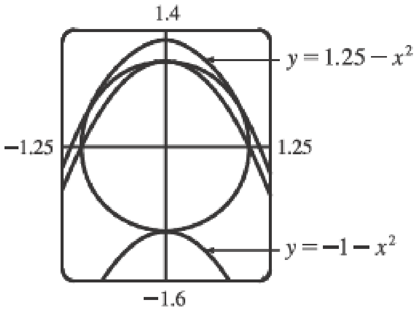
1. At the extreme values of f, the level curves of f just touch the curve $g(x, y) = 8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x, y) = c$ with the largest value of c which still intersects the curve $g(x, y) = 8$ is approximately $c = 59$, and the smallest value of c corresponding to a level curve which intersects $g(x, y) = 8$ appears to be $c = 30$. Thus we estimate the maximum value of f subject to the constraint $g(x, y) = 8$ to be about 59 and the minimum to be 30.

2. (a) The values $c = \pm 1$ and $c = 1.25$ seem to give curves which are tangent to the circle. These values represent possible extreme values of the function $x^2 + y$ subject to the constraint $x^2 + y^2 = 1$.



(b) $\nabla f = \langle 2x, 1 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. So $2x = 2\lambda x \Rightarrow$ either $\lambda = 1$ or $x = 0$. If $\lambda = 1$, then $y = \frac{1}{2}$ and so $x = \pm\frac{\sqrt{3}}{2}$ (from the constraint). If $x = 0$, then $y = \pm 1$. Therefore f has possible extreme values at the points $(0, \pm 1)$ and $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. We calculate $f\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{5}{4}$ (the maximum value), $f(0, 1) = 1$, and $f(0, -1) = -1$ (the minimum value). These are our answers from part (a).

3. $f(x, y) = x^2 + y^2$, $g(x, y) = xy = 1$, and $\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2y \rangle = \langle \lambda y, \lambda x \rangle$, so $2x = \lambda y$, $2y = \lambda x$, and $xy = 1$. From the last equation, $x \neq 0$ and $y \neq 0$, so $2x = \lambda y \Rightarrow \lambda = 2x/y$. Substituting, we have $2y = (2x/y)\,x \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$. But $xy = 1$, so $x = y = \pm 1$ and the possible points for the extreme values of f are $(1, 1)$ and $(-1, -1)$. Here there is no maximum value, since the constraint $xy = 1$ allows x or y to become arbitrarily large, and hence $f(x, y) = x^2 + y^2$ can be made arbitrarily large. The minimum value is $f(1, 1) = f(-1, -1) = 2$.

4. $f(x, y) = 4x + 6y$, $g(x, y) = x^2 + y^2 = 13 \Rightarrow \nabla f = \langle 4, 6 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2\lambda x = 4$ and $2\lambda y = 6$ imply $x = \dfrac{2}{\lambda}$ and $y = \dfrac{3}{\lambda}$. But $13 = x^2 + y^2 = \left(\dfrac{2}{\lambda}\right)^2 + \left(\dfrac{3}{\lambda}\right)^2 \Rightarrow 13 = \dfrac{13}{\lambda^2} \Rightarrow \lambda = \pm 1$, so f has possible extreme values at the points $(2, 3)$, $(-2, -3)$. We compute $f(2, 3) = 26$ and $f(-2, -3) = -26$, so the maximum value of f on $x^2 + y^2 = 13$ is $f(2, 3) = 26$ and the minimum value is $f(-2, -3) = -26$.

5. $f(x, y) = x^2 y$, $g(x, y) = x^2 + 2y^2 = 6 \Rightarrow \nabla f = \langle 2xy, x^2 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 4\lambda y \rangle$. Then $2xy = 2\lambda x$ implies $x = 0$ or $\lambda = y$. If $x = 0$, then $x^2 = 4\lambda y$ implies $\lambda = 0$ or $y = 0$. However, if $y = 0$ then $g(x, y) = 0$, a contradiction. So $\lambda = 0$ and then $g(x, y) = 6 \Rightarrow y = \pm\sqrt{3}$. If $\lambda = y$, then $x^2 = 4\lambda y$ implies $x^2 = 4y^2$, and so $g(x, y) = 6 \Rightarrow 4y^2 + 2y^2 = 6 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$. Thus f has possible extreme values at the points $(0, \pm\sqrt{3})$, $(\pm 2, 1)$, and $(\pm 2, -1)$. After evaluating f at these points, we find the maximum value to be $f(\pm 2, 1) = 4$ and the minimum to be $f(\pm 2, -1) = -4$.

6. $f(x,y)=e^{xy}$, $g(x,y)=x^3+y^3=16$, and $\nabla f=\lambda\nabla g \quad\Rightarrow\quad \langle ye^{xy}, xe^{xy}\rangle=\langle 3\lambda x^2, 3\lambda y^2\rangle$, so $ye^{xy}=3\lambda x^2$ and $xe^{xy}=3\lambda y^2$. Note that $x=0 \quad\Leftrightarrow\quad y=0$ which contradicts $x^3+y^3=16$, so we may assume $x\neq 0$, $y\neq 0$, and then $\lambda=ye^{xy}/(3x^2)=xe^{xy}/(3y^2) \quad\Rightarrow\quad x^3=y^3 \quad\Rightarrow\quad x=y$. But $x^3+y^3=16$, so $2x^3=16 \quad\Rightarrow\quad x=2=y$. Here there is no minimum value, since we can choose points satisfying the constraint $x^3+y^3=16$ that make $f(x,y)=e^{xy}$ arbitrarily close to 0 (but never equal to 0). The maximum value is $f(2,2)=e^4$.

7. $f(x,y,z)=2x+6y+10z$, $g(x,y,z)=x^2+y^2+z^2=35 \quad\Rightarrow\quad \nabla f=\langle 2,6,10\rangle$, $\lambda\nabla g=\langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $2\lambda x=2$, $2\lambda y=6$, $2\lambda z=10$ imply $x=\dfrac{1}{\lambda}$, $y=\dfrac{3}{\lambda}$, and $z=\dfrac{5}{\lambda}$. But $35=x^2+y^2+z^2=\left(\dfrac{1}{\lambda}\right)^2+\left(\dfrac{3}{\lambda}\right)^2+\left(\dfrac{5}{\lambda}\right)^2 \quad\Rightarrow$ $35=\dfrac{35}{\lambda^2} \quad\Rightarrow\quad \lambda=\pm 1$, so f has possible extreme values at the points $(1,3,5)$, $(-1,-3,-5)$. The maximum value of f on $x^2+y^2+z^2=35$ is $f(1,3,5)=70$, and the minimum is $f(-1,-3,-5)=-70$.

