1. (a) First we compute $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$ by the Second Derivatives Test.

   (b) $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1,1) < 0$, f has a saddle point at $(1,1)$ by the Second Derivatives Test.

2. (a) $D = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D < 0$, g has a saddle point at $(0,2)$ by the Second Derivatives Test.

   (b) $D = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D > 0$ and $g_{xx}(0,2) < 0$, g has a local maximum at $(0,2)$ by the Second Derivatives Test.

   (c) $D = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about g at the point $(0,2)$.

3. In the figure, a point at approximately $(1,1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1,1)$. The level curves near $(0,0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

   To verify our predictions, we have $f(x,y) = 4 + x^3 + y^3 - 3xy \quad\Rightarrow\quad f_x(x,y) = 3x^2 - 3y$, $f_y(x,y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3(x^2)^2 - 3x = 0 \quad\Rightarrow\quad 3x(x^3 - 1) = 0 \quad\Rightarrow\quad x = 0$ or $x = 1$. Then we have two critical points, $(0,0)$ and $(1,1)$. The second partial derivatives are $f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = -3$, and $f_{yy}(x,y) = 6y$, so $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then $D(0,0) = 36(0)(0) - 9 = -9$, and $D(1,1) = 36(1)(1) - 9 = 27$. Since $D(0,0) < 0$, f has a saddle point at $(0,0)$ by the Second Derivatives Test. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$.

4. In the figure, points at approximately $(-1,1)$ and $(-1,-1)$ are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minima at or near $(-1,\pm1)$. Similarly, the point $(1,0)$ appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points $(-1,0)$, $(1,1)$, and $(1,-1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

To confirm our predictions, we have $f(x,y) = 3x - x^3 - 2y^2 + y^4 \Rightarrow f_x(x,y) = 3 - 3x^2$, $f_y(x,y) = -4y + 4y^3$. Setting these partial derivatives equal to 0, we have $3 - 3x^2 = 0 \Rightarrow x = \pm1$ and $-4y + 4y^3 = 0 \Rightarrow y(y^2-1) = 0 \Rightarrow y = 0, \pm1$. So our critical points are $(\pm1,0)$, $(\pm1,\pm1)$. The second partial derivatives are $f_{xx}(x,y) = -6x$, $f_{xy}(x,y) = 0$, and $f_{yy}(x,y) = 12y^2 - 4$, so

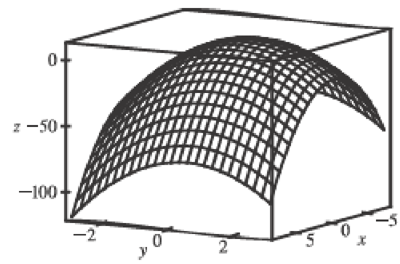
$D(x,y) = f_{xx}(x,y)\,f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (-6x)(12y^2-4) - (0)^2 = -72xy^2 + 24x$.

We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	D	f_{xx}	Conclusion
$(1,0)$	24	-6	$D>0$, $f_{xx}<0 \Rightarrow f$ has a local maximum at $(1,0)$
$(1,1)$	-48		$D<0 \Rightarrow f$ has a saddle point at $(1,1)$
$(1,-1)$	-48		$D<0 \Rightarrow f$ has a saddle point at $(1,-1)$
$(-1,0)$	-24		$D<0 \Rightarrow f$ has a saddle point at $(-1,0)$
$(-1,1)$	48	6	$D>0$, $f_{xx}>0 \Rightarrow f$ has a local minimum at $(-1,1)$
$(-1,-1)$	48	6	$D>0$, $f_{xx}>0 \Rightarrow f$ has a local minimum at $(-1,-1)$

5. $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2 \Rightarrow f_x = -2 - 2x$, $f_y = 4 - 8y$, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -8$. Then $f_x = 0$ and $f_y = 0$ imply $x = -1$ and $y = \frac{1}{2}$, and the only critical point is $\left(-1, \frac{1}{2}\right)$.

$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-8) - 0^2 = 16$, and since $D\left(-1,\frac{1}{2}\right) = 16 > 0$ and $f_{xx}\left(-1,\frac{1}{2}\right) = -2 < 0$, $f\left(-1,\frac{1}{2}\right) = 11$ is a local maximum by the Second Derivatives Test.



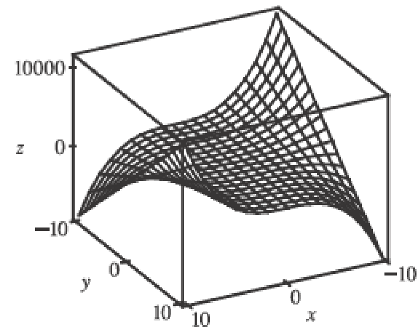
6. $f(x,y) = x^3y + 12x^2 - 8y \quad \Rightarrow \quad f_x = 3x^2y + 24x$,

$f_y = x^3 - 8$, $f_{xx} = 6xy + 24$, $f_{xy} = 3x^2$, $f_{yy} = 0$.

Then $f_y = 0$ implies $x = 2$, and substitution into $f_x = 0$ gives

$12y + 48 = 0 \quad \Rightarrow \quad y = -4$. Thus, the only critical point is $(2, -4)$.

$D(2, -4) = (-24)(0) - 12^2 = -144 < 0$, so $(2, -4)$ is a saddle point.



7. $f(x,y) = x^4 + y^4 - 4xy + 2 \quad \Rightarrow \quad f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$,

$f_{xx} = 12x^2$, $f_{xy} = -4$, $f_{yy} = 12y^2$. Then $f_x = 0$ implies $y = x^3$,

and substitution into $f_y = 0 \quad \Rightarrow \quad x = y^3$ gives $x^9 - x = 0 \quad \Rightarrow$

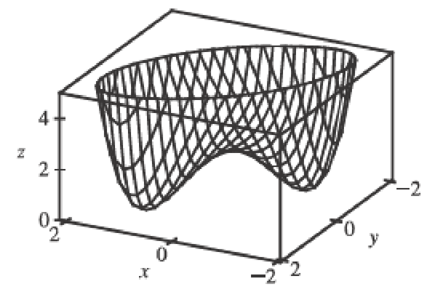
$x(x^8 - 1) = 0 \quad \Rightarrow \quad x = 0$ or $x = \pm 1$. Thus the critical points are $(0,0)$,

$(1,1)$, and $(-1,-1)$. Now $D(0,0) = 0 \cdot 0 - (-4)^2 = -16 < 0$,

so $(0,0)$ is a saddle point. $D(1,1) = (12)(12) - (-4)^2 > 0$ and

$f_{xx}(1,1) = 12 > 0$, so $f(1,1) = 0$ is a local minimum. $D(-1,-1) = (12)(12) - (-4)^2 > 0$ and

$f_{xx} = (-1,-1) = 12 > 0$, so $f(-1,-1) = 0$ is also a local minimum.



8. $f(x,y) = e^{4y - x^2 - y^2} \quad \Rightarrow \quad f_x = -2xe^{4y-x^2-y^2}$,

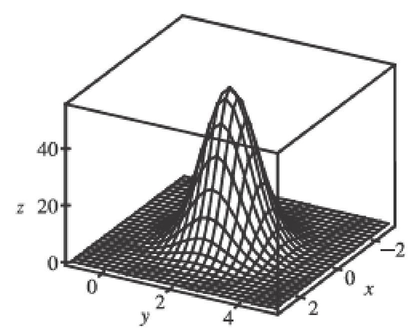
$f_y = (4 - 2y)e^{4y-x^2-y^2}$, $f_{xx} = (4x^2 - 2)e^{4y-x^2-y^2}$,

$f_{xy} = -2x(4-2y)e^{4y-x^2-y^2}$, $f_{yy} = (4y^2 - 16y + 14)e^{4y-x^2-y^2}$.

Then $f_x = 0$ and $f_y = 0$ implies $x = 0$ and $y = 2$, so the only critical

point is $(0,2)$. Now $D(0,2) = (-2e^4)(-2e^4) - 0^2 = 4e^8 > 0$ and

$f_{xx}(0,2) = -2e^4 < 0$, so $f(0,2) = e^4$ is a local maximum.



9. $f(x,y) = (1 + xy)(x + y) = x + y + x^2y + xy^2 \quad \Rightarrow$

$f_x = 1 + 2xy + y^2$, $f_y = 1 + x^2 + 2xy$, $f_{xx} = 2y$, $f_{xy} = 2x + 2y$,

$f_{yy} = 2x$. Then $f_x = 0$ implies $1 + 2xy + y^2 = 0$ and $f_y = 0$ implies

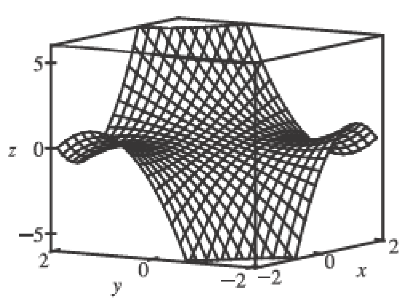
$1 + x^2 + 2xy = 0$. Subtracting the second equation from the first gives

$y^2 - x^2 = 0 \quad \Rightarrow \quad y = \pm x$, but if $y = x$ then $1 + 2xy + y^2 = 0 \quad \Rightarrow$

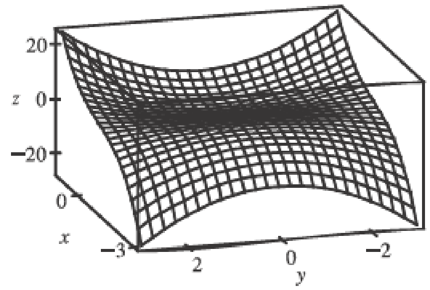
$1 + 3x^2 = 0$ which has no real solution. If $y = -x$ then

$1 + 2xy + y^2 = 0 \quad \Rightarrow \quad 1 - x^2 = 0 \quad \Rightarrow \quad x = \pm 1$, so critical points are $(1,-1)$ and $(-1,1)$.

