51. $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-1}{c} \right\rangle$, so an equation of the tangent plane is $\dfrac{2x_0}{a^2}x + \dfrac{2y_0}{b^2}y - \dfrac{1}{c}z = \dfrac{2x_0^2}{a^2} + \dfrac{2y_0^2}{b^2} - \dfrac{z_0}{c}$ or $\dfrac{2x_0}{a^2}x + \dfrac{2y_0}{b^2}y = \dfrac{z}{c} + 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}\right) - \dfrac{z_0}{c}$. But $\dfrac{z_0}{c} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}$, so the equation can be written as $\dfrac{2x_0}{a^2}x + \dfrac{2y_0}{b^2}y = \dfrac{z + z_0}{c}$.

52. Let $F(x, y, z) = x^2 + z^2 - y$; then the paraboloid $y = x^2 + z^2$ is a level surface of F. $\nabla F(x, y, z) = \langle 2x, -1, 2z \rangle$ is a normal vector to the surface at (x, y, z) and so it is a normal vector for the tangent plane there. The tangent plane is parallel to the plane $x + 2y + 3z = 1$ when the normal vectors of the planes are parallel, so we need a point (x_0, y_0, z_0) on the paraboloid where $\langle 2x_0, -1, 2z_0 \rangle = k\langle 1, 2, 3 \rangle$. Comparing y-components we have $k = -\frac{1}{2}$, so $\langle 2x_0, -1, 2z_0 \rangle = \left\langle -\frac{1}{2}, -1, -\frac{3}{2} \right\rangle$ and $2x_0 = -\frac{1}{2} \quad\Rightarrow\quad x_0 = -\frac{1}{4}$, $2z_0 = -\frac{3}{2} \quad\Rightarrow\quad z_0 = -\frac{3}{4}$. Then $y_0 = x_0^2 + z_0^2 = \left(-\frac{1}{4}\right)^2 + \left(-\frac{3}{4}\right)^2 = \frac{5}{8}$ and the point is $\left(-\frac{1}{4}, \frac{5}{8}, -\frac{3}{4}\right)$.

53. The hyperboloid $x^2 - y^2 - z^2 = 1$ is a level surface of $F(x, y, z) = x^2 - y^2 - z^2$ and $\nabla F(x, y, z) = \langle 2x, -2y, -2z \rangle$ is a normal vector to the surface and hence a normal vector for the tangent plane at (x, y, z). The tangent plane is parallel to the plane $z = x + y$ or $x + y - z = 0$ if and only if the corresponding normal vectors are parallel, so we need a point (x_0, y_0, z_0) on the hyperboloid where $\langle 2x_0, -2y_0, -2z_0 \rangle = c\langle 1, 1, -1 \rangle$ or equivalently $\langle x_0, -y_0, -z_0 \rangle = k\langle 1, 1, -1 \rangle$ for some $k \neq 0$. Then we must have $x_0 = k$, $y_0 = -k$, $z_0 = k$ and substituting into the equation of the hyperboloid gives $k^2 - (-k)^2 - k^2 = 1 \quad\Leftrightarrow\quad -k^2 = 1$, an impossibility. Thus there is no such point on the hyperboloid.

54. First note that the point $(1, 1, 2)$ is on both surfaces. For the ellipsoid, an equation of the tangent plane at $(1, 1, 2)$ is $6x + 4y + 4z = 18$ or $3x + 2y + 2z = 9$, and for the sphere, an equation of the tangent plane at $(1, 1, 2)$ is $(2 - 8)x + (2 - 6)y + (4 - 8)z = -18$ or $-6x - 4y - 4z = -18$ or $3x + 2y + 2z = 9$. Since these tangent planes are the same, the surfaces are tangent to each other at the point $(1, 1, 2)$.

55. Let (x_0, y_0, z_0) be a point on the cone [other than $(0, 0, 0)$]. Then an equation of the tangent plane to the cone at this point is $2x_0x + 2y_0y - 2z_0z = 2\left(x_0^2 + y_0^2 - z_0^2\right)$. But $x_0^2 + y_0^2 = z_0^2$ so the tangent plane is given by $x_0x + y_0y - z_0z = 0$, a plane which always contains the origin.

56. Let (x_0, y_0, z_0) be a point on the sphere. Then the normal line is given by $\dfrac{x - x_0}{2x_0} = \dfrac{y - y_0}{2y_0} = \dfrac{z - z_0}{2z_0}$. For the center $(0, 0, 0)$ to be on the line, we need $-\dfrac{x_0}{2x_0} = -\dfrac{y_0}{2y_0} = -\dfrac{z_0}{2z_0}$ or equivalently $1 = 1 = 1$, which is true.