8. $f(x,y,z)=8x-4z$, $g(x,y,z)=x^2+10y^2+z^2=5 \quad\Rightarrow\quad \nabla f=\langle 8,0,-4\rangle$, $\lambda\nabla g=\langle 2\lambda x, 20\lambda y, 2\lambda z\rangle$. Then $2\lambda x=8$, $20\lambda y=0$, $2\lambda z=-4$ imply $x=\dfrac{4}{\lambda}$, $y=0$, and $z=-\dfrac{2}{\lambda}$. But $5=x^2+10y^2+z^2=\left(\dfrac{4}{\lambda}\right)^2+10\,(0)^2+\left(-\dfrac{2}{\lambda}\right)^2 \quad\Rightarrow$ $5=\dfrac{20}{\lambda^2} \quad\Rightarrow\quad \lambda=\pm 2$, so f has possible extreme values at the points $(2,0,-1)$, $(-2,0,1)$. The maximum of f on $x^2+10y^2+z^2=5$ is $f(2,0,-1)=20$, and the minimum is $f(-2,0,1)=-20$.

9. $f(x,y,z)=xyz$, $g(x,y,z)=x^2+2y^2+3z^2=6 \quad\Rightarrow\quad \nabla f=\langle yz, xz, xy\rangle$, $\lambda\nabla g=\langle 2\lambda x, 4\lambda y, 6\lambda z\rangle$. If $\lambda=0$ then at least one of the coordinates is 0, in which case $f(x,y,z)=0$. (None of these ends up giving a maximum or minimum.) If $\lambda\neq 0$, then $\nabla f=\lambda\nabla g$ implies $\lambda=(yz)/(2x)=(xz)/(4y)=(xy)/(6z)$ or $x^2=2y^2$ and $z^2=\frac{2}{3}y^2$. Thus $x^2+2y^2+3z^2=6$ implies $6y^2=6$ or $y=\pm 1$. Thus the possible remaining points are $\left(\sqrt{2},\pm 1,\sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2},\pm 1,-\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2},\pm 1,\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2},\pm 1,-\sqrt{\frac{2}{3}}\right)$. The maximum value of f on the ellipsoid is $\frac{2}{\sqrt{3}}$, occurring when all coordinates are positive or exactly two are negative and the minimum is $-\frac{2}{\sqrt{3}}$ occurring when 1 or 3 of the coordinates are negative.

10. $f(x,y,z)=x^2y^2z^2$, $g(x,y,z)=x^2+y^2+z^2=1 \quad\Rightarrow\quad \nabla f=\langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2\rangle$, $\lambda\nabla g=\langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $\nabla f=\lambda\nabla g$ implies **(1)** $\lambda=y^2z^2=x^2z^2=x^2y^2$ and $\lambda\neq 0$, or **(2)** $\lambda=0$ and one or two (but not three) of the coordinates are 0. If **(1)** then $x^2=y^2=z^2=\frac{1}{3}$. The minimum value of f on the sphere occurs in case **(2)** with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from **(1)**, that is, the points $\left(\pm\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$.

11. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x^4 + y^4 + z^4 = 1 \quad \Rightarrow \quad \nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \langle 4\lambda x^3, 4\lambda y^3, 4\lambda z^3 \rangle$.

Case 1: If $x \neq 0$, $y \neq 0$ and $z \neq 0$, then $\nabla f = \lambda \nabla g$ implies $\lambda = 1/(2x^2) = 1/(2y^2) = 1/(2z^2)$ or $x^2 = y^2 = z^2$ and $3x^4 = 1$ or $x = \pm\frac{1}{\sqrt[4]{3}}$ giving the points $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$ all with an f-value of $\sqrt{3}$.

Case 2: If one of the variables equals zero and the other two are not zero, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{\sqrt{2}}$ and corresponding f value of $\sqrt{2}$.

Case 3: If exactly two of the variables are zero, then the third variable has value ± 1 with the corresponding f value of 1. Thus on $x^4 + y^4 + z^4 = 1$, the maximum value of f is $\sqrt{3}$ and the minimum value is 1.

12. $f(x,y,z) = x^4 + y^4 + z^4$, $g(x,y,z) = x^2 + y^2 + z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 4x^3, 4y^3, 4z^3 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$.

Case 1: If $x \neq 0$, $y \neq 0$ and $z \neq 0$ then $\nabla f = \lambda \nabla g$ implies $\lambda = 2x^2 = 2y^2 = 2z^2$ or $x^2 = y^2 = z^2 = \frac{1}{3}$ yielding 8 points each with an f-value of $\frac{1}{3}$.

Case 2: If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f-value is $\frac{1}{2}$.

Case 3: If exactly two of the variables are 0, then the third variable has value ± 1 with corresponding f-value of 1. Thus on $x^2 + y^2 + z^2 = 1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.

13. $f(x,y,z,t) = x + y + z + t$, $g(x,y,z,t) = x^2 + y^2 + z^2 + t^2 = 1 \quad \Rightarrow \quad \langle 1,1,1,1 \rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z, 2\lambda t \rangle$, so $\lambda = 1/(2x) = 1/(2y) = 1/(2z) = 1/(2t)$ and $x = y = z = t$. But $x^2 + y^2 + z^2 + t^2 = 1$, so the possible points are $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$. Thus the maximum value of f is $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2$ and the minimum value is $f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = -2$.

14. $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$, $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \quad \Rightarrow$ $\langle 1, 1, \ldots, 1 \rangle = \langle 2\lambda x_1, 2\lambda x_2, \ldots, 2\lambda x_n \rangle$, so $\lambda = 1/(2x_1) = 1/(2x_2) = \cdots = 1/(2x_n)$ and $x_1 = x_2 = \cdots = x_n$. But $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, so $x_i = \pm 1/\sqrt{n}$ for $i = 1, \ldots, n$. Thus the maximum value of f is $f(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n}) = \sqrt{n}$ and the minimum value is $f(-1/\sqrt{n}, -1/\sqrt{n}, \ldots, -1/\sqrt{n}) = -\sqrt{n}$.

15. $f(x,y,z) = x + 2y$, $g(x,y,z) = x + y + z = 1$, $h(x,y,z) = y^2 + z^2 = 4 \quad \Rightarrow \quad \nabla f = \langle 1, 2, 0 \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$ and $\mu \nabla h = \langle 0, 2\mu y, 2\mu z \rangle$. Then $1 = \lambda$, $2 = \lambda + 2\mu y$ and $0 = \lambda + 2\mu z$ so $\mu y = \frac{1}{2} = -\mu z$ or $y = 1/(2\mu)$, $z = -1/(2\mu)$. Thus $x + y + z = 1$ implies $x = 1$ and $y^2 + z^2 = 4$ implies $\mu = \pm\frac{1}{2\sqrt{2}}$. Then the possible points are $\left(1, \pm\sqrt{2}, \mp\sqrt{2}\right)$ and the maximum value is $f\left(1, \sqrt{2}, -\sqrt{2}\right) = 1 + 2\sqrt{2}$ and the minimum value is $f\left(1, -\sqrt{2}, \sqrt{2}\right) = 1 - 2\sqrt{2}$.