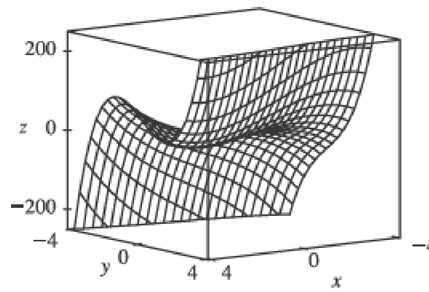
$D(1,-1) = (-2)(2) - 0 < 0$ and $D(-1,1) = (2)(-2) - 0 < 0$, so $(-1,1)$ and $(1,-1)$ are saddle points.



10. $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2 \Rightarrow \quad f_x = 6x^2 + y^2 + 10x$, $f_y = 2xy + 2y$, $f_{xx} = 12x + 10$, $f_{yy} = 2x + 2$, $f_{xy} = 2y$. Then $f_y = 0$ implies $y = 0$ or $x = -1$. Substituting into $f_x = 0$ gives the critical points $(0, 0)$, $\left(-\frac{5}{3}, 0\right)$, $(-1, \pm 2)$. Now $D(0, 0) = 20 > 0$ and $f_{xx}(0, 0) = 10 > 0$, so $f(0, 0) = 0$ is a local minimum. Also $f_{xx}\left(-\frac{5}{3}, 0\right) < 0$, $D\left(-\frac{5}{3}, 0\right) > 0$, and $D(-1, \pm 2) < 0$. Hence $f\left(-\frac{5}{3}, 0\right) = \frac{125}{27}$ is a local maximum while $(-1, \pm 2)$ are saddle points.



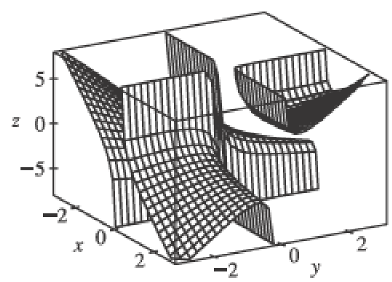
11. $f(x, y) = x^3 - 12xy + 8y^3 \quad \Rightarrow \quad f_x = 3x^2 - 12y$, $f_y = -12x + 24y^2$, $f_{xx} = 6x$, $f_{xy} = -12$, $f_{yy} = 48y$. Then $f_x = 0$ implies $x^2 = 4y$ and $f_y = 0$ implies $x = 2y^2$. Substituting the second equation into the first gives $(2y^2)^2 = 4y \quad \Rightarrow \quad 4y^4 = 4y \quad \Rightarrow \quad 4y(y^3 - 1) = 0 \quad \Rightarrow \quad y = 0$ or $y = 1$. If $y = 0$ then $x = 0$ and if $y = 1$ then $x = 2$, so the critical points are $(0, 0)$ and $(2, 1)$. $\quad D(0, 0) = (0)(0) - (-12)^2 = -144 < 0$, so $(0, 0)$ is a saddle point. $D(2, 1) = (12)(48) - (-12)^2 = 432 > 0$ and $f_{xx}(2, 1) = 12 > 0$ so $f(2, 1) = -8$ is a local minimum.



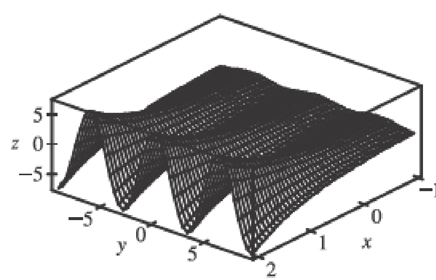
12. $f(x, y) = xy + \dfrac{1}{x} + \dfrac{1}{y} \quad \Rightarrow \quad f_x = y - \dfrac{1}{x^2}$, $f_y = x - \dfrac{1}{y^2}$, $f_{xx} = \dfrac{2}{x^3}$, $f_{xy} = 1$, $f_{yy} = \dfrac{2}{y^3}$. Then $f_x = 0$ implies $y = \dfrac{1}{x^2}$ and $f_y = 0$ implies $x = \dfrac{1}{y^2}$. Substituting the first equation into the second gives

$$x = \frac{1}{(1/x^2)^2} \quad \Rightarrow \quad x = x^4 \quad \Rightarrow \quad x(x^3 - 1) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = 1.$$

f is not defined when $x = 0$, and when $x = 1$ we have $y = 1$, so the only critical point is $(1, 1)$. $D(1, 1) = (2)(2) - 1^2 = 3 > 0$ and $f_{xx}(1, 1) = 2 > 0$, so $f(1, 1) = 3$ is a local minimum.

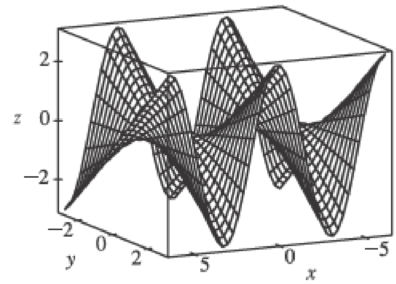


13. $f(x, y) = e^x \cos y \quad \Rightarrow \quad f_x = e^x \cos y$, $f_y = -e^x \sin y$. Now $f_x = 0$ implies $\cos y = 0$ or $y = \frac{\pi}{2} + n\pi$ for n an integer. But $\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so there are no critical points.



14. $f(x,y) = y\cos x \quad\Rightarrow\quad f_x = -y\sin x,\ f_y = \cos x,\ f_{xx} = -y\cos x,$ $f_{xy} = -\sin x,\ f_{yy} = 0$. Then $f_y = 0$ if and only if $x = \frac{\pi}{2} + n\pi$ for n an integer. But $\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so $f_x = 0 \quad\Rightarrow\quad y = 0$ and the critical points are $\left(\frac{\pi}{2} + n\pi, 0\right)$, n an integer.

$D\left(\frac{\pi}{2} + n\pi, 0\right) = (0)(0) - (\pm 1)^2 = -1 < 0$, so each critical point is a saddle point.



15. $f(x,y) = (x^2 + y^2)e^{y^2 - x^2} \quad\Rightarrow$

$f_x = (x^2 + y^2)e^{y^2-x^2}(-2x) + 2xe^{y^2-x^2} = 2xe^{y^2-x^2}(1 - x^2 - y^2),$

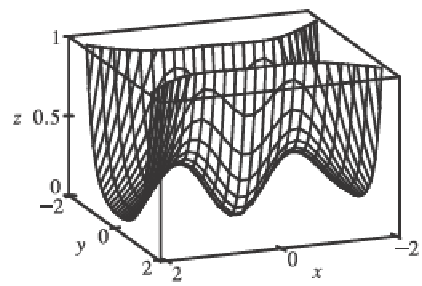
$f_y = (x^2 + y^2)e^{y^2-x^2}(2y) + 2ye^{y^2-x^2} = 2ye^{y^2-x^2}(1 + x^2 + y^2),$

$f_{xx} = 2xe^{y^2-x^2}(-2x) + (1 - x^2 - y^2)\left(2x\left(-2xe^{y^2-x^2}\right) + 2e^{y^2-x^2}\right) = 2e^{y^2-x^2}((1 - x^2 - y^2)(1 - 2x^2) - 2x^2),$

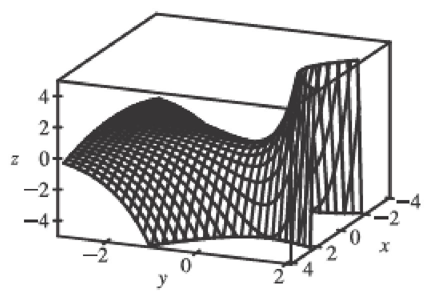
$f_{xy} = 2xe^{y^2-x^2}(-2y) + 2x(2y)e^{y^2-x^2}(1 - x^2 - y^2) = -4xye^{y^2-x^2}(x^2 + y^2),$

$f_{yy} = 2ye^{y^2-x^2}(2y) + (1 + x^2 + y^2)\left(2y\left(2ye^{y^2-x^2}\right) + 2e^{y^2-x^2}\right) = 2e^{y^2-x^2}((1 + x^2 + y^2)(1 + 2y^2) + 2y^2).$

$f_y = 0$ implies $y = 0$, and substituting into $f_x = 0$ gives $2xe^{-x^2}(1 - x^2) = 0 \quad\Rightarrow\quad x = 0$ or $x = \pm 1$. Thus the critical points are $(0,0)$ and $(\pm 1, 0)$. Now $D(0,0) = (2)(2) - 0 > 0$ and $f_{xx}(0,0) = 2 > 0$, so $f(0,0) = 0$ is a local minimum. $D(\pm 1, 0) = (-4e^{-1})(4e^{-1}) - 0 < 0$ so $(\pm 1, 0)$ are saddle points.

