\theta = 0 \;\Rightarrow\; \cos\theta(17\cos\theta + 8) = 0 \;\Rightarrow\; \cos\theta = 0$ or $\cos\theta = -\frac{8}{17}$. If $\cos\theta = 0$ then $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ but $\frac{3\pi}{2}$ does not satisfy the original equation. If $\cos\theta = -\frac{8}{17}$ then $\theta = \cos^{-1}\left(-\frac{8}{17}\right)$ or $\theta = 2\pi - \cos^{-1}\left(-\frac{8}{17}\right)$ but $\theta = \cos^{-1}\left(-\frac{8}{17}\right)$ is not a solution of the original equation. Thus the directions are $\theta = \frac{\pi}{2}$ or $\theta = 2\pi - \cos^{-1}\left(-\frac{8}{17}\right) \approx 4.22$ rad.

29. The direction of fastest change is $\nabla f(x, y) = (2x - 2)\,\mathbf{i} + (2y - 4)\,\mathbf{j}$, so we need to find all points (x, y) where $\nabla f(x, y)$ is parallel to $\mathbf{i} + \mathbf{j} \;\Leftrightarrow\; (2x - 2)\,\mathbf{i} + (2y - 4)\,\mathbf{j} = k\,(\mathbf{i} + \mathbf{j}) \;\Leftrightarrow\; k = 2x - 2$ and $k = 2y - 4$. Then $2x - 2 = 2y - 4 \;\Rightarrow\; y = x + 1$, so the direction of fastest change is $\mathbf{i} + \mathbf{j}$ at all points on the line $y = x + 1$.

30. The fisherman is traveling in the direction $\langle -80, -60\rangle$. A unit vector in this direction is $\mathbf{u} = \frac{1}{100}\langle -80, -60\rangle = \left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle$, and if the depth of the lake is given by $f(x, y) = 200 + 0.02x^2 - 0.001y^3$, then $\nabla f(x, y) = \left\langle 0.04x, -0.003y^2\right\rangle$. $D_\mathbf{u}\,f(80, 60) = \nabla f(80, 60)\cdot\mathbf{u} = \langle 3.2, -10.8\rangle\cdot\left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle = 3.92$. Since $D_\mathbf{u}\,f(80, 60)$ is positive, the depth of the lake is increasing near $(80, 60)$ in the direction toward the buoy.

31. $T = \dfrac{k}{\sqrt{x^2 + y^2 + z^2}}$ and $120 = T(1, 2, 2) = \dfrac{k}{3}$ so $k = 360$.

(a) $\mathbf{u} = \dfrac{\langle 1, -1, 1\rangle}{\sqrt{3}}$,

$$D_\mathbf{u}T(1, 2, 2) = \nabla T(1, 2, 2)\cdot\mathbf{u} = \left[-360\left(x^2 + y^2 + z^2\right)^{-3/2}\langle x, y, z\rangle\right]_{(1,2,2)}\cdot\mathbf{u} = -\tfrac{40}{3}\langle 1, 2, 2\rangle\cdot\tfrac{1}{\sqrt{3}}\langle 1, -1, 1\rangle = -\tfrac{40}{3\sqrt{3}}$$

(b) From (a), $\nabla T = -360\left(x^2 + y^2 + z^2\right)^{-3/2}\langle x, y, z\rangle$, and since $\langle x, y, z\rangle$ is the position vector of the point (x, y, z), the vector $-\langle x, y, z\rangle$, and thus ∇T, always points toward the origin.

32. $\nabla T = -400e^{-x^2-3y^2-9z^2}\langle x, 3y, 9z\rangle$

(a) $\mathbf{u} = \frac{1}{\sqrt{6}}\langle 1, -2, 1\rangle$, $\nabla T(2, -1, 2) = -400e^{-43}\langle 2, -3, 18\rangle$ and

$$D_\mathbf{u}\,T(2, -1, 2) = \left(-\frac{400e^{-43}}{\sqrt{6}}\right)(26) = -\frac{5200\sqrt{6}}{3e^{43}}\;{}^\circ\mathrm{C/m}.$$

(b) $\nabla T(2, -1, 2) = 400e^{-43}\langle -2, 3, -18\rangle$ or equivalently $\langle -2, 3, -18\rangle$.