16. $f(x, y, z) = 3x - y - 3z,\ g(x, y, z) = x + y - z = 0,\ h(x, y, z) = x^2 + 2z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 3, -1, -3\rangle$, $\lambda \nabla g = \langle \lambda, \lambda, -\lambda\rangle$, $\mu \nabla h = (2\mu x, 0, 4\mu z)$. Then $3 = \lambda + 2\mu x$, $-1 = \lambda$ and $-3 = -\lambda + 4\mu z$, so $\lambda = -1$, $\mu z = -1$, $\mu x = 2$. Thus $h(x, y, z) = 1$ implies $\dfrac{4}{\mu^2} + 2\left(\dfrac{1}{\mu^2}\right) = 1$ or $\mu = \pm\sqrt{6}$, so $z = \mp\frac{1}{\sqrt{6}}$; $x = \pm\frac{2}{\sqrt{6}}$; and $g(x, y, z) = 0$ implies $y = \mp\frac{3}{\sqrt{6}}$. Hence the maximum of f subject to the constraints is $f\left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right) = 2\sqrt{6}$ and the minimum is $f\left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right) = -2\sqrt{6}$.

17. $f(x, y, z) = yz + xy,\ g(x, y, z) = xy = 1,\ h(x, y, z) = y^2 + z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle y, x + z, y\rangle$, $\lambda \nabla g = \langle \lambda y, \lambda x, 0\rangle$, $\mu \nabla h = \langle 0, 2\mu y, 2\mu z\rangle$. Then $y = \lambda y$ implies $\lambda = 1$ [$y \neq 0$ since $g(x, y, z) = 1$], $x + z = \lambda x + 2\mu y$ and $y = 2\mu z$. Thus $\mu = z/(2y) = y/(2y)$ or $y^2 = z^2$, and so $y^2 + z^2 = 1$ implies $y = \pm\frac{1}{\sqrt{2}}$, $z = \pm\frac{1}{\sqrt{2}}$. Then $xy = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Hence the maximum of f subject to the constraints is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{3}{2}$ and the minimum is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$.

Note: Since $xy = 1$ is one of the constraints we could have solved the problem by solving $f(y, z) = yz + 1$ subject to $y^2 + z^2 = 1$.

18. $f(x, y) = 2x^2 + 3y^2 - 4x - 5 \quad \Rightarrow \quad \nabla f = \langle 4x - 4, 6y\rangle = \langle 0, 0\rangle \quad \Rightarrow \quad x = 1, y = 0$. Thus $(1, 0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x, y) = x^2 + y^2 = 16 \quad \Rightarrow \quad \lambda \nabla g = \langle 2\lambda x, 2\lambda y\rangle$, so $6y = 2\lambda y \quad \Rightarrow \quad$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \quad \Rightarrow \quad x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1, 0) = -7$, $f(4, 0) = 11$, $f(-4, 0) = 43$, and $f\left(-2, \pm 2\sqrt{3}\right) = 47$. Thus the maximum value of $f(x, y)$ on the disk $x^2 + y^2 \leq 16$ is $f\left(-2, \pm 2\sqrt{3}\right) = 47$, and the minimum value is $f(1, 0) = -7$.

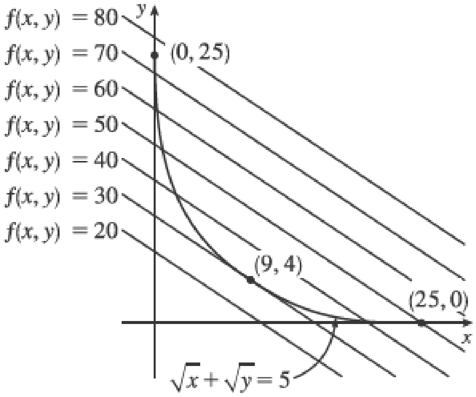
19. $f(x, y) = e^{-xy}$. For the interior of the region, we find the critical points: $f_x = -ye^{-xy}$, $f_y = -xe^{-xy}$, so the only critical point is $(0, 0)$, and $f(0, 0) = 1$. For the boundary, we use Lagrange multipliers. $g(x, y) = x^2 + 4y^2 = 1 \quad \Rightarrow$ $\lambda \nabla g = \langle 2\lambda x, 8\lambda y\rangle$, so setting $\nabla f = \lambda \nabla g$ we get $-ye^{-xy} = 2\lambda x$ and $-xe^{-xy} = 8\lambda y$. The first of these gives $e^{-xy} = -2\lambda x/y$, and then the second gives $-x(-2\lambda x/y) = 8\lambda y \quad \Rightarrow \quad x^2 = 4y^2$. Solving this last equation with the constraint $x^2 + 4y^2 = 1$ gives $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{2\sqrt{2}}$. Now $f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{1/4} \approx 1.284$ and $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-1/4} \approx 0.779$. The former are the maxima on the region and the latter are the minima.

20. (a) $f(x,y) = 2x + 3y$, $g(x,y) = \sqrt{x} + \sqrt{y} = 5 \quad \Rightarrow \quad \nabla f = \langle 2, 3 \rangle = \lambda \nabla g = \lambda \left\langle \dfrac{1}{2\sqrt{x}}, \dfrac{1}{2\sqrt{y}} \right\rangle$. Then

$2 = \dfrac{\lambda}{2\sqrt{x}}$ and $3 = \dfrac{\lambda}{2\sqrt{y}}$ so $4\sqrt{x} = \lambda = 6\sqrt{y} \quad \Rightarrow \quad \sqrt{y} = \frac{2}{3}\sqrt{x}$. With $\sqrt{x} + \sqrt{y} = 5$ we have $\sqrt{x} + \frac{2}{3}\sqrt{x} = 5 \quad \Rightarrow$ $\sqrt{x} = 3 \quad \Rightarrow \quad x = 9$. Substituting into $\sqrt{y} = \frac{2}{3}\sqrt{x}$ gives $\sqrt{y} = 2$ or $y = 4$. Thus the only possible extreme value subject to the constraint is $f(9,4) = 30$. (The question remains whether this is indeed the maximum of f.)

(b) $f(25,0) = 50$ which is larger than the result of part (a).

(c)



We can see from the level curves of f that the maximum occurs at the left endpoint $(0, 25)$ of the constraint curve g. The maximum value is $f(0, 25) = 75$.

(d) Here ∇g does not exist if $x = 0$ or $y = 0$, so the method will not locate any associated points. Also, the method of Lagrange multipliers identifies points where the level curves of f share a common tangent line with the constraint curve g. This normally does not occur at an endpoint, although an absolute maximum or minimum may occur there.

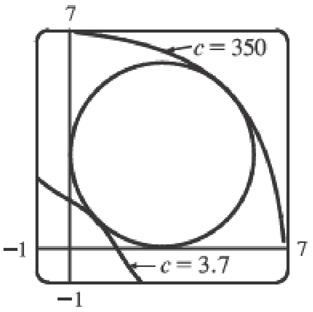
(e) Here $f(9, 4)$ is the absolute *minimum* of f subject to g.

21. (a) $f(x,y) = x, \;\; g(x,y) = y^2 + x^4 - x^3 = 0 \quad \Rightarrow \quad \nabla f = \langle 1, 0 \rangle = \lambda \nabla g = \lambda \left\langle 4x^3 - 3x^2, 2y \right\rangle$. Then $1 = \lambda(4x^3 - 3x^2)$ **(1)** and $0 = 2\lambda y$ **(2)**. We have $\lambda \neq 0$ from **(1)**, so **(2)** gives $y = 0$. Then, from the constraint equation, $x^4 - x^3 = 0 \quad \Rightarrow \quad x^3(x-1) = 0 \quad \Rightarrow \quad x = 0$ or $x = 1$. But $x = 0$ contradicts **(1)**, so the only possible extreme value subject to the constraint is $f(1, 0) = 1$. (The question remains whether this is indeed the minimum of f.)