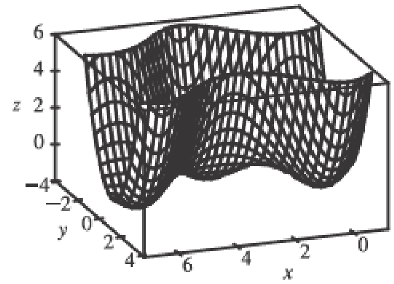


16. $f(x,y) = e^y(y^2 - x^2) \quad\Rightarrow\quad f_x = -2xe^y,\ f_y = (2y + y^2 - x^2)e^y,$ $f_{xx} = -2e^y,\ f_{xy} = -2xe^y,\ f_{yy} = (2 + 4y + y^2 - x^2)e^y$. Then $f_x = 0$ implies $x = 0$ and substituting into $f_y = 0$ gives $(2y + y^2)e^y = 0 \quad\Rightarrow$ $y(2 + y) = 0 \quad\Rightarrow\quad y = 0$ or $y = -2$, so the critical points are $(0,0)$ and $(0,-2)$. $D(0,0) = (-2)(2) - (0)^2 = -4 < 0$ so $(0,0)$ is a saddle point.



$D(0,-2) = (-2e^{-2})(-2e^{-2}) - (0)^2 = 4e^{-4} > 0$ and $f_{xx}(0,-2) = -2e^{-2} < 0$, so $f(0,-2) = 4e^{-2}$ is a local maximum.

17. $f(x,y) = y^2 - 2y\cos x \quad\Rightarrow\quad f_x = 2y\sin x,\ f_y = 2y - 2\cos x,$



$f_{xx} = 2y\cos x,\ f_{xy} = 2\sin x,\ f_{yy} = 2$. Then $f_x = 0$ implies $y = 0$ or $\sin x = 0 \quad\Rightarrow\quad x = 0, \pi,$ or 2π for $-1 \le x \le 7$. Substituting $y = 0$ into $f_y = 0$ gives $\cos x = 0 \quad\Rightarrow\quad x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, substituting $x = 0$ or $x = 2\pi$ into $f_y = 0$ gives $y = 1$, and substituting $x = \pi$ into $f_y = 0$ gives $y = -1$. Thus the critical points are $(0,1)$, $\left(\frac{\pi}{2},0\right)$, $(\pi,-1)$, $\left(\frac{3\pi}{2},0\right)$, and $(2\pi,1)$.

$D\left(\frac{\pi}{2},0\right) = D\left(\frac{3\pi}{2},0\right) = -4 < 0$ so $\left(\frac{\pi}{2},0\right)$ and $\left(\frac{3\pi}{2},0\right)$ are saddle points. $D(0,1) = D(\pi,-1) = D(2\pi,1) = 4 > 0$ and $f_{xx}(0,1) = f_{xx}(\pi,-1) = f_{xx}(2\pi,1) = 2 > 0$, so $f(0,1) = f(\pi,-1) = f(2\pi,1) = -1$ are local minima.

18. $f(x,y) = \sin x\,\sin y \quad\Rightarrow\quad f_x = \cos x\,\sin y,\ f_y = \sin x\,\cos y,\ f_{xx} = -\sin x\,\sin y,\ f_{xy} = \cos x\,\cos y,$ $f_{yy} = -\sin x\,\sin y$. Here we have $-\pi < x < \pi$ and $-\pi < y < \pi$, so $f_x = 0$ implies $\cos x = 0$ or $\sin y = 0$. If $\cos x = 0$ then $x = -\frac{\pi}{2}$ or $\frac{\pi}{2}$, and if $\sin y = 0$ then $y = 0$. Substituting $x = \pm\frac{\pi}{2}$ into $f_y = 0$ gives $\cos y = 0 \quad\Rightarrow\quad y = -\frac{\pi}{2}$ or $\frac{\pi}{2}$, and substituting $y = 0$ into $f_y = 0$ gives $\sin x = 0 \quad\Rightarrow\quad x = 0$. Thus the critical points are $\left(-\frac{\pi}{2},\pm\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2},\pm\frac{\pi}{2}\right)$, and $(0,0)$.

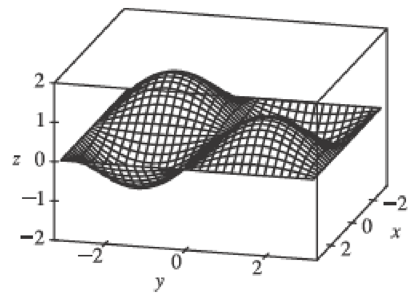
$D(0,0) = -1 < 0$ so $(0,0)$ is a saddle point.

$D\left(-\frac{\pi}{2},\pm\frac{\pi}{2}\right) = D\left(\frac{\pi}{2},\pm\frac{\pi}{2}\right) = 1 > 0$ and

$f_{xx}\left(-\frac{\pi}{2},-\frac{\pi}{2}\right) = f_{xx}\left(\frac{\pi}{2},\frac{\pi}{2}\right) = -1 < 0$ while

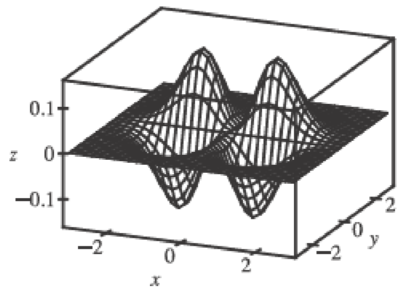
$f_{xx}\left(-\frac{\pi}{2},\frac{\pi}{2}\right) = f_{xx}\left(\frac{\pi}{2},-\frac{\pi}{2}\right) = 1 > 0$, so $f\left(-\frac{\pi}{2},-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2},\frac{\pi}{2}\right) = 1$

are local maxima and $f\left(-\frac{\pi}{2},\frac{\pi}{2}\right) = f\left(\frac{\pi}{2},-\frac{\pi}{2}\right) = 1$ are local minima.



19. $f(x,y) = x^2 + 4y^2 - 4xy + 2 \quad\Rightarrow\quad f_x = 2x - 4y,\ f_y = 8y - 4x,\ f_{xx} = 2,\ f_{xy} = -4,\ f_{yy} = 8$. Then $f_x = 0$ and $f_y = 0$ each implies $y = \frac{1}{2}x$, so all points of the form $\left(x_0, \frac{1}{2}x_0\right)$ are critical points and for each of these we have $D\left(x_0,\frac{1}{2}x_0\right) = (2)(8) - (-4)^2 = 0$. The Second Derivatives Test gives no information, but $f(x,y) = x^2 + 4y^2 - 4xy + 2 = (x-2y)^2 + 2 \ge 2$ with equality if and only if $y = \frac{1}{2}x$. Thus $f\left(x_0,\frac{1}{2}x_0\right) = 2$ are all local (and absolute) minima.

20. $f(x,y) = x^2ye^{-x^2-y^2} \quad \Rightarrow$



$f_x = x^2ye^{-x^2-y^2}(-2x) + 2xye^{-x^2-y^2} = 2xy(1-x^2)e^{-x^2-y^2}$,

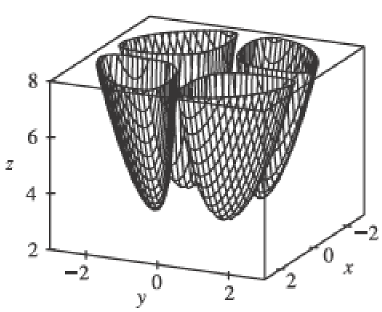
$f_y = x^2ye^{-x^2-y^2}(-2y) + x^2e^{-x^2-y^2} = x^2(1-2y^2)e^{-x^2-y^2}$,

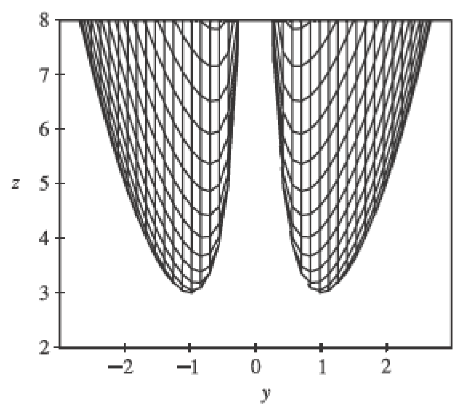
$f_{xx} = 2y(2x^4 - 5x^2 + 1)e^{-x^2-y^2}$,

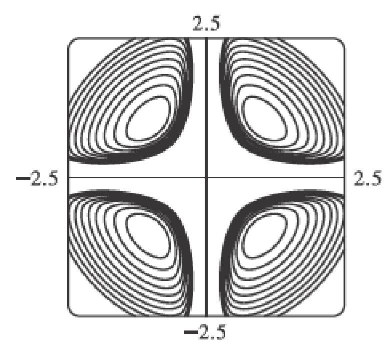
$f_{xy} = 2x(1-x^2)(1-2y^2)e^{-x^2-y^2}$, $f_{yy} = 2x^2y(2y^2-3)e^{-x^2-y^2}$.

$f_x = 0$ implies $x = 0$, $y = 0$, or $x = \pm 1$. If $x = 0$ then $f_y = 0$ for any y-value, so all points of the form $(0, y)$ are critical points. If $y = 0$ then $f_y = 0 \quad \Rightarrow \quad x^2e^{-x^2} = 0 \quad \Rightarrow \quad x = 0$, so $(0,0)$ (already included above) is a critical point. If $x = \pm 1$ then $(1-2y^2)e^{-1-y^2} = 0 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{2}}$, so $\left(\pm 1, \frac{1}{\sqrt{2}}\right)$ and $\left(\pm 1, -\frac{1}{\sqrt{2}}\right)$ are critical points. Now $D\left(\pm 1, \frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0$, $f_{xx}\left(\pm 1, \frac{1}{\sqrt{2}}\right) = -2\sqrt{2}\,e^{-3/2} < 0$ and $D\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0$, $f_{xx}\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}\,e^{-3/2} > 0$, so $f\left(\pm 1, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-3/2}$ are local maximum points while $f\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}e^{-3/2}$ are local minimum points. At all critical points $(0, y)$ we have $D(0,y) = 0$, so the Second Derivatives Test gives no information. However, if $y > 0$ then $x^2ye^{-x^2-y^2} \geq 0$ with equality only when $x = 0$, so we have local minimum values $f(0,y) = 0$, $y > 0$. Similarly, if $y < 0$ then $x^2ye^{-x^2-y^2} \leq 0$ with equality when $x = 0$ so $f(0,y) = 0$, $y < 0$ are local maximum values, and $(0,0)$ is a saddle point.