(c) $|\nabla T| = 400e^{-x^2 - 3y^2 - 9z^2}\sqrt{x^2 + 9y^2 + 81z^2}$ °C/m is the maximum rate of increase. At $(2, -1, 2)$ the maximum rate of increase is $400e^{-43}\sqrt{337}$ °C/m.

33. $\nabla V(x,y,z) = \langle 10x - 3y + yz, xz - 3x, xy \rangle$, $\nabla V(3,4,5) = \langle 38, 6, 12 \rangle$

(a) $D_{\mathbf{u}} V(3,4,5) = \langle 38, 6, 12 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, -1 \rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3,4,5) = \langle 38, 6, 12 \rangle$, or equivalently, $\langle 19, 3, 6 \rangle$.

(c) $|\nabla V(3,4,5)| = \sqrt{38^2 + 6^2 + 12^2} = \sqrt{1624} = 2\sqrt{406}$

34. $z = f(x,y) = 1000 - 0.005x^2 - 0.01y^2 \quad \Rightarrow \quad \nabla f(x,y) = \langle -0.01x, -0.02y \rangle$ and $\nabla f(60,40) = \langle -0.6, -0.8 \rangle$.

(a) Due south is in the direction of the unit vector $\mathbf{u} = -\mathbf{j}$ and

$D_{\mathbf{u}} f(60,40) = \nabla f(60,40) \cdot \langle 0, -1 \rangle = \langle -0.6, -0.8 \rangle \cdot \langle 0, -1 \rangle = 0.8$. Thus, if you walk due south from $(60, 40, 966)$ you will ascend at a rate of 0.8 vertical meters per horizontal meter.

(b) Northwest is in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}} \langle -1, 1 \rangle$ and

$D_{\mathbf{u}} f(60,40) = \nabla f(60,40) \cdot \frac{1}{\sqrt{2}} \langle -1, 1 \rangle = \langle -0.6, -0.8 \rangle \cdot \frac{1}{\sqrt{2}} \langle -1, 1 \rangle = -\frac{0.2}{\sqrt{2}} \approx -0.14$. Thus, if you walk northwest from $(60, 40, 966)$ you will descend at a rate of approximately 0.14 vertical meters per horizontal meter.

(c) $\nabla f(60,40) = \langle -0.6, -0.8 \rangle$ is the direction of largest slope with a rate of ascent given by

$|\nabla f(60,40)| = \sqrt{(-0.6)^2 + (-0.8)^2} = 1$. The angle above the horizontal in which the path begins is given by

$\tan\theta = 1 \quad \Rightarrow \quad \theta = 45^\circ$.

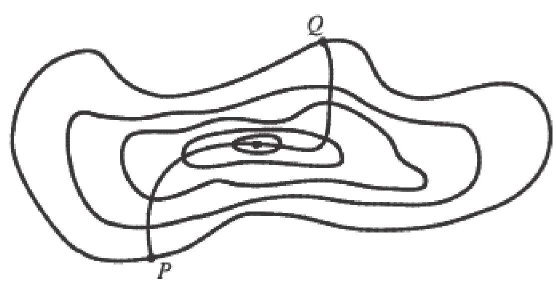
35. A unit vector in the direction of $\overrightarrow{AB}$ is $\mathbf{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\mathbf{j}$. Thus $D_{\overrightarrow{AB}} f(1,3) = f_x(1,3) = 3$ and

$D_{\overrightarrow{AC}} f(1,3) = f_y(1,3) = 26$. Therefore $\nabla f(1,3) = \langle f_x(1,3), f_y(1,3) \rangle = \langle 3, 26 \rangle$, and by definition,

$D_{\overrightarrow{AD}} f(1,3) = \nabla f \cdot \mathbf{u}$ where $\mathbf{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\langle \frac{5}{13}, \frac{12}{13} \rangle$. Therefore,

$D_{\overrightarrow{AD}} f(1,3) = \langle 3, 26 \rangle \cdot \langle \frac{5}{13}, \frac{12}{13} \rangle = 3 \cdot \frac{5}{13} + 26 \cdot \frac{12}{13} = \frac{327}{13}$.