57. Let (x_0, y_0, z_0) be a point on the surface. Then an equation of the tangent plane at the point is

$\dfrac{x}{2\sqrt{x_0}} + \dfrac{y}{2\sqrt{y_0}} + \dfrac{z}{2\sqrt{z_0}} = \dfrac{\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}}{2}$. But $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{c}$, so the equation is

$\dfrac{x}{\sqrt{x_0}} + \dfrac{y}{\sqrt{y_0}} + \dfrac{z}{\sqrt{z_0}} = \sqrt{c}$. The x-, y-, and z-intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$ and $\sqrt{cz_0}$ respectively. (The x-intercept is found by setting $y = z = 0$ and solving the resulting equation for x, and the y- and z-intercepts are found similarly.) So the sum of the intercepts is $\sqrt{c}\left(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}\right) = c$, a constant.

58. The surface $xyz = 1$ is a level surface of $F(x, y, z) = xyz$ and $\nabla F(x, y, z) = \langle yz, xz, xy \rangle$ is normal to the surface, so a normal vector for the tangent plane to the surface at (x_0, y_0, z_0) is $\langle y_0 z_0, x_0 z_0, x_0 y_0 \rangle$. An equation for the tangent plane is

$y_0 z_0 (x - x_0) + x_0 z_0 (y - y_0) + x_0 y_0 (z - z_0) = 0 \quad \Rightarrow \quad y_0 z_0 x + x_0 z_0 y + x_0 y_0 z = 3 x_0 y_0 z_0$ or $\dfrac{x}{x_0} + \dfrac{y}{y_0} + \dfrac{z}{z_0} = 3$.

If (x_0, y_0, z_0) is in the first octant, then the tangent plane cuts off a pyramid in the first octant with vertices $(0, 0, 0)$, $(3x_0, 0, 0)$, $(0, 3y_0, 0)$, $(0, 0, 3z_0)$. The base in the xy-plane is a triangle with area $\frac{1}{2}(3x_0)(3y_0)$ and the height (along the z-axis) of the pyramid is $3z_0$. The volume of the pyramid for any (x_0, y_0, z_0) on the surface $xyz = 1$ in the first octant is $\frac{1}{3}(\text{base})(\text{height}) = \frac{1}{3} \cdot \frac{1}{2}(3x_0)(3y_0) \cdot 3z_0 = \frac{9}{2} x_0 y_0 z_0 = \frac{9}{2}$ since $x_0 y_0 z_0 = 1$.

59. If $f(x, y, z) = z - x^2 - y^2$ and $g(x, y, z) = 4x^2 + y^2 + z^2$, then the tangent line is perpendicular to both ∇f and ∇g at $(-1, 1, 2)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ will therefore be parallel to the tangent line.

We have $\nabla f(x, y, z) = \langle -2x, -2y, 1 \rangle \quad \Rightarrow \quad \nabla f(-1, 1, 2) = \langle 2, -2, 1 \rangle$, and $\nabla g(x, y, z) = \langle 8x, 2y, 2z \rangle \quad \Rightarrow$

$\nabla g(-1, 1, 2) = \langle -8, 2, 4 \rangle$. Hence $\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = -10\,\mathbf{i} - 16\,\mathbf{j} - 12\,\mathbf{k}$.

Parametric equations are: $x = -1 - 10t,\ y = 1 - 16t,\ z = 2 - 12t$.

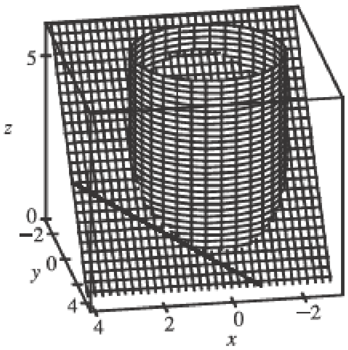
60. (a) Let $f(x, y, z) = y + z$ and $g(x, y, z) = x^2 + y^2$. Then the required tangent line is perpendicular to both ∇f and ∇g at $(1, 2, 1)$ and the vector $\mathbf{v} = \nabla f \times \nabla g$ is parallel to the tangent line. We have

$\nabla f(x, y, z) = \langle 0, 1, 1 \rangle \quad \Rightarrow \quad \nabla f(1, 2, 1) = \langle 0, 1, 1 \rangle$, and

$\nabla g(x, y, z) = \langle 2x, 2y, 0 \rangle \quad \Rightarrow \quad \nabla g(1, 2, 1) = \langle 2, 4, 0 \rangle$. Hence

$\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{vmatrix} = -4\,\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}$. So parametric equations of the desired tangent line are $x = 1 - 4t,\ y = 2 + 2t,\ z = 1 - 2t$.

(b)



61. (a) The direction of the normal line of F is given by ∇F, and that of G by ∇G. Assuming that $\nabla F \neq \mathbf{0} \neq \nabla G$, the two normal lines are perpendicular at P if $\nabla F \cdot \nabla G = 0$ at $P \quad \Leftrightarrow$ $\langle \partial F/\partial x, \partial F/\partial y, \partial F/\partial z \rangle \cdot \langle \partial G/\partial x, \partial G/\partial y, \partial G/\partial z \rangle = 0$ at $P \quad \Leftrightarrow \quad F_x G_x + F_y G_y + F_z G_z = 0$ at P.