21. $f(x,y) = x^2 + y^2 + x^{-2}y^{-2}$

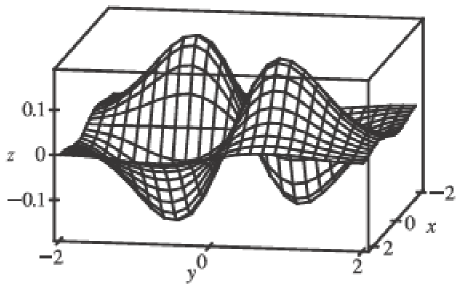


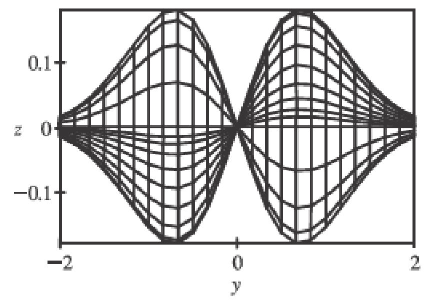


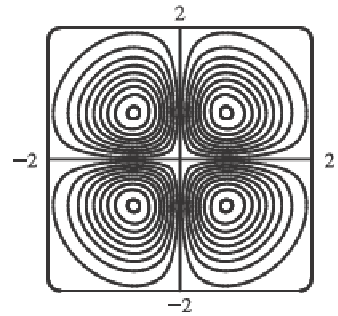


From the graphs, there appear to be local minima of about $f(1, \pm 1) = f(-1, \pm 1) \approx 3$ (and no local maxima or saddle points). $f_x = 2x - 2x^{-3}y^{-2}$, $f_y = 2y - 2x^{-2}y^{-3}$, $f_{xx} = 2 + 6x^{-4}y^{-2}$, $f_{xy} = 4x^{-3}y^{-3}$, $f_{yy} = 2 + 6x^{-2}y^{-4}$. Then $f_x = 0$ implies $2x^4y^2 - 2 = 0$ or $x^4y^2 = 1$ or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $f_y = 0$ implies $2x^2y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies $2x^{-6} - 2 = 0$ or $x^6 = 1$. Thus $x = \pm 1$ and if $x = 1$, $y = \pm 1$; if $x = -1$, $y = \pm 1$. So the critical points are $(1,1)$, $(1,-1)$,$(-1,1)$ and $(-1,-1)$. Now $D(1, \pm 1) = D(-1, \pm 1) = 64 - 16 > 0$ and $f_{xx} > 0$ always, so $f(1, \pm 1) = f(-1, \pm 1) = 3$ are local minima.

22. $f(x,y) = xye^{-x^2-y^2}$







There appear to be local maxima of about $f(\pm 0.7, \pm 0.7) \approx 0.18$ and local minima of about $f(\pm 0.7, \mp 0.7) \approx -0.18$. Also, there seems to be a saddle point at the origin.

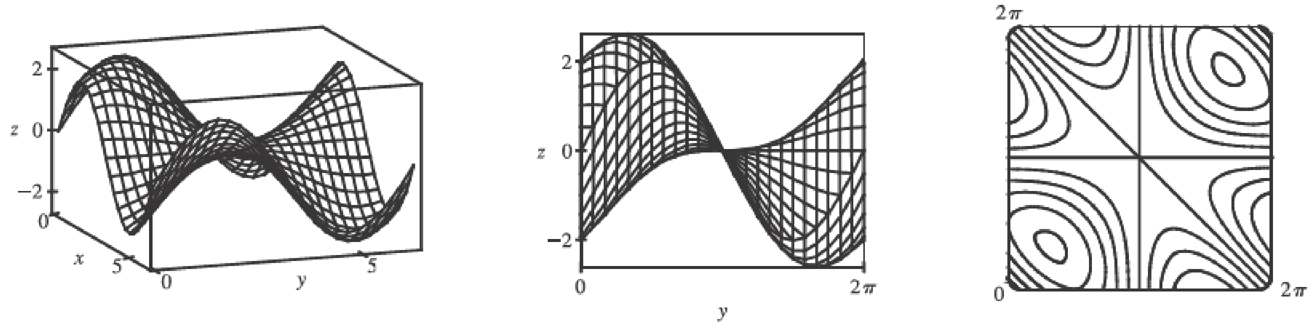
$f_x = ye^{-x^2-y^2}(1-2x^2)$, $f_y = xe^{-x^2-y^2}(1-2y^2)$, $f_{xx} = 2xye^{-x^2-y^2}(2x^2-3)$, $f_{yy} = 2xye^{-x^2-y^2}(2y^2-3)$, $f_{xy} = (1-2x^2)e^{-x^2-y^2}(1-2y^2)$. Then $f_x = 0$ implies $y = 0$ or $x = \pm\frac{1}{\sqrt{2}}$.

Substituting these values into $f_y = 0$ gives the critical points $(0,0)$, $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. Then

$D(x,y) = e^{2(-x^2-y^2)}\left[4x^2y^2(2x^2-3)(2y^2-3) - (1-2x^2)^2(1-2y^2)^2\right]$, so $D(0,0) = -1$, while $D\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$ and $D\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$. But $f_{xx}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) < 0$, $f_{xx}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) > 0$, $f_{xx}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) > 0$, $f_{xx}\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) < 0$.

Hence $(0,0)$ is a saddle point; $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2e}$ are local minima and $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2e}$ are local maxima.

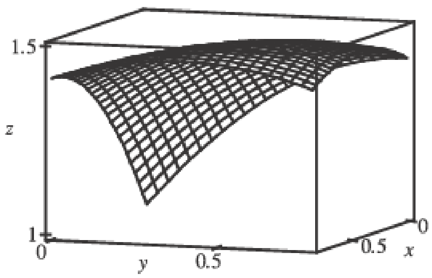
23. $f(x,y) = \sin x + \sin y + \sin(x+y),\ 0 \le x \le 2\pi,\ 0 \le y \le 2\pi$

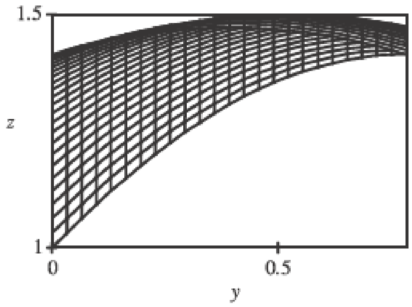


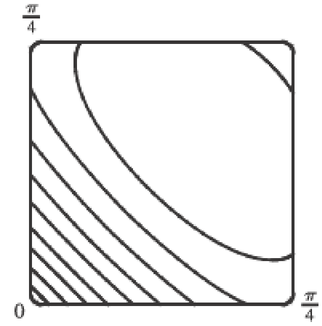
From the graphs it appears that f has a local maximum at about $(1,1)$ with value approximately 2.6, a local minimum at about $(5,5)$ with value approximately -2.6, and a saddle point at about $(3,3)$.

$f_x = \cos x + \cos(x+y),\ f_y = \cos y + \cos(x+y),\ f_{xx} = -\sin x - \sin(x+y),\ f_{yy} = -\sin y - \sin(x+y),$ $f_{xy} = -\sin(x+y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x - \cos y = 0$ or $\cos x = \cos y$. Thus $x = y$ or $x = 2\pi - y$. If $x = y$, $f_x = 0$ becomes $\cos x + \cos 2x = 0$ or $2\cos^2 x + \cos x - 1 = 0$, a quadratic in $\cos x$. Thus $\cos x = -1$ or $\frac{1}{2}$ and $x = \pi, \frac{\pi}{3}$, or $\frac{5\pi}{3}$, yielding the critical points (π,π), $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)$. Similarly if $x = 2\pi - y$, $f_x = 0$ becomes $(\cos x) + 1 = 0$ and the resulting critical point is (π,π). Now $D(x,y) = \sin x\ \sin y + \sin x\ \sin(x+y) + \sin y\ \sin(x+y)$. So $D(\pi,\pi) = 0$ and the Second Derivatives Test doesn't apply. However, along the line $y = x$ we have $f(x,x) = 2\sin x + \sin 2x = 2\sin x + 2\sin x\ \cos x = 2\sin x(1+\cos x)$, and $f(x,x) > 0$ for $0 < x < \pi$ while $f(x,x) < 0$ for $\pi < x < 2\pi$. Thus every disk with center (π,π) contains points where f is positive as well as points where f is negative, so the graph crosses its tangent plane ($z = 0$) there and (π,π) is a saddle point. $D\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{\pi}{3},\frac{\pi}{3}\right) < 0$ so $f\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$ is a local maximum while $D\left(\frac{5\pi}{3},\frac{5\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{5\pi}{3},\frac{5\pi}{3}\right) > 0$, so $f\left(\frac{5\pi}{3},\frac{5\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$ is a local minimum.