(b) The constraint is $y^2 + x^4 - x^3 = 0 \quad \Leftrightarrow \quad y^2 = x^3 - x^4$. The left side is non-negative, so we must have $x^3 - x^4 \geq 0$ which is true only for $0 \leq x \leq 1$. Therefore the minimum possible value for $f(x,y) = x$ is 0 which occurs for $x = y = 0$. However, $\lambda \nabla g(0,0) = \lambda \langle 0 - 0, 0 \rangle = \langle 0, 0 \rangle$ and $\nabla f(0,0) = \langle 1, 0 \rangle$, so $\nabla f(0,0) \neq \lambda \nabla g(0,0)$ for all values of λ.

(c) Here $\nabla g(0,0) = \mathbf{0}$ but the method of Lagrange multipliers requires that $\nabla g \neq \mathbf{0}$ everywhere on the constraint curve.

22. (a) The graphs of $f(x, y) = 3.7$ and $f(x, y) = 350$ seem to be tangent to the circle, and so 3.7 and 350 are the approximate minimum and maximum values of the function $f(x, y)$ subject to the constraint $(x-3)^2 + (y-3)^2 = 9$.



(b) Let $g(x, y) = (x-3)^2 + (y-3)^2$. We calculate $f_x(x, y) = 3x^2 + 3y$, $f_y(x, y) = 3y^2 + 3x$, $g_x(x, y) = 2x - 6$, and $g_y(x, y) = 2y - 6$, and use a CAS to search for solutions to the equations $g(x, y) = (x-3)^2 + (y-3)^2 = 9$, $f_x = \lambda g_x$, and $f_y = \lambda g_y$. The solutions are $(x, y) = \left(3 - \frac{3}{2}\sqrt{2}, 3 - \frac{3}{2}\sqrt{2}\right) \approx (0.879, 0.879)$ and $(x, y) = \left(3 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}\right) \approx (5.121, 5.121)$. These give $f\left(3 - \frac{3}{2}\sqrt{2}, 3 - \frac{3}{2}\sqrt{2}\right) = \frac{351}{2} - \frac{243}{2}\sqrt{2} \approx 3.673$ and $f\left(3 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}\right) = \frac{351}{2} + \frac{243}{2}\sqrt{2} \approx 347.33$, in accordance with part (a).

23. $P(L, K) = bL^\alpha K^{1-\alpha}$, $g(L, K) = mL + nK = p \quad \Rightarrow \quad \nabla P = \left\langle \alpha bL^{\alpha-1}K^{1-\alpha}, (1-\alpha)bL^\alpha K^{-\alpha}\right\rangle$, $\lambda \nabla g = \langle \lambda m, \lambda n\rangle$. Then $\alpha b(K/L)^{1-\alpha} = \lambda m$ and $(1-\alpha)b(L/K)^\alpha = \lambda n$ and $mL + nK = p$, so $\alpha b(K/L)^{1-\alpha}/m = (1-\alpha)b(L/K)^\alpha/n$ or $n\alpha/[m(1-\alpha)] = (L/K)^\alpha(L/K)^{1-\alpha}$ or $L = Kn\alpha/[m(1-\alpha)]$. Substituting into $mL + nK = p$ gives $K = (1-\alpha)p/n$ and $L = \alpha p/m$ for the maximum production.

24. $C(L, K) = mL + nK$, $g(L, K) = bL^\alpha K^{1-\alpha} = Q \quad \Rightarrow \quad \nabla C = \langle m, n\rangle$, $\lambda \nabla g = \left\langle \lambda \alpha bL^{\alpha-1}K^{1-\alpha}, \lambda(1-\alpha)bL^\alpha K^{-\alpha}\right\rangle$. Then $\dfrac{m}{\alpha b}\left(\dfrac{L}{K}\right)^{1-\alpha} = \dfrac{n}{(1-\alpha)b}\left(\dfrac{K}{L}\right)^\alpha$ and $bL^\alpha K^{1-\alpha} = Q \quad \Rightarrow \quad \dfrac{n\alpha}{m(1-\alpha)} = \left(\dfrac{L}{K}\right)^{1-\alpha}\left(\dfrac{L}{K}\right)^\alpha \quad \Rightarrow$ $L = \dfrac{Kn\alpha}{m(1-\alpha)}$ and so $b\left[\dfrac{Kn\alpha}{m(1-\alpha)}\right]^\alpha K^{1-\alpha} = Q$. Hence $K = \dfrac{Q}{b(n\alpha/[m(1-\alpha)])^\alpha} = \dfrac{Qm^\alpha(1-\alpha)^\alpha}{bn^\alpha\alpha^\alpha}$ and $L = \dfrac{Qm^{\alpha-1}(1-\alpha)^{\alpha-1}}{bn^{\alpha-1}\alpha^{\alpha-1}} = \dfrac{Qn^{1-\alpha}\alpha^{1-\alpha}}{bm^{1-\alpha}(1-\alpha)^{1-\alpha}}$ minimizes cost.

25. Let the sides of the rectangle be x and y. Then $f(x, y) = xy$, $g(x, y) = 2x + 2y = p \quad \Rightarrow \quad \nabla f(x, y) = \langle y, x\rangle$, $\lambda \nabla g = \langle 2\lambda, 2\lambda\rangle$. Then $\lambda = \frac{1}{2}y = \frac{1}{2}x$ implies $x = y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

26. Let $f(x, y, z) = s(s-x)(s-y)(s-z)$, $g(x, y, z) = x + y + z$. Then
$\nabla f = \langle -s(s-y)(s-z), -s(s-x)(s-z), -s(s-x)(s-y)\rangle$, $\lambda \nabla g = \langle \lambda, \lambda, \lambda\rangle$. Thus
$(s-y)(s-z) = (s-x)(s-z)$ **(1)**, and $(s-x)(s-z) = (s-x)(s-y)$ **(2)**.
(1) implies $x = y$ while **(2)** implies $y = z$, so $x = y = z = p/3$ and the triangle with maximum area is equilateral.

27. Let $f(x, y, z) = d^2 = (x-2)^2 + (y-1)^2 + (z+1)^2$, then we want to minimize f subject to the constraint $g(x, y, z) = x + y - z = 1$. $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 2(x-2), 2(y-1), 2(z+1)\rangle = \lambda\langle 1, 1, -1\rangle$, so $x = (\lambda+4)/2$, $y = (\lambda+2)/2$, $z = -(\lambda+2)/2$. Substituting into the constraint equation gives $\dfrac{\lambda+4}{2} + \dfrac{\lambda+2}{2} + \dfrac{\lambda+2}{2} = 1 \quad \Rightarrow$ $3\lambda + 8 = 2 \quad \Rightarrow \quad \lambda = -2$, so $x = 1$, $y = 0$, and $z = 0$. This must correspond to a minimum, so the shortest distance is $d = \sqrt{(1-2)^2 + (0-1)^2 + (0+1)^2} = \sqrt{3}$.