(b) Here $F = x^2 + y^2 - z^2$ and $G = x^2 + y^2 + z^2 - r^2$, so $\nabla F \cdot \nabla G = \langle 2x, 2y, -2z \rangle \cdot \langle 2x, 2y, 2z \rangle = 4x^2 + 4y^2 - 4z^2 = 4F = 0$, since the point (x, y, z) lies on the graph of $F = 0$. To see that this is true without using calculus, note that $G = 0$ is the equation of a sphere centered at the origin and $F = 0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere's normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F = 0$ and $G = 0$ are everywhere orthogonal.

62. (a) The function $f(x, y) = (xy)^{1/3}$ is continuous on $\mathbb{R}^2$ since it is a composition of a polynomial and the cube root function, both of which are continuous. (See the text just after Example 8 in Section 15.2 [ET 14.2].)

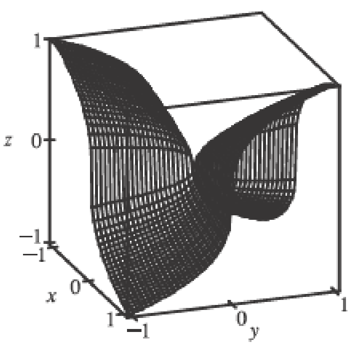
$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{(h \cdot 0)^{1/3} - 0}{h} = 0,$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \to 0} \frac{(0 \cdot h)^{1/3} - 0}{h} = 0.$$

Therefore, $f_x(0,0)$ and $f_y(0,0)$ do exist and are equal to 0. Now let $\mathbf{u}$ be any unit vector other than $\mathbf{i}$ and $\mathbf{j}$ (these correspond to f_x and f_y respectively.) Then $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j}$ where $a \neq 0$ and $b \neq 0$. Thus

$D_{\mathbf{u}}\, f(0,0) = \lim_{h \to 0} \dfrac{f(0+ha, 0+hb) - f(0,0)}{h} = \lim_{h \to 0} \dfrac{\sqrt[3]{(ha)(hb)}}{h} = \lim_{h \to 0} \dfrac{\sqrt[3]{ab}}{h^{1/3}}$ and this limit does not exist, so $D_{\mathbf{u}}\, f(0,0)$ does not exist.

(b)



Notice that if we start at the origin and proceed in the direction of the x- or y-axis, then the graph is flat. But if we proceed in any other direction, then the graph is extremely steep.

63. Let $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$. Then we know that at the given point, $D_{\mathbf{u}}\, f = \nabla f \cdot \mathbf{u} = a f_x + b f_y$ and $D_{\mathbf{v}}\, f = \nabla f \cdot \mathbf{v} = c f_x + d f_y$. But these are just two linear equations in the two unknowns f_x and f_y, and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we can solve the equations to find $\nabla f = \langle f_x, f_y \rangle$ at the given point. In fact,

$$\nabla f = \left\langle \frac{d\, D_{\mathbf{u}}\, f - b\, D_{\mathbf{v}}\, f}{ad - bc}, \frac{a\, D_{\mathbf{v}}\, f - c\, D_{\mathbf{u}}\, f}{ad - bc} \right\rangle.$$

64. Since $z = f(x, y)$ is differentiable at $\mathbf{x_0} = (x_0, y_0)$, by Definition 15.4.7 [ET 14.4.7] we have

$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$ where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Now $\Delta z = f(\mathbf{x}) - f(\mathbf{x_0})$, $\langle \Delta x, \Delta y \rangle = \mathbf{x} - \mathbf{x_0}$ so $(\Delta x, \Delta y) \to (0, 0)$ is equivalent to $\mathbf{x} \to \mathbf{x_0}$ and $\langle f_x(x_0, y_0)\,, f_y(x_0, y_0) \rangle = \nabla f(\mathbf{x_0})$. Substituting into (15.4.7) [ET (14.4.7)] gives

$f(\mathbf{x}) - f(\mathbf{x_0}) = \nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0}) + \langle \varepsilon_1, \varepsilon_2 \rangle \cdot \langle \Delta x, \Delta y \rangle$ or $\langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x_0}) = f(\mathbf{x}) - f(\mathbf{x_0}) - \nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0})$,

and so $\dfrac{f(\mathbf{x}) - f(\mathbf{x_0}) - \nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0})}{|\mathbf{x} - \mathbf{x_0}|} = \dfrac{\langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x_0})}{|\mathbf{x} - \mathbf{x_0}|}$. But $\dfrac{\mathbf{x} - \mathbf{x_0}}{|\mathbf{x} - \mathbf{x_0}|}$ is a unit vector so

$\displaystyle\lim_{\mathbf{x} \to \mathbf{x_0}} \frac{\langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x_0})}{|\mathbf{x} - \mathbf{x_0}|} = 0$ since $\varepsilon_1, \varepsilon_2 \to 0$ as $\mathbf{x} \to \mathbf{x_0}$. Hence $\displaystyle\lim_{\mathbf{x} \to \mathbf{x_0}} \frac{f(\mathbf{x}) - f(\mathbf{x_0}) - \nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0})}{|\mathbf{x} - \mathbf{x_0}|} = 0$.