24. $f(x,y) = \sin x + \sin y + \cos(x+y),\ 0 \le x \le \frac{\pi}{4},\ 0 \le y \le \frac{\pi}{4}$

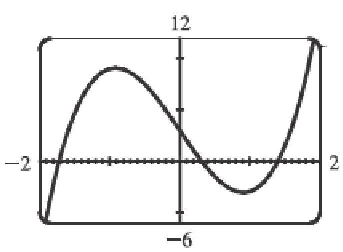


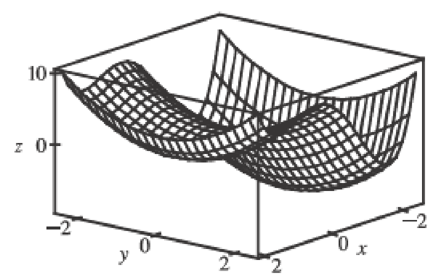




From the graphs, it seems that f has a local maximum at about $(0.5, 0.5)$. $f_x = \cos x - \sin(x+y)$, $f_y = \cos y - \sin(x+y)$, $f_{xx} = -\sin x - \cos(x+y)$, $f_{yy} = -\sin y - \cos(x+y)$, $f_{xy} = -\cos(x+y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x = \cos y$. Thus $x = y$. Substituting $x = y$ into $f_x = 0$ gives $\cos x - \sin 2x = 0$ or $\cos x(1 - 2\sin x) = 0$. But $\cos x \ne 0$ for $0 \le x \le \frac{\pi}{4}$ and $1 - 2\sin x = 0$ implies $x = \frac{\pi}{6}$, so the only critical point is $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$. Here $f_{xx}\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = -1 < 0$ and $D\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = (-1)^2 - \frac{1}{4} > 0$. Thus $f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{3}{2}$ is a local maximum.

25. $f(x,y) = x^4 - 5x^2 + y^2 + 3x + 2 \quad \Rightarrow \quad f_x(x,y) = 4x^3 - 10x + 3$ and $f_y(x,y) = 2y$. $f_y = 0 \quad \Rightarrow \quad y = 0$, and the graph of f_x shows that the roots of $f_x = 0$ are approximately $x = -1.714$, 0.312 and 1.402. (Alternatively, we could have used a calculator or a CAS to find these roots.) So to three decimal places, the critical points are $(-1.714, 0)$, $(1.402, 0)$, and $(0.312, 0)$. Now since $f_{xx} = 12x^2 - 10$, $f_{xy} = 0$, $f_{yy} = 2$, and $D = 24x^2 - 20$, we have $D(-1.714, 0) > 0$, $f_{xx}(-1.714, 0) > 0$, $D(1.402, 0) > 0$, $f_{xx}(1.402, 0) > 0$, and $D(0.312, 0) < 0$. Therefore $f(-1.714, 0) \approx -9.200$ and $f(1.402, 0) \approx 0.242$ are local minima, and $(0.312, 0)$ is a saddle point. The lowest point on the graph is approximately $(-1.714, 0, -9.200)$.

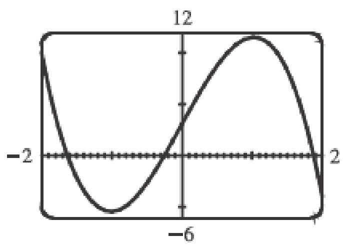


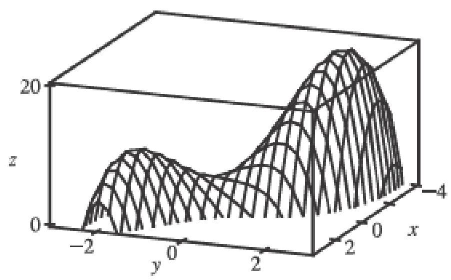


26. $f(x,y) = 5 - 10xy - 4x^2 + 3y - y^4 \quad \Rightarrow \quad f_x(x,y) = -10y - 8x,\ f_y(x,y) = -10x + 3 - 4y^3.$

Now $f_x = 0 \quad \Rightarrow \quad x = -\frac{5}{4}y$, so using a graph, we find solutions to

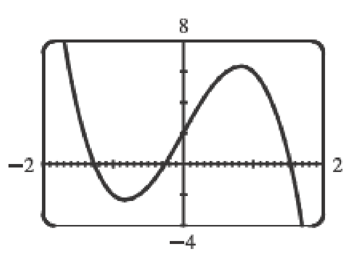
$0 = f_y\left(-\frac{5}{4}y, y\right) = -10\left(-\frac{5}{4}y\right) + 3 - 4y^3 = -4y^3 + \frac{25}{2}y + 3$. (Alternatively, we could have found the roots of $f_x = f_y = 0$ directly, using a calculator or a CAS.) To three decimal places, the solutions are $y \approx 1.877, -0.245$, and -1.633, so f has critical points at approximately $(-2.347, 1.877)$, $(0.306, -0.245)$, and $(2.041, -1.633)$. Now since $f_{xx} = -8$, $f_{xy} = -10$, $f_{yy} = -12y^2$, and $D = 96y^2 - 100$, we have $D(-2.347, 1.877) > 0$, $D(0.306, -0.245) < 0$, and $D(2.041, -1.633) > 0$. Therefore, since $f_{xx} < 0$ everywhere, $f(-2.347, 1.877) \approx 20.238$ and $f(2.041, -1.633) \approx 9.657$ are local maxima, and $(0.306, -0.245)$ is a saddle point. The highest point on the graph is approximately $(-2.347, 1.877, 20.238)$.

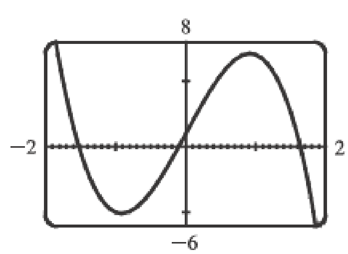


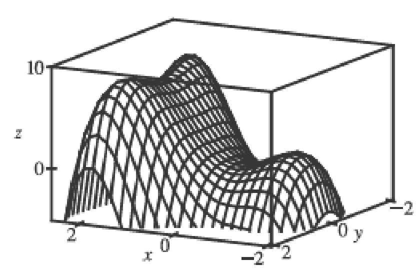


27. $f(x,y) = 2x + 4x^2 - y^2 + 2xy^2 - x^4 - y^4 \quad \Rightarrow \quad f_x(x,y) = 2 + 8x + 2y^2 - 4x^3,\ f_y(x,y) = -2y + 4xy - 4y^3.$

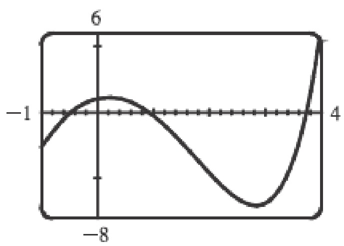
Now $f_y = 0 \quad \Leftrightarrow \quad 2y(2y^2 - 2x + 1) = 0 \quad \Leftrightarrow \quad y = 0$ or $y^2 = x - \frac{1}{2}$. The first of these implies that $f_x = -4x^3 + 8x + 2$, and the second implies that $f_x = 2 + 8x + 2\left(x - \frac{1}{2}\right) - 4x^3 = -4x^3 + 10x + 1$. From the graphs, we see that the first possibility for f_x has roots at approximately -1.267, -0.259, and 1.526, and the second has a root at approximately 1.629 (the negative roots do not give critical points, since $y^2 = x - \frac{1}{2}$ must be positive). So to three decimal places, f has critical points at $(-1.267, 0)$, $(-0.259, 0)$, $(1.526, 0)$, and $(1.629, \pm 1.063)$. Now since $f_{xx} = 8 - 12x^2$, $f_{xy} = 4y$, $f_{yy} = 4x - 12y^2$, and $D = (8 - 12x^2)(4x - 12y^2) - 16y^2$, we have $D(-1.267, 0) > 0$, $f_{xx}(-1.267, 0) > 0$, $D(-0.259, 0) < 0$, $D(1.526, 0) < 0$, $D(1.629, \pm 1.063) > 0$, and $f_{xx}(1.629, \pm 1.063) < 0$. Therefore, to three decimal places, $f(-1.267, 0) \approx 1.310$ and $f(1.629, \pm 1.063) \approx 8.105$ are local maxima, and $(-0.259, 0)$ and $(1.526, 0)$ are saddle points. The highest points on the graph are approximately $(1.629, \pm 1.063, 8.105)$.

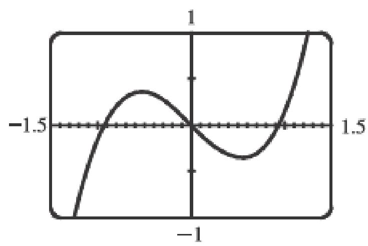


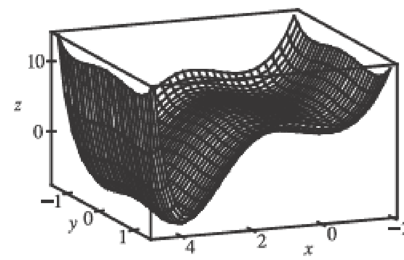




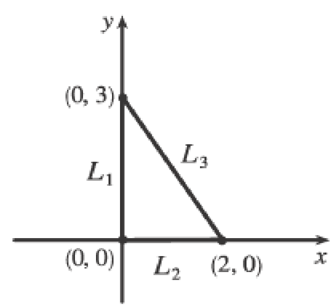
28. $f(x,y) = e^x + y^4 - x^3 + 4\cos y \quad \Rightarrow \quad f_x(x,y) = e^x - 3x^2$ and $f_y(x,y) = 4y^3 - 4\sin y$. From the graphs, we see that to three decimal places, $f_x = 0$ when $x \approx -0.459$, 0.910, or 3.733, and $f_y = 0$ when $y \approx 0$ or ± 0.929. (Alternatively, we could have used a calculator or a CAS to find the roots of $f_x = 0$ and $f_y = 0$.) So, to three decimal places, f has critical points at $(-0.459, 0)$, $(-0.459, \pm 0.929)$, $(0.910, 0)$, $(0.910, \pm 0.929)$, $(3.733, 0)$, and $(3.733, \pm 0.929)$. Now $f_{xx} = e^x - 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4\cos y$, and $D = (e^x - 6x)(12y^2 - 4\cos y)$. Therefore $D(-0.459, 0) < 0$, $D(-0.459, \pm 0.929) > 0$, $f_{xx}(-0.459, \pm 0.929) > 0$, $D(0.910, 0) > 0$, $f_{xx}(0.910, 0) < 0$, $D(0.910, \pm 0.929) < 0$, $D(3.733, 0) < 0$, $D(3.733, \pm 0.929) > 0$, and $f_{xx}(3.733, \pm 0.929) > 0$. So $f(-0.459, \pm 0.929) \approx 3.868$ and $f(3.733, \pm 0.929) \approx -7.077$ are local minima, $f(0.910, 0) \approx 5.731$ is a local maximum, and $(-0.459, 0)$, $(0.910, \pm 0.929)$, and $(3.733, 0)$ are saddle points. The lowest points on the graph are approximately $(3.733, \pm 0.929, -7.077)$.