36. The curve of steepest ascent is perpendicular to all of the contour lines.



37. (a) $\nabla(au+bv) = \left\langle \dfrac{\partial(au+bv)}{\partial x}, \dfrac{\partial(au+bv)}{\partial y} \right\rangle = \left\langle a\,\dfrac{\partial u}{\partial x} + b\,\dfrac{\partial v}{\partial x}, a\,\dfrac{\partial u}{\partial y} + b\,\dfrac{\partial v}{\partial y} \right\rangle = a\left\langle \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y} \right\rangle + b\left\langle \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y} \right\rangle$

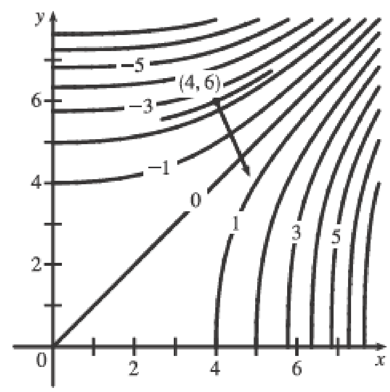
$= a\,\nabla u + b\,\nabla v$

(b) $\nabla(uv) = \left\langle v\,\dfrac{\partial u}{\partial x} + u\,\dfrac{\partial v}{\partial x}, v\,\dfrac{\partial u}{\partial y} + u\,\dfrac{\partial v}{\partial y} \right\rangle = v\left\langle \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y} \right\rangle + u\left\langle \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y} \right\rangle = v\,\nabla u + u\,\nabla v$

(c) $\nabla\left(\dfrac{u}{v}\right) = \left\langle \dfrac{v\,\frac{\partial u}{\partial x} - u\,\frac{\partial v}{\partial x}}{v^2}, \dfrac{v\,\frac{\partial u}{\partial y} - u\,\frac{\partial v}{\partial y}}{v^2} \right\rangle = \dfrac{v\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle - u\left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle}{v^2} = \dfrac{v\,\nabla u - u\,\nabla v}{v^2}$

(d) $\nabla u^n = \left\langle \dfrac{\partial(u^n)}{\partial x}, \dfrac{\partial(u^n)}{\partial y} \right\rangle = \left\langle nu^{n-1}\,\dfrac{\partial u}{\partial x}, nu^{n-1}\,\dfrac{\partial u}{\partial y} \right\rangle = nu^{n-1}\,\nabla u$

38. If we place the initial point of the gradient vector $\nabla f(4,6)$ at $(4,6)$, the vector is perpendicular to the level curve of f that includes $(4,6)$, so we sketch a portion of the level curve through $(4,6)$ (using the nearby level curves as a guideline) and draw a line perpendicular to the curve at $(4,6)$. The gradient vector is parallel to this line, pointing in the direction of increasing function values, and with length equal to the maximum value of the directional derivative of f at $(4,6)$. We can estimate this length by finding the average rate of change in the direction of the gradient. The line intersects the contour lines corresponding to -2 and -3 with an estimated distance of 0.5 units. Thus the rate of change is approximately $\dfrac{-2-(-3)}{0.5} = 2$, and we sketch the gradient vector with length 2.



39. Let $F(x,y,z) = 2(x-2)^2 + (y-1)^2 + (z-3)^2$. Then $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10$ is a level surface of F. $F_x(x,y,z) = 4(x-2) \Rightarrow F_x(3,3,5) = 4$, $F_y(x,y,z) = 2(y-1) \Rightarrow F_y(3,3,5) = 4$, and $F_z(x,y,z) = 2(z-3) \Rightarrow F_z(3,3,5) = 4$.

(a) Equation 19 gives an equation of the tangent plane at $(3,3,5)$ as $4(x-3) + 4(y-3) + 4(z-5) = 0 \Leftrightarrow 4x + 4y + 4z = 44$ or equivalently $x + y + z = 11$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-3}{4} = \dfrac{y-3}{4} = \dfrac{z-5}{4}$ or equivalently $x - 3 = y - 3 = z - 5$. Corresponding parametric equations are $x = 3 + t$, $y = 3 + t$, $z = 5 + t$.

40. Let $F(x,y,z) = x^2 - z^2 - y$. Then $y = x^2 - z^2 \Leftrightarrow x^2 - z^2 - y = 0$ is a level surface of F. $F_x(x,y,z) = 2x \Rightarrow F_x(4,7,3) = 8$, $F_y(x,y,z) = -1 \Rightarrow F_y(4,7,3) = -1$, and $F_z(x,y,z) = -2z \Rightarrow F_z(4,7,3) = -6$.