28. Let $f(x,y,z) = d^2 = (x-1)^2 + (y-2)^2 + (z-3)^2$, then we want to minimize f subject to the constraint $g(x,y,z) = x - y + z = 4$. $\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-1), 2(y-2), 2(z-3)\rangle = \lambda \langle 1,-1,1\rangle$, so $x = (\lambda+2)/2$, $y = (4-\lambda)/2$, $z = (\lambda+6)/2$. Substituting into the constraint equation gives $\dfrac{\lambda+2}{2} - \dfrac{4-\lambda}{2} + \dfrac{\lambda+6}{2} = 4 \Rightarrow \lambda = \frac{4}{3}$, so $x = \frac{5}{3}$, $y = \frac{4}{3}$, and $z = \frac{11}{3}$. This must correspond to a minimum, so the point on the plane closest to the point $(1,2,3)$ is $\left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right)$.

29. Let $f(x,y,z) = d^2 = (x-4)^2 + (y-2)^2 + z^2$. Then we want to minimize f subject to the constraint $g(x,y,z) = x^2 + y^2 - z^2 = 0$. $\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-4), 2(y-2), 2z\rangle = \langle 2\lambda x, 2\lambda y, -2\lambda z\rangle$, so $x - 4 = \lambda x$, $y - 2 = \lambda y$, and $z = -\lambda z$. From the last equation we have $z + \lambda z = 0 \Rightarrow z(1+\lambda) = 0$, so either $z = 0$ or $\lambda = -1$. But from the constraint equation we have $z = 0 \Rightarrow x^2 + y^2 = 0 \Rightarrow x = y = 0$ which is not possible from the first two equations. So $\lambda = -1$ and $x - 4 = \lambda x \Rightarrow x = 2$, $y - 2 = \lambda y \Rightarrow y = 1$, and $x^2 + y^2 - z^2 = 0 \Rightarrow 4 + 1 - z^2 = 0 \Rightarrow z = \pm\sqrt{5}$. This must correspond to a minimum, so the points on the cone closest to $(4,2,0)$ are $\left(2, 1, \pm\sqrt{5}\right)$.

30. Let $f(x,y,z) = d^2 = x^2 + y^2 + z^2$. Then we want to minimize f subject to the constraint $g(x,y,z) = y^2 - xz = 9$. $\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2y, 2z\rangle = \langle -\lambda z, 2\lambda y, -\lambda x\rangle$, so $2x = -\lambda z$, $y = \lambda y$, and $2z = -\lambda x$. If $x = 0$ then the last equation implies $z = 0$, and from the constraint $y^2 - xz = 9$ we have $y = \pm 3$. If $x \neq 0$, then the first and third equations give $\lambda = -2x/z = -2z/x \Rightarrow x^2 = z^2$. From the second equation we have $y = 0$ or $\lambda = 1$. If $y = 0$ then $y^2 - xz = 9 \Rightarrow z = -9/x$ and $x^2 = z^2 \Rightarrow x^2 = 81/x^2 \Rightarrow x = \pm 3$. Since $z = -9/x$, $x = 3 \Rightarrow z = -3$ and $x = -3 \Rightarrow z = 3$. If $\lambda = 1$, then $2x = -z$ and $2z = -x$ which implies $x = z = 0$, contradicting the assumption that $x \neq 0$. Thus the possible points are $(0,\pm 3,0)$, $(3,0,-3)$, $(-3,0,3)$. We have $f(0,\pm 3,0) = 9$ and $f(3,0,-3) = f(-3,0,3) = 18$, so the points on the surface that are closest to the origin are $(0,\pm 3,0)$.

31. $f(x,y,z) = xyz$, $g(x,y,z) = x + y + z = 100 \Rightarrow \nabla f = \langle yz, xz, xy\rangle = \lambda \nabla g = \langle \lambda, \lambda, \lambda\rangle$. Then $\lambda = yz = xz = xy$ implies $x = y = z = \frac{100}{3}$.

32. Minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to $g(x,y,z) = x + y + z = 12$ with $x > 0$, $y > 0$, $z > 0$. Then $\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2y, 2z\rangle = \lambda \langle 1,1,1\rangle \Rightarrow 2x = \lambda, 2y = \lambda, 2z = \lambda \Rightarrow x = y = z$, so $x + y + z = 12 \Rightarrow 3x = 12 \Rightarrow x = 4 = y = z$. By comparing nearby values we can confirm that this gives a minimum and not a maximum. Thus the three numbers are 4, 4, and 4.

33. If the dimensions are $2x$, $2y$, and $2z$, then maximize $f(x,y,z)=(2x)(2y)(2z)=8xyz$ subject to $g(x,y,z)=x^2+y^2+z^2=r^2$ ($x>0$, $y>0$, $z>0$). Then $\nabla f=\lambda\nabla g \quad\Rightarrow\quad \langle 8yz,8xz,8xy\rangle=\lambda\,\langle 2x,2y,2z\rangle \quad\Rightarrow$ $8yz=2\lambda x$, $8xz=2\lambda y$, and $8xy=2\lambda z$, so $\lambda=\dfrac{4yz}{x}=\dfrac{4xz}{y}=\dfrac{4xy}{z}$. This gives $x^2z=y^2z \quad\Rightarrow\quad x^2=y^2$ (since $z\neq 0$) and $xy^2=xz^2 \quad\Rightarrow\quad z^2=y^2$, so $x^2=y^2=z^2 \quad\Rightarrow\quad x=y=z$, and substituting into the constraint equation gives $3x^2=r^2 \quad\Rightarrow\quad x=r/\sqrt{3}=y=z$. Thus the largest volume of such a box is

$$f\left(\tfrac{r}{\sqrt{3}},\tfrac{r}{\sqrt{3}},\tfrac{r}{\sqrt{3}}\right)=8\left(\tfrac{r}{\sqrt{3}}\right)\left(\tfrac{r}{\sqrt{3}}\right)\left(\tfrac{r}{\sqrt{3}}\right)=\frac{8}{3\sqrt{3}}r^3.$$

34. If the dimensions of the box are x, y, and z then minimize $f(x,y,z)=2xy+2xz+2yz$ subject to $g(x,y,z)=xyz=1000$ ($x>0$, $y>0$, $z>0$). Then $\nabla f=\lambda\nabla g \quad\Rightarrow\quad \langle 2y+2z,2x+2z,2x+2y\rangle=\lambda\,\langle yz,xz,xy\rangle \quad\Rightarrow\quad 2y+2z=\lambda yz$, $2x+2z=\lambda xz$, $2x+2y=\lambda xy$. Solving for λ in each equation gives $\lambda=\dfrac{2}{z}+\dfrac{2}{y}=\dfrac{2}{z}+\dfrac{2}{x}=\dfrac{2}{y}+\dfrac{2}{x} \quad\Rightarrow\quad x=y=z$. From $xyz=1000$ we have $x^3=1000 \quad\Rightarrow\quad x=10$ and the dimensions of the box are $x=y=z=10$ cm.