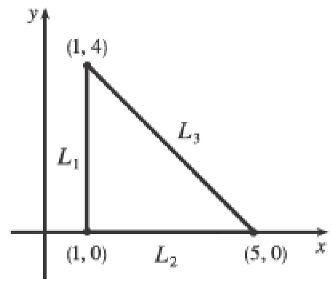




29. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. Here $f_x = 4$, $f_y = -5$ so there are no critical points inside D. Thus the absolute extrema must both occur on the boundary. Along L_1: $x = 0$ and $f(0,y) = 1 - 5y$ for $0 \le y \le 3$, a decreasing function in y, so the maximum value is $f(0,0) = 1$ and the minimum value is $f(0,3) = -14$. Along L_2: $y = 0$ and $f(x,0) = 1 + 4x$ for $0 \le x \le 2$, an increasing function in x, so the minimum value is $f(0,0) = 1$ and the maximum value is $f(2,0) = 9$. Along L_3: $y = -\frac{3}{2}x + 3$ and $f\left(x, -\frac{3}{2}x + 3\right) = \frac{23}{2}x - 14$ for $0 \le x \le 2$, an increasing function in x, so the minimum value is $f(0,3) = -14$ and the maximum value is $f(2,0) = 9$. Thus the absolute maximum of f on D is $f(2,0) = 9$ and the absolute minimum is $f(0,3) = -14$.



30. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. $f_x = y - 1$, $f_y = x - 2$, and setting $f_x = f_y = 0$ gives $(2, 1)$ as the only critical point, where $f(2, 1) = 1$. Along L_1: $x = 1$ and $f(1, y) = 2 - y$ for $0 \le y \le 4$, a decreasing function in y, so the maximum value is $f(1, 0) = 2$ and the minimum value is $f(1, 4) = -2$. Along L_2: $y = 0$ and $f(x, 0) = 3 - x$ for $1 \le x \le 5$, a decreasing function in x, so the maximum value is $f(1, 0) = 2$ and the minimum value is $f(5, 0) = -2$. Along L_3: $y = 5 - x$ and $f(x, 5 - x) = -x^2 + 6x - 7 = -(x - 3)^2 + 2$ for $1 \le x \le 5$, which has a maximum at $x = 3$ where $f(3, 2) = 2$ and a minimum at both $x = 1$ and $x = 5$, where $f(1, 4) = f(5, 0) = -2$. Thus the absolute maximum of f on D is $f(1, 0) = f(3, 2) = 2$ and the absolute minimum is $f(1, 4) = f(5, 0) = -2$.

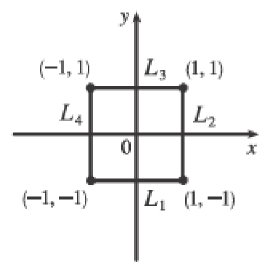


31. $f_x(x, y) = 2x + 2xy$, $f_y(x, y) = 2y + x^2$, and setting $f_x = f_y = 0$ gives $(0, 0)$ as the only critical point in D, with $f(0, 0) = 4$.

On L_1: $y = -1$, $f(x, -1) = 5$, a constant.

On L_2: $x = 1$, $f(1, y) = y^2 + y + 5$, a quadratic in y which attains its maximum at $(1, 1)$, $f(1, 1) = 7$ and its minimum at $\left(1, -\frac{1}{2}\right)$, $f\left(1, -\frac{1}{2}\right) = \frac{19}{4}$.

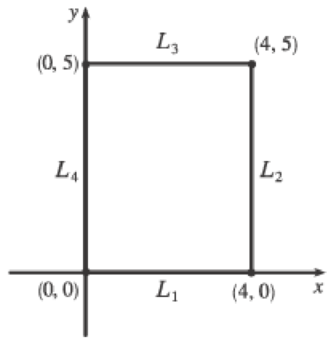
On L_3: $f(x, 1) = 2x^2 + 5$ which attains its maximum at $(-1, 1)$ and $(1, 1)$ with $f(\pm 1, 1) = 7$ and its minimum at $(0, 1)$, $f(0, 1) = 5$.



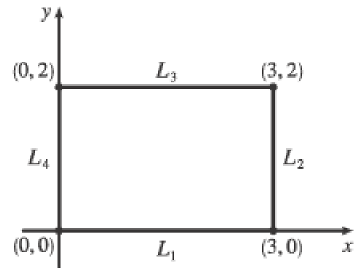
On L_4: $f(-1, y) = y^2 + y + 5$ with maximum at $(-1, 1)$, $f(-1, 1) = 7$ and minimum at $\left(-1, -\frac{1}{2}\right)$, $f\left(-1, -\frac{1}{2}\right) = \frac{19}{4}$.

Thus the absolute maximum is attained at both $(\pm 1, 1)$ with $f(\pm 1, 1) = 7$ and the absolute minimum on D is attained at $(0, 0)$ with $f(0, 0) = 4$.

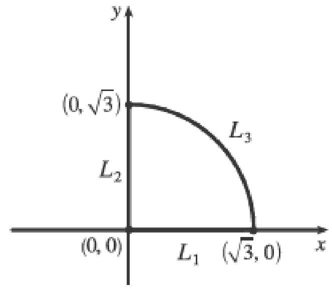
32. $f_x(x,y) = 4 - 2x$ and $f_y(x,y) = 6 - 2y$, so the only critical point is $(2,3)$ (which is in D) where $f(2,3) = 13$. Along L_1: $y = 0$, so $f(x,0) = 4x - x^2 = -(x-2)^2 + 4$, $0 \le x \le 4$, which has a maximum value when $x = 2$ where $f(2,0) = 4$ and a minimum value both when $x = 0$ and $x = 4$, where $f(0,0) = f(4,0) = 0$. Along L_2: $x = 4$, so $f(4,y) = 6y - y^2 = -(y-3)^2 + 9$, $0 \le y \le 5$, which has a maximum value when $y = 3$ where $f(4,3) = 9$ and a minimum value when $y = 0$ where $f(4,0) = 0$. Along L_3: $y = 5$, so $f(x,5) = -x^2 + 4x + 5 = -(x-2)^2 + 9$, $0 \le x \le 4$, which has a maximum value when $x = 2$ where $f(2,5) = 9$ and a minimum value both when $x = 0$ and $x = 4$, where $f(0,5) = f(4,5) = 5$. Along L_4: $x = 0$, so $f(0,y) = 6y - y^2 = -(y-3)^2 + 9$, $0 \le y \le 5$, which has a maximum value when $y = 3$ where $f(0,3) = 9$ and a minimum value when $y = 0$ where $f(0,0) = 0$. Thus the absolute maximum is $f(2,3) = 13$ and the absolute minimum is attained at both $(0,0)$ and $(4,0)$, where $f(0,0) = f(4,0) = 0$.



33. $f(x,y) = x^4 + y^4 - 4xy + 2$ is a polynomial and hence continuous on D, so it has an absolute maximum and minimum on D. In Exercise 7, we found the critical points of f; only $(1,1)$ with $f(1,1) = 0$ is inside D. On L_1: $y = 0$, $f(x,0) = x^4 + 2$, $0 \le x \le 3$, a polynomial in x which attains its maximum at $x = 3$, $f(3,0) = 83$, and its minimum at $x = 0$, $f(0,0) = 2$. On L_2: $x = 3$, $f(3,y) = y^4 - 12y + 83$, $0 \le y \le 2$, a polynomial in y which attains its minimum at $y = \sqrt[3]{3}$, $f(3, \sqrt[3]{3}) = 83 - 9\sqrt[3]{3} \approx 70.0$, and its maximum at $y = 0$, $f(3,0) = 83$. On L_3: $y = 2$, $f(x,2) = x^4 - 8x + 18$, $0 \le x \le 3$, a polynomial in x which attains its minimum at $x = \sqrt[3]{2}$, $f(\sqrt[3]{2}, 2) = 18 - 6\sqrt[3]{2} \approx 10.4$, and its maximum at $x = 3$, $f(3,2) = 75$. On L_4: $x = 0$, $f(0,y) = y^4 + 2$, $0 \le y \le 2$, a polynomial in y which attains its maximum at $y = 2$, $f(0,2) = 18$, and its minimum at $y = 0$, $f(0,0) = 2$. Thus the absolute maximum of f on D is $f(3,0) = 83$ and the absolute minimum is $f(1,1) = 0$.