(a) An equation of the tangent plane at $(4,7,3)$ is $8(x-4) - 1(y-7) - 6(z-3) = 0$ or $8x - y - 6z = 7$.

(b) The normal line has symmetric equations $\dfrac{x-4}{8} = \dfrac{y-7}{-1} = \dfrac{z-3}{-6}$ and parametric equations $x = 4 + 8t$, $y = 7 - t$, $z = 3 - 6t$.

41. Let $F(x, y, z) = x^2 - 2y^2 + z^2 + yz$. Then $x^2 - 2y^2 + z^2 + yz = 2$ is a level surface of F and $\nabla F(x, y, z) = \langle 2x, -4y + z, 2z + y \rangle$.

(a) $\nabla F(2, 1, -1) = \langle 4, -5, -1 \rangle$ is a normal vector for the tangent plane at $(2, 1, -1)$, so an equation of the tangent plane is $4(x - 2) - 5(y - 1) - 1(z + 1) = 0$ or $4x - 5y - z = 4$.

(b) The normal line has direction $\langle 4, -5, -1 \rangle$, so parametric equations are $x = 2 + 4t,\ y = 1 - 5t,\ z = -1 - t$, and symmetric equations are $\dfrac{x - 2}{4} = \dfrac{y - 1}{-5} = \dfrac{z + 1}{-1}$.

42. Let $F(x, y, z) = x - z - 4\arctan(yz)$. Then $x - z = 4\arctan(yz)$ is the level surface $F(x, y, z) = 0$, and $\nabla F(x, y, z) = \left\langle 1, -\dfrac{4z}{1 + y^2z^2}, -1 - \dfrac{4y}{1 + y^2z^2} \right\rangle$.

(a) $\nabla F(1 + \pi, 1, 1) = \langle 1, -2, -3 \rangle$ and an equation of the tangent plane is $1(x - (1 + \pi)) - 2(y - 1) - 3(z - 1) = 0$ or $x - 2y - 3z = -4 + \pi$.

(b) The normal line has direction $\langle 1, -2, -3 \rangle$, so parametric equations are $x = 1 + \pi + t,\ y = 1 - 2t,\ z = 1 - 3t$, and symmetric equations are $x - 1 - \pi = \dfrac{y - 1}{-2} = \dfrac{z - 1}{-3}$.

43. $F(x, y, z) = -z + xe^y \cos z \quad \Rightarrow \quad \nabla F(x, y, z) = \langle e^y \cos z, xe^y \cos z, -1 - xe^y \sin z \rangle$ and $\nabla F(1, 0, 0) = \langle 1, 1, -1 \rangle$.

(a) $1(x - 1) + 1(y - 0) - 1(z - 0) = 0$ or $x + y - z = 1$

(b) $x - 1 = y = -z$

44. $F(x, y, z) = yz - \ln(x + z) \quad \Rightarrow \quad \nabla F(x, y, z) = \left\langle -\dfrac{1}{x + z}, z, y - \dfrac{1}{x + z} \right\rangle$ and $\nabla F(0, 0, 1) = \langle -1, 1, -1 \rangle$.

(a) $(-1)(x - 0) + (1)(y - 0) - 1(z - 1) = 0$ or $x - y + z = 1$

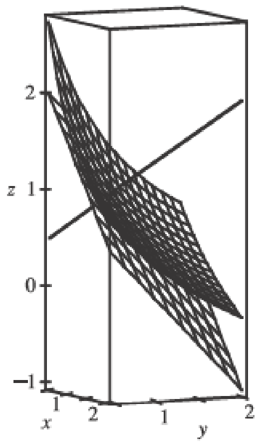
(b) Parametric equations are $x = -t,\ y = t,\ z = 1 - t$ and symmetric equations are $\dfrac{x}{-1} = \dfrac{y}{1} = \dfrac{z - 1}{-1}$ or $-x = y = 1 - z$.