35. $f(x,y,z)=xyz,\ g(x,y,z)=x+2y+3z=6 \quad\Rightarrow\quad \nabla f=\langle yz,xz,xy\rangle=\lambda\nabla g=\langle\lambda,2\lambda,3\lambda\rangle$. Then $\lambda=yz=\frac{1}{2}xz=\frac{1}{3}xy$ implies $x=2y$, $z=\frac{2}{3}y$. But $2y+2y+2y=6$ so $y=1$, $x=2$, $z=\frac{2}{3}$ and the volume is $V=\frac{4}{3}$.

36. $f(x,y,z)=xyz,\ g(x,y,z)=xy+yz+xz=32 \quad\Rightarrow\quad \nabla f=\langle yz,xz,xy\rangle=\lambda\nabla g=\langle\lambda(y+z),\lambda(x+z),\lambda(x+y)\rangle$. Then $\lambda(y+z)=yz$ **(1)**, $\lambda(x+z)=xz$ **(2)**, and $\lambda(x+y)=xy$ **(3)**. And (1) minus (2) implies $\lambda(y-x)=z(y-x)$ so $x=y$ or $\lambda=z$. If $\lambda=z$, then (1) implies $z(y+z)=yz$ or $z=0$ which is false. Thus $x=y$. Similarly (2) minus (3) implies $\lambda(z-y)=x(z-y)$ so $y=z$ or $\lambda=x$. As above, $\lambda\neq x$, so $x=y=z$ and $3x^2=32$ or $x=y=z=\frac{8}{\sqrt{6}}$ cm.

37. $f(x,y,z)=xyz,\ g(x,y,z)=4(x+y+z)=c \quad\Rightarrow\quad \nabla f=\langle yz,xz,xy\rangle$, $\lambda\nabla g=\langle 4\lambda,4\lambda,4\lambda\rangle$. Thus $4\lambda=yz=xz=xy$ or $x=y=z=\frac{1}{12}c$ are the dimensions giving the maximum volume.

38. $C(x,y,z)=5xy+2xz+2yz,\ g(x,y,z)=xyz=V \quad\Rightarrow$ $\nabla C=\langle 5y+2z,5x+2z,2x+2y\rangle=\lambda\nabla g=\langle\lambda yz,\lambda xz,\lambda xy\rangle$. Then $\lambda yz=5y+2z$ **(1)**, $\lambda xz=5x+2z$ **(2)**, $\lambda xy=2(x+y)$ **(3)**, and $xyz=V$ **(4)**. Now (1) $-$ (2) implies $\lambda z(y-x)=5(y-x)$, so $x=y$ or $\lambda=5/z$, but z can't be 0, so $x=y$. Then twice (2) minus five times (3) together with $x=y$ implies $\lambda y(2x-5y)=2(2z-5y)$ which gives $z=\frac{5}{2}y$ [again $\lambda\neq 2/y$ or else (3) implies $y=0$]. Hence $\frac{5}{2}y^3=V$ and the dimensions which minimize cost are $x=y=\sqrt[3]{\frac{2}{5}V}$ units, $z=V^{1/3}\left(\frac{5}{2}\right)^{2/3}$ units.

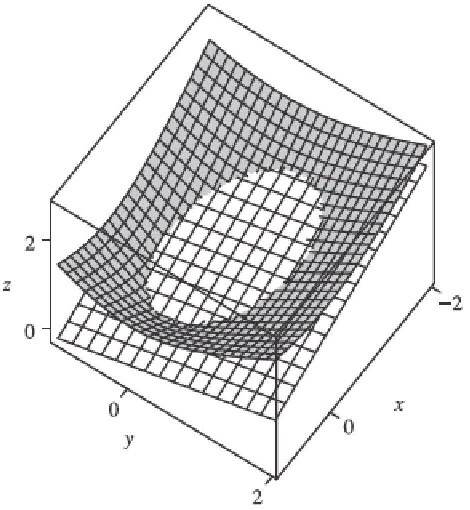
39. If the dimensions of the box are given by x, y, and z, then we need to find the maximum value of $f(x, y, z) = xyz$ $[x, y, z > 0]$ subject to the constraint $L = \sqrt{x^2 + y^2 + z^2}$ or $g(x, y, z) = x^2 + y^2 + z^2 = L^2$. $\nabla f = \lambda \nabla g \quad \Rightarrow$ $\langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$, so $yz = 2\lambda x \quad \Rightarrow \quad \lambda = \dfrac{yz}{2x}$, $xz = 2\lambda y \quad \Rightarrow \quad \lambda = \dfrac{xz}{2y}$, and $xy = 2\lambda z \quad \Rightarrow \quad \lambda = \dfrac{xy}{2z}$. Thus $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{2y} \quad \Rightarrow \quad x^2 = y^2 \quad [\text{since } z \neq 0] \quad \Rightarrow \quad x = y$ and $\lambda = \dfrac{yz}{2x} = \dfrac{xy}{2z} \quad \Rightarrow \quad x = z \quad [\text{since } y \neq 0]$.

Substituting into the constraint equation gives $x^2 + x^2 + x^2 = L^2 \quad \Rightarrow \quad x^2 = L^2/3 \quad \Rightarrow \quad x = L/\sqrt{3} = y = z$ and the maximum volume is $\left(L/\sqrt{3}\right)^3 = L^3/\left(3\sqrt{3}\right)$.

40. Let the dimensions of the box be x, y, and z, so its volume is $f(x, y, z) = xyz$, its surface area is $2xy + 2yz + 2xz = 1500$ and its total edge length is $4x + 4y + 4z = 200$. We find the extreme values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = xy + yz + xz = 750$ and $h(x, y, z) = x + y + z = 50$. Then
$\nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda(y + z), \lambda(x + z), \lambda(x + y) \rangle + \langle \mu, \mu, \mu \rangle$. So $yz = \lambda(y + z) + \mu$ **(1)**, $xz = \lambda(x + z) + \mu$ **(2)**, and $xy = \lambda(x + y) + \mu$ **(3)**. Notice that the box can't be a cube or else $x = y = z = \frac{50}{3}$ but then $xy + yz + xz = \frac{2500}{3} \neq 750$. Assume x is the distinct side, that is, $x \neq y$, $x \neq z$. Then **(1)** minus **(2)** implies $z\,(y - x) = \lambda(y - x)$ or $\lambda = z$, and **(1)** minus **(3)** implies $y(z - x) = \lambda(z - x)$ or $\lambda = y$. So $y = z = \lambda$ and $x + y + z = 50$ implies $x = 50 - 2\lambda$; also $xy + yz + xz = 750$ implies $x(2\lambda) + \lambda^2 = 750$. Hence $50 - 2\lambda = \dfrac{750 - \lambda^2}{2\lambda}$ or $3\lambda^2 - 100\lambda + 750 = 0$ and $\lambda = \dfrac{50 \pm 5\sqrt{10}}{3}$, giving the points $\left(\frac{1}{3}\left(50 \mp 10\sqrt{10}\right), \frac{1}{3}\left(50 \pm 5\sqrt{10}\right), \frac{1}{3}\left(50 \pm 5\sqrt{10}\right)\right)$.
Thus the minimum of f is $f\left(\frac{1}{3}\left(50 - 10\sqrt{3}\right), \frac{1}{3}\left(50 + 5\sqrt{10}\right), \frac{1}{3}\left(50 + 5\sqrt{10}\right)\right) = \frac{1}{27}\left(87{,}500 - 2500\sqrt{10}\right)$, and its maximum is $f\left(\frac{1}{3}\left(50 + 10\sqrt{10}\right), \frac{1}{3}\left(50 - 5\sqrt{10}\right), \frac{1}{3}\left(50 - 5\sqrt{10}\right)\right) = \frac{1}{27}\left(87{,}500 + 2500\sqrt{10}\right)$.
Note: If either y or z is the distinct side, then symmetry gives the same result.