34. $f_x = y^2$ and $f_y = 2xy$, and since $f_x = 0 \quad \Leftrightarrow \quad y = 0$, there are no critical points in the interior of D. Along L_1: $y = 0$ and $f(x, 0) = 0$.

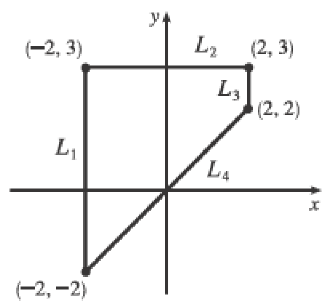


Along L_2: $x = 0$ and $f(0, y) = 0$. Along L_3: $y = \sqrt{3 - x^2}$, so let $g(x) = f\left(x, \sqrt{3 - x^2}\right) = 3x - x^3$ for $0 \le x \le \sqrt{3}$. Then $g'(x) = 3 - 3x^2 = 0 \quad \Leftrightarrow \quad x = 1$. The maximum value is $f\left(1, \sqrt{2}\right) = 2$ and the minimum occurs both at $x = 0$ and $x = \sqrt{3}$ where $f\left(0, \sqrt{3}\right) = f\left(\sqrt{3}, 0\right) = 0$. Thus the absolute maximum of f on D is $f\left(1, \sqrt{2}\right) = 2$, and the absolute minimum is 0 which occurs at all points along L_1 and L_2.

35. $f_x(x, y) = 6x^2$ and $f_y(x, y) = 4y^3$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0, 0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$ so let $g(x) = f(x, y) = 2x^3 + (1 - x^2)^2 = x^4 + 2x^3 - 2x^2 + 1, -1 \le x \le 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \quad \Rightarrow \quad x = 0$, -2, or $\frac{1}{2}$. $f(0, \pm 1) = g\,(0) = 1$, $f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$, and $(-2, -3)$ is not in D. Checking the endpoints, we get $f(-1, 0) = g(-1) = -2$ and $f(1, 0) = g(1) = 2$. Thus the absolute maximum and minimum of f on D are $f(1, 0) = 2$ and $f(-1, 0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos\theta$, $y = \sin\theta$, so $f(\cos\theta, \sin\theta) = 2\cos^3\theta + \sin^4\theta$, $0 \le \theta \le 2\pi$.

36. $f_x(x, y) = 3x^2 - 3$ and $f_y(x, y) = -3y^2 + 12$ and the critical points are $(1, 2)$, $(1, -2)$, $(-1, 2)$, and $(-1, -2)$. But only $(1, 2)$ and $(-1, 2)$ are in D and $f(1, 2) = 14$, $f(-1, 2) = 18$. Along L_1: $x = -2$ and $f(-2, y) = -2 - y^3 + 12y$, $-2 \le y \le 3$, which has a maximum at $y = 2$ where $f(-2, 2) = 14$ and a minimum at $y = -2$ where $f(-2, -2) = -18$. Along L_2: $x = 2$ and



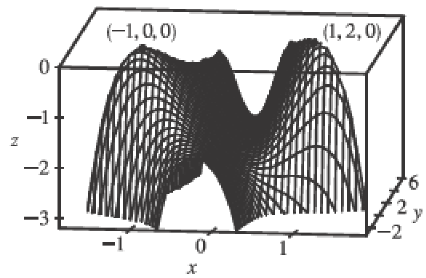
$f(2, y) = 2 - y^3 + 12y$, $2 \le y \le 3$, which has a maximum at $y = 2$ where $f\,(2, 2) = 18$ and a minimum at $y = 3$ where $f(2, 3) = 11$. Along L_3: $y = 3$ and $f(x, 3) = x^3 - 3x + 9$, $-2 \le x \le 2$, which has a maximum at $x = -1$ and $x = 2$ where $f(-1, 3) = f\,(2, 3) = 11$ and a minimum at $x = 1$ and $x = -2$ where $f(1, 3) = f(-2, 3) = 7$.

Along L_4: $y = x$ and $f(x, x) = 9x$, $-2 \le x \le 2$, which has a maximum at $x = 2$ where $f(2, 2) = 18$ and a minimum at $x = -2$ where $f(-2, -2) = -18$. So the absolute maximum value of f on D is $f\,(2, 2) = 18$ and the minimum is $f(-2, -2) = -18$.

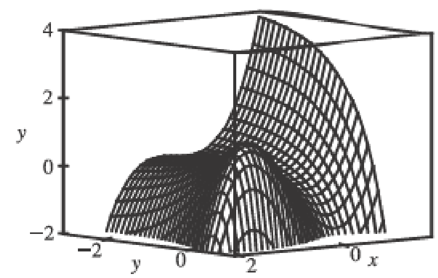
37. $f(x,y) = -(x^2-1)^2 - (x^2y-x-1)^2 \quad \Rightarrow \quad f_x(x,y) = -2(x^2-1)(2x) - 2(x^2y-x-1)(2xy-1)$ and $f_y(x,y) = -2(x^2y-x-1)x^2$. Setting $f_y(x,y) = 0$ gives either $x = 0$ or $x^2y - x - 1 = 0$.

There are no critical points for $x = 0$, since $f_x(0,y) = -2$, so we set $x^2y - x - 1 = 0 \quad \Leftrightarrow \quad y = \dfrac{x+1}{x^2} \quad [x \neq 0]$,

so $f_x\left(x, \dfrac{x+1}{x^2}\right) = -2(x^2-1)(2x) - 2\left(x^2\,\dfrac{x+1}{x^2} - x - 1\right)\left(2x\,\dfrac{x+1}{x^2} - 1\right) = -4x(x^2-1)$. Therefore $f_x(x,y) = f_y(x,y) = 0$ at the points $(1,2)$ and $(-1,0)$. To classify these critical points, we calculate $f_{xx}(x,y) = -12x^2 - 12x^2y^2 + 12xy + 4y + 2$, $f_{yy}(x,y) = -2x^4$, and $f_{xy}(x,y) = -8x^3y + 6x^2 + 4x$. In order to use the Second Derivatives Test we calculate $D(-1,0) = f_{xx}(-1,0)\,f_{yy}(-1,0) - [f_{xy}(-1,0)]^2 = 16 > 0$, $f_{xx}(-1,0) = -10 < 0$, $D(1,2) = 16 > 0$, and $f_{xx}(1,2) = -26 < 0$, so both $(-1,0)$ and $(1,2)$ give local maxima.



38. $f(x,y) = 3xe^y - x^3 - e^{3y}$ is differentiable everywhere, so the requirement for critical points is that $f_x = 3e^y - 3x^2 = 0 \quad (1)$ and $f_y = 3xe^y - 3e^{3y} = 0 \quad (2)$. From (1) we obtain $e^y = x^2$, and then (2) gives $3x^3 - 3x^6 = 0 \quad \Rightarrow \quad x = 1$ or 0, but only $x = 1$ is valid, since $x = 0$ makes (1) impossible. So substituting $x = 1$ into (1) gives $y = 0$, and the only critical point is $(1,0)$.



The Second Derivatives Test shows that this gives a local maximum, since $D(1,0) = \left[-6x(3xe^y - 9e^{3y}) - (3e^y)^2\right]_{(1,0)} = 27 > 0$ and $f_{xx}(1,0) = [-6x]_{(1,0)} = -6 < 0$. But $f(1,0) = 1$ is not an absolute maximum because, for instance, $f(-3,0) = 17$. This can also be seen from the graph.

39. Let d be the distance from $(2,1,-1)$ to any point (x,y,z) on the plane $x + y - z = 1$, so $d = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2}$ where $z = x + y - 1$, and we minimize $d^2 = f(x,y) = (x-2)^2 + (y-1)^2 + (x+y)^2$. Then $f_x(x,y) = 2(x-2) + 2(x+y) = 4x + 2y - 4$, $f_y(x,y) = 2(y-1) + 2(x+y) = 2x + 4y - 2$. Solving $4x + 2y - 4 = 0$ and $2x + 4y - 2 = 0$ simultaneously gives $x = 1$, $y = 0$. An absolute minimum exists (since there is a minimum distance from the point to the plane) and it must occur at a critical point, so the shortest distance occurs for $x = 1$, $y = 0$ for which $d = \sqrt{(1-2)^2 + (0-1)^2 + (0+1)^2} = \sqrt{3}$.

40. Here the distance d from a point on the plane to the point $(1, 2, 3)$ is $d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$, where $z = 4 - x + y$. We can minimize $d^2 = f(x, y) = (x-1)^2 + (y-2)^2 + (1-x+y)^2$, so $f_x(x, y) = 2(x-1) + 2(1-x+y)(-1) = 4x - 2y - 4$ and $f_y(x, y) = 2(y-2) + 2(1-x+y) = 4y - 2x - 2$. Solving $4x - 2y - 4 = 0$ and $4y - 2x - 2 = 0$ simultaneously gives $x = \frac{5}{3}$ and $y = \frac{4}{3}$, so the only critical point is $\left(\frac{5}{3}, \frac{4}{3}\right)$. This point must correspond to the minimum distance, so the point on the plane closest to $(1, 2, 3)$ is $\left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right)$.

41. Let d be the distance from the point $(4, 2, 0)$ to any point (x, y, z) on the cone, so $d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$ where $z^2 = x^2 + y^2$, and we minimize $d^2 = (x-4)^2 + (y-2)^2 + x^2 + y^2 = f(x, y)$. Then $f_x(x, y) = 2(x-4) + 2x = 4x - 8$, $f_y(x, y) = 2(y-2) + 2y = 4y - 4$, and the critical points occur when $f_x = 0 \Rightarrow x = 2$, $f_y = 0 \Rightarrow y = 1$. Thus the only critical point is $(2, 1)$. An absolute minimum exists (since there is a minimum distance from the cone to the point) which must occur at a critical point, so the points on the cone closest to $(4, 2, 0)$ are $\left(2, 1, \pm\sqrt{5}\right)$.