45. $F(x, y, z) = xy + yz + zx$,

$\nabla F(x, y, z) = \langle y + z, x + z, y + x \rangle$,

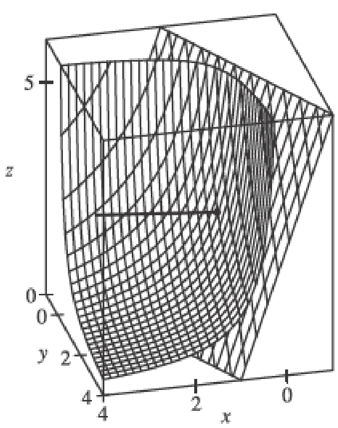
$\nabla F(1, 1, 1) = \langle 2, 2, 2 \rangle$, so an equation of the tangent plane is $2x + 2y + 2z = 6$ or $x + y + z = 3$, and the normal line is given by $x - 1 = y - 1 = z - 1$ or $x = y = z$. To graph the surface we solve for z:

$z = \dfrac{3 - xy}{x + y}$.

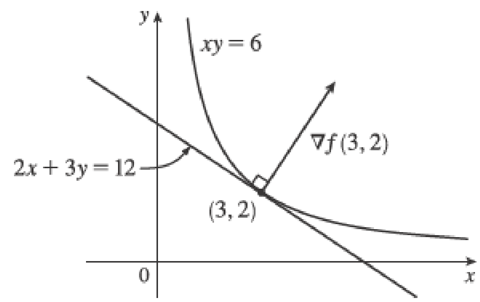


46. $F(x, y, z) = xyz$,

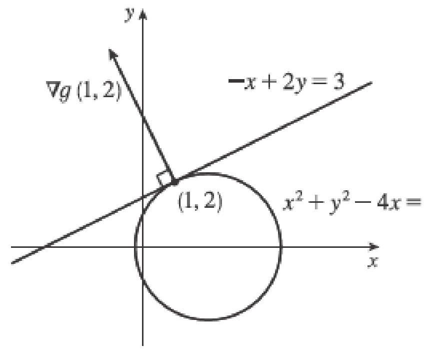
$\nabla F(x, y, z) = \langle yz, xz, yx \rangle$,$\nabla F(1, 2, 3) = \langle 6, 3, 2 \rangle$, so an equation of the tangent plane is $6x + 3y + 2z = 18$, and the normal line is given by $\dfrac{x - 1}{6} = \dfrac{y - 2}{3} = \dfrac{z - 3}{2}$ or $x = 1 + 6t,\ y = 2 + 3t,\ z = 3 + 2t$. To graph the surface we solve for z: $z = \dfrac{6}{xy}$.



47. $f(x,y) = xy \Rightarrow \nabla f(x,y) = \langle y, x \rangle$, $\nabla f(3,2) = \langle 2, 3 \rangle$. $\nabla f(3,2)$ is perpendicular to the tangent line, so the tangent line has equation $\nabla f(3,2) \cdot \langle x-3, y-2 \rangle = 0 \Rightarrow \langle 2,3 \rangle \cdot \langle x-3, x-2 \rangle = 0 \Rightarrow$ $2(x-3) + 3(y-2) = 0$ or $2x + 3y = 12$.



48. $g(x,y) = x^2 + y^2 - 4x \Rightarrow \nabla g(x,y) = \langle 2x - 4, 2y \rangle$, $\nabla g(1,2) = \langle -2, 4 \rangle$. $\nabla g(1,2)$ is perpendicular to the tangent line, so the tangent line has equation $\nabla g(1,2) \cdot \langle x-1, y-2 \rangle = 0 \Rightarrow$ $\langle -2, 4 \rangle \cdot \langle x-1, y-2 \rangle = 0 \Rightarrow -2(x-1) + 4(y-2) = 0 \Leftrightarrow$ $-2x + 4y = 6$ or equivalently $-x + 2y = 3$.



49. $\nabla F(x_0, y_0, z_0) = \left\langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2} \right\rangle$. Thus an equation of the tangent plane at (x_0, y_0, z_0) is $\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y + \frac{2z_0}{c^2}z = 2\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}\right) = 2(1) = 2$ since (x_0, y_0, z_0) is a point on the ellipsoid. Hence $\frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z = 1$ is an equation of the tangent plane.

50. $\nabla F(x_0, y_0, z_0) = \left\langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{-2z_0}{c^2} \right\rangle$, so an equation of the tangent plane at (x_0, y_0, z_0) is $\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y - \frac{2z_0}{c^2}z = 2\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2}\right) = 2$ or $\frac{x_0}{a^2}x + \frac{y_0}{b^2}y - \frac{z_0}{c^2}z = 1$.