41. We need to find the extreme values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the two constraints $g(x, y, z) = x + y + 2z = 2$ and $h(x, y, z) = x^2 + y^2 - z = 0$. $\nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, 2\lambda \rangle$ and $\mu \nabla h = \langle 2\mu x, 2\mu y, -\mu \rangle$. Thus we need $2x = \lambda + 2\mu x$ **(1)**, $2y = \lambda + 2\mu y$ **(2)**, $2z = 2\lambda - \mu$ **(3)**, $x + y + 2z = 2$ **(4)**, and $x^2 + y^2 - z = 0$ **(5)**.
From **(1)** and **(2)**, $2(x - y) = 2\mu(x - y)$, so if $x \neq y$, $\mu = 1$. Putting this in **(3)** gives $2z = 2\lambda - 1$ or $\lambda = z + \frac{1}{2}$, but putting $\mu = 1$ into **(1)** says $\lambda = 0$. Hence $z + \frac{1}{2} = 0$ or $z = -\frac{1}{2}$. Then **(4)** and **(5)** become $x + y - 3 = 0$ and $x^2 + y^2 + \frac{1}{2} = 0$. The last equation cannot be true, so this case gives no solution. So we must have $x = y$. Then **(4)** and **(5)** become $2x + 2z = 2$ and $2x^2 - z = 0$ which imply $z = 1 - x$ and $z = 2x^2$. Thus $2x^2 = 1 - x$ or $2x^2 + x - 1 = (2x - 1)(x + 1) = 0$ so $x = \frac{1}{2}$ or $x = -1$. The two points to check are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$: $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$ and $f(-1, -1, 2) = 6$. Thus $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the point on the ellipse nearest the origin and $(-1, -1, 2)$ is the one farthest from the origin.

42. (a) After plotting $z = \sqrt{x^2 + y^2}$, the top half of the cone, and the plane $z = (5 - 4x + 3y)/8$ we see the ellipse formed by the intersection of the surfaces. The ellipse can be plotted explicitly using cylindrical coordinates (see Section 16.7 [ET 15.7]): The cone is given by $z = r$, and the plane is $4r\cos\theta - 3r\sin\theta + 8z = 5$. Substituting $z = r$ into the plane equation gives $4r\cos\theta - 3r\sin\theta + 8r = 5 \quad \Rightarrow \quad r = \dfrac{5}{4\cos\theta - 3\sin\theta + 8}$. Since $z = r$ on the ellipse, parametric equations (in cylindrical coordinates) are $\theta = t,\ r = z = \dfrac{5}{4\cos t - 3\sin t + 8},\ 0 \le t \le 2\pi$.



(b) We need to find the extreme values of $f(x, y, z) = z$ subject to the two constraints $g(x, y, z) = 4x - 3y + 8z = 5$ and $h(x, y, z) = x^2 + y^2 - z^2 = 0$. $\nabla f = \lambda\nabla g + \mu\nabla h \quad \Rightarrow \quad \langle 0, 0, 1\rangle = \lambda\langle 4, -3, 8\rangle + \mu\langle 2x, 2y, -2z\rangle$, so we need $4\lambda + 2\mu x = 0 \quad \Rightarrow \quad x = -\frac{2\lambda}{\mu}$ (1), $\quad -3\lambda + 2\mu y = 0 \quad \Rightarrow \quad y = \frac{3\lambda}{2\mu}$ (2), $\quad 8\lambda - 2\mu z = 1 \quad \Rightarrow \quad z = \frac{8\lambda - 1}{2\mu}$ (3), $4x - 3y + 8z = 5$ (4), and $x^2 + y^2 = z^2$ (5). [Note that $\mu \neq 0$, else $\lambda = 0$ from (1), but substitution into (3) gives a contradiction.] Substituting (1), (2), and (3) into (4) gives $4\left(-\frac{2\lambda}{\mu}\right) - 3\left(\frac{3\lambda}{2\mu}\right) + 8\left(\frac{8\lambda - 1}{2\mu}\right) = 5 \quad \Rightarrow \quad \mu = \frac{39\lambda - 8}{10}$ and into (5) gives $\left(-\frac{2\lambda}{\mu}\right)^2 + \left(\frac{3\lambda}{2\mu}\right)^2 = \left(\frac{8\lambda - 1}{2\mu}\right)^2 \quad \Rightarrow \quad 16\lambda^2 + 9\lambda^2 = (8\lambda - 1)^2 \quad \Rightarrow \quad 39\lambda^2 - 16\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = \frac{1}{13}$ or $\lambda = \frac{1}{3}$. If $\lambda = \frac{1}{13}$ then $\mu = -\frac{1}{2}$ and $x = \frac{4}{13}, y = -\frac{3}{13}, z = \frac{5}{13}$. If $\lambda = \frac{1}{3}$ then $\mu = \frac{1}{2}$ and $x = -\frac{4}{3}, y = 1, z = \frac{5}{3}$. Thus the highest point on the ellipse is $\left(-\frac{4}{3}, 1, \frac{5}{3}\right)$ and the lowest point is $\left(\frac{4}{13}, -\frac{3}{13}, \frac{5}{13}\right)$.

43. $f(x, y, z) = ye^{x-z},\ g(x, y, z) = 9x^2 + 4y^2 + 36z^2 = 36,\ h(x, y, z) = xy + yz = 1.\quad \nabla f = \lambda\nabla g + \mu\nabla h \quad \Rightarrow$ $\langle ye^{x-z}, e^{x-z}, -ye^{x-z}\rangle = \lambda\langle 18x, 8y, 72z\rangle + \mu\langle y, x + z, y\rangle$, so $ye^{x-z} = 18\lambda x + \mu y$, $e^{x-z} = 8\lambda y + \mu(x + z)$, $-ye^{x-z} = 72\lambda z + \mu y$, $9x^2 + 4y^2 + 36z^2 = 36$, $xy + yz = 1$. Using a CAS to solve these 5 equations simultaneously for x, y, z, λ, and μ (in Maple, use the `allvalues` command), we get 4 real-valued solutions:

$$x \approx 0.222444, \quad y \approx -2.157012, \quad z \approx -0.686049, \quad \lambda \approx -0.200401, \quad \mu \approx 2.108584$$

$$x \approx -1.951921, \quad y \approx -0.545867, \quad z \approx 0.119973, \quad \lambda \approx 0.003141, \quad \mu \approx -0.076238$$

$$x \approx 0.155142, \quad y \approx 0.904622, \quad z \approx 0.950293, \quad \lambda \approx -0.012447, \quad \mu \approx 0.489938$$

$$x \approx 1.138731, \quad y \approx 1.768057, \quad z \approx -0.573138, \quad \lambda \approx 0.317141, \quad \mu \approx 1.862675$$

Substituting these values into f gives $f(0.222444, -2.157012, -0.686049) \approx -5.3506$, $f(-1.951921, -0.545867, 0.119973) \approx -0.0688$, $f(0.155142, 0.904622, 0.950293) \approx 0.4084$, $f(1.138731, 1.768057, -0.573138) \approx 9.7938$. Thus the maximum is approximately 9.7938, and the mininum is approximately -5.3506.