42. The distance from the origin to a point (x, y, z) on the surface is $d = \sqrt{x^2 + y^2 + z^2}$ where $y^2 = 9 + xz$, so we minimize $d^2 = x^2 + 9 + xz + z^2 = f(x, z)$. Then $f_x = 2x + z$, $f_z = x + 2z$, and $f_x = 0$, $f_z = 0 \Rightarrow x = 0, z = 0$, so the only critical point is $(0, 0)$. $D(0, 0) = (2)(2) - 1 = 3 > 0$ with $f_{xx}(0, 0) = 2 > 0$, so this is a minimum. Thus $y^2 = 9 + 0 \Rightarrow y = \pm 3$ and the points on the surface closest to the origin are $(0, \pm 3, 0)$.

43. $x + y + z = 100$, so maximize $f(x, y) = xy(100 - x - y)$. $f_x = 100y - 2xy - y^2$, $f_y = 100x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 100 - 2x - 2y$. Then $f_x = 0$ implies $y = 0$ or $y = 100 - 2x$. Substituting $y = 0$ into $f_y = 0$ gives $x = 0$ or $x = 100$ and substituting $y = 100 - 2x$ into $f_y = 0$ gives $3x^2 - 100x = 0$ so $x = 0$ or $\frac{100}{3}$. Thus the critical points are $(0, 0)$, $(100, 0)$, $(0, 100)$ and $\left(\frac{100}{3}, \frac{100}{3}\right)$.
$D(0, 0) = D(100, 0) = D(0, 100) = -10{,}000$ while $D\left(\frac{100}{3}, \frac{100}{3}\right) = \frac{10{,}000}{3}$ and $f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = -\frac{200}{3} < 0$. Thus $(0, 0)$, $(100, 0)$ and $(0, 100)$ are saddle points whereas $f\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local maximum. Thus the numbers are $x = y = z = \frac{100}{3}$.

44. Let x, y, z, be the positive numbers. Then $x + y + z = 12$ and we want to minimize $x^2 + y^2 + z^2 = x^2 + y^2 + (12 - x - y)^2 = f(x, y)$ for $0 < x, y < 12$. $f_x = 2x + 2(12 - x - y)(-1) = 4x + 2y - 24$, $f_y = 2y + 2(12 - x - y)(-1) = 2x + 4y - 24$, $f_{xx} = 4$, $f_{xy} = 2$, $f_{yy} = 4$. Then $f_x = 0$ implies $4x + 2y = 24$ or $y = 12 - 2x$ and substituting into $f_y = 0$ gives $2x + 4(12 - 2x) = 24 \Rightarrow 6x = 24 \Rightarrow x = 4$ and then $y = 4$, so the only critical point is $(4, 4)$. $D(4, 4) = 16 - 4 > 0$ and $f_{xx}(4, 4) = 4 > 0$, so $f(4, 4)$ is a local minimum. $f(4, 4)$ is also the absolute minimum [compare to the values of f as $x, y \to 0$ or 12] so the numbers are $x = y = z = 4$.

45. Center the sphere at the origin so that its equation is $x^2 + y^2 + z^2 = r^2$, and orient the inscribed rectangular box so that its edges are parallel to the coordinate axes. Any vertex of the box satisfies $x^2 + y^2 + z^2 = r^2$, so take (x, y, z) to be the vertex in the first octant. Then the box has length $2x$, width $2y$, and height $2z = 2\sqrt{r^2 - x^2 - y^2}$ with volume given by

$V(x, y) = (2x)(2y)\left(2\sqrt{r^2 - x^2 - y^2}\right) = 8xy\sqrt{r^2 - x^2 - y^2}$ for $0 < x < r$, $0 < y < r$. Then

$V_x = (8xy) \cdot \frac{1}{2}(r^2 - x^2 - y^2)^{-1/2}(-2x) + \sqrt{r^2 - x^2 - y^2} \cdot 8y = \dfrac{8y(r^2 - 2x^2 - y^2)}{\sqrt{r^2 - x^2 - y^2}}$ and $V_y = \dfrac{8x(r^2 - x^2 - 2y^2)}{\sqrt{r^2 - x^2 - y^2}}$.

Setting $V_x = 0$ gives $y = 0$ or $2x^2 + y^2 = r^2$, but $y > 0$ so only the latter solution applies. Similarly, $V_y = 0$ with $x > 0$ implies $x^2 + 2y^2 = r^2$. Substituting, we have $2x^2 + y^2 = x^2 + 2y^2 \Rightarrow x^2 = y^2 \Rightarrow y = x$. Then $x^2 + 2y^2 = r^2 \Rightarrow 3x^2 = r^2 \Rightarrow x = \sqrt{r^2/3} = r/\sqrt{3} = y$. Thus the only critical point is $\left(r/\sqrt{3}, r/\sqrt{3}\right)$. There must be a maximum volume and here it must occur at a critical point, so the maximum volume occurs when $x = y = r/\sqrt{3}$ and the maximum volume is $V\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right) = 8\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)\sqrt{r^2 - \left(\frac{r}{\sqrt{3}}\right)^2 - \left(\frac{r}{\sqrt{3}}\right)^2} = \dfrac{8}{3\sqrt{3}}r^3$.

46. Let x, y, and z be the dimensions of the box. We wish to minimize surface area $= 2xy + 2xz + 2yz$, but we have volume $= xyz = 1000 \Rightarrow z = \dfrac{1000}{xy}$ so we minimize

$f(x, y) = 2xy + 2x\left(\dfrac{1000}{xy}\right) + 2y\left(\dfrac{1000}{xy}\right) = 2xy + \dfrac{2000}{y} + \dfrac{2000}{x}$. Then $f_x = 2y - \dfrac{2000}{x^2}$ and $f_y = 2x - \dfrac{2000}{y^2}$. Setting $f_x = 0$ implies $y = \dfrac{1000}{x^2}$ and substituting into $f_y = 0$ gives $x - \dfrac{x^4}{1000} = 0 \Rightarrow x^3 = 1000$ [since $x \neq 0$] $\Rightarrow x = 10$.

The surface area has a minimum but no maximum and it must occur at a critical point, so the minimal surface area occurs for a box with dimensions $x = 10$ cm, $y = 1000/10^2 = 10$ cm, $z = 1000/10^2 = 10$ cm.

47. Maximize $f(x, y) = \dfrac{xy}{3}(6 - x - 2y)$, then the maximum volume is $V = xyz$.

$f_x = \frac{1}{3}(6y - 2xy - y^2) = \frac{1}{3}y(6 - 2x - 2y)$ and $f_y = \frac{1}{3}x(6 - x - 4y)$. Setting $f_x = 0$ and $f_y = 0$ gives the critical point $(2, 1)$ which geometrically must yield a maximum. Thus the volume of the largest such box is $V = (2)(1)\left(\frac{2}{3}\right) = \frac{4}{3}$.

48. Surface area $= 2(xy + xz + yz) = 64\text{ cm}^2$, so $xy + xz + yz = 32$ or $z = \dfrac{32 - xy}{x + y}$. Maximize the volume

$f(x, y) = xy\dfrac{32 - xy}{x + y}$. Then $f_x = \dfrac{32y^2 - 2xy^3 - x^2y^2}{(x + y)^2} = y^2\dfrac{32 - 2xy - x^2}{(x + y)^2}$ and $f_y = x^2\dfrac{32 - 2xy - y^2}{(x + y)^2}$. Setting $f_x = 0$ implies $y = \dfrac{32 - x^2}{2x}$ and substituting into $f_y = 0$ gives $32(4x^2) - (32 - x^2)(4x^2) - (32 - x^2)^2 = 0$ or $3x^4 + 64x^2 - (32)^2 = 0$. Thus $x^2 = \frac{64}{6}$ or $x = \frac{8}{\sqrt{6}}$, $y = \frac{64/3}{16/\sqrt{6}} = \frac{8}{\sqrt{6}}$ and $z = \frac{8}{\sqrt{6}}$. Thus the box is a cube with edge length $\frac{8}{\sqrt{6}}$ cm.

49. Let the dimensions be x, y, and z; then $4x + 4y + 4z = c$ and the volume is $V = xyz = xy\left(\frac{1}{4}c - x - y\right) = \frac{1}{4}cxy - x^2y - xy^2$, $x > 0$, $y > 0$. Then $V_x = \frac{1}{4}cy - 2xy - y^2$ and $V_y = \frac{1}{4}cx - x^2 - 2xy$, so $V_x = 0 = V_y$ when $2x + y = \frac{1}{4}c$ and $x + 2y = \frac{1}{4}c$. Solving, we get $x = \frac{1}{12}c$, $y = \frac{1}{12}c$ and $z = \frac{1}{4}c - x - y = \frac{1}{12}c$. From the geometrical nature of the problem, this critical point must give an absolute maximum. Thus the box is a cube with edge length $\frac{1}{12}c$.

50. The cost equals $5xy + 2(xz + yz)$ and $xyz = V$, so $C(x, y) = 5xy + 2V(x + y)/(xy) = 5xy + 2V(x^{-1} + y^{-1})$. Then $C_x = 5y - 2Vx^{-2}$, $C_y = 5x - 2Vy^{-2}$, $f_x = 0$ implies $y = 2V/(5x^2)$, $f_y = 0$ implies $x = \sqrt[3]{\frac{2}{5}V} = y$. Thus the dimensions of the aquarium which minimize the cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.