44. $f(x,y,z) = x + y + z,\ g(x,y,z) = x^2 - y^2 - z = 0,\ h(x,y,z) = x^2 + z^2 = 4.$

$\nabla f = \lambda \nabla g + \mu \nabla h \quad \Rightarrow \quad \langle 1,1,1\rangle = \lambda\langle 2x,-2y,-1\rangle + \mu\langle 2x,0,2z\rangle$, so $1 = 2\lambda x + 2\mu x$, $1 = -2\lambda y$, $1 = -\lambda + 2\mu z$, $x^2 - y^2 = z$, $x^2 + z^2 = 4$. Using a CAS to solve these 5 equations simultaneously for x, y, z, λ, and μ, we get 4 real-valued solutions:

$$\begin{array}{lllll}
x \approx -1.652878, & y \approx -1.964194, & z \approx -1.126052, & \lambda \approx 0.254557, & \mu \approx -0.557060 \\
x \approx -1.502800, & y \approx 0.968872, & z \approx 1.319694, & \lambda \approx -0.516064, & \mu \approx 0.183352 \\
x \approx -0.992513, & y \approx 1.649677, & z \approx -1.736352, & \lambda \approx -0.303090, & \mu \approx -0.200682 \\
x \approx 1.895178, & y \approx 1.718347, & z \approx 0.638984, & \lambda \approx -0.290977, & \mu \approx 0.554805
\end{array}$$

Substituting these values into f gives $f(-1.652878,-1.964194,-1.126052) \approx -4.7431$, $f(-1.502800, 0.968872, 1.319694) \approx 0.7858$, $f(-0.992513, 1.649677, -1.736352) \approx -1.0792$, $f(1.895178, 1.718347, 0.638984) \approx 4.2525$. Thus the maximum is approximately 4.2525, and the mininum is approximately -4.7431.

45. (a) We wish to maximize $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$ subject to $g(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n = c$ and $x_i > 0$.

$$\nabla f = \left\langle \tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n),\ \tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n),\ \ldots,\ \tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) \right\rangle$$

and $\lambda \nabla g = \langle \lambda, \lambda, \ldots, \lambda \rangle$, so we need to solve the system of equations

$$\begin{aligned}
\tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n) = \lambda \quad &\Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_1 \\
\tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n) = \lambda \quad &\Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_2 \\
&\ \ \vdots \\
\tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) = \lambda \quad &\Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_n
\end{aligned}$$

This implies $n\lambda x_1 = n\lambda x_2 = \cdots = n\lambda x_n$. Note $\lambda \neq 0$, otherwise we can't have all $x_i > 0$. Thus $x_1 = x_2 = \cdots = x_n$. But $x_1 + x_2 + \cdots + x_n = c \quad \Rightarrow \quad nx_1 = c \quad \Rightarrow \quad x_1 = \dfrac{c}{n} = x_2 = x_3 = \cdots = x_n$. Then the only point where f can have an extreme value is $\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right)$. Since we can choose values for $(x_1, x_2, \ldots, x_n)$ that make f as close to zero (but not equal) as we like, f has no minimum value. Thus the maximum value is

$$f\left(\frac{c}{n}, \frac{c}{n}, \ldots, \frac{c}{n}\right) = \sqrt[n]{\frac{c}{n} \cdot \frac{c}{n} \cdot \cdots \cdot \frac{c}{n}} = \frac{c}{n}.$$

(b) From part (a), $\dfrac{c}{n}$ is the maximum value of f. Thus $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n} \leq \dfrac{c}{n}$. But $x_1 + x_2 + \cdots + x_n = c$, so $\sqrt[n]{x_1 x_2 \cdots x_n} \leq \dfrac{x_1 + x_2 + \cdots + x_n}{n}$. These two means are equal when f attains its maximum value $\dfrac{c}{n}$, but this can occur only at the point $\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right)$ we found in part (a). So the means are equal only when $x_1 = x_2 = x_3 = \cdots = x_n = \dfrac{c}{n}$.

46. (a) Let $f(x_1,\ldots,x_n,y_1,\ldots,y_n)=\sum\limits_{i=1}^{n}x_iy_i$, $g(x_1,\ldots,x_n)=\sum\limits_{i=1}^{n}x_i^2$, and $h(x_1,\ldots,x_n)=\sum\limits_{i=1}^{n}y_i^2$. Then

$\nabla f=\nabla\sum\limits_{i=1}^{n}x_iy_i=\langle y_1,y_2,\ldots,y_n,x_1,x_2,\ldots,x_n\rangle$, $\nabla g=\nabla\sum\limits_{i=1}^{n}x_i^2=\langle 2x_1,2x_2,\ldots,2x_n,0,0,\ldots,0\rangle$ and

$\nabla h=\nabla\sum\limits_{i=1}^{n}y_i^2=\langle 0,0,\ldots,0,2y_1,2y_2,\ldots,2y_n\rangle$. So $\nabla f=\lambda\nabla g+\mu\nabla h \quad\Leftrightarrow\quad y_i=2\lambda x_i$ and $x_i=2\mu y_i$,

$1\le i\le n$. Then $1=\sum\limits_{i=1}^{n}y_i^2=\sum\limits_{i=1}^{n}4\lambda^2x_i^2=4\lambda^2\sum\limits_{i=1}^{n}x_i^2=4\lambda^2 \quad\Rightarrow\quad \lambda=\pm\frac{1}{2}$. If $\lambda=\frac{1}{2}$ then $y_i=2\left(\frac{1}{2}\right)x_i=x_i$,

$1\le i\le n$. Thus $\sum\limits_{i=1}^{n}x_iy_i=\sum\limits_{i=1}^{n}x_i^2=1$. Similarly if $\lambda=-\frac{1}{2}$ we get $y_i=-x_i$ and $\sum\limits_{i=1}^{n}x_iy_i=-1$. Similarly we get

$\mu=\pm\frac{1}{2}$ giving $y_i=\pm x_i$, $1\le i\le n$, and $\sum\limits_{i=1}^{n}x_iy_i=\pm 1$. Thus the maximum value of $\sum\limits_{i=1}^{n}x_iy_i$ is 1.

(b) Here we assume $\sum\limits_{i=1}^{n}a_i^2\neq 0$ and $\sum\limits_{i=1}^{n}b_i^2\neq 0$. (If $\sum\limits_{i=1}^{n}a_i^2=0$, then each $a_i=0$ and so the inequality is trivially true.)

$x_i=\dfrac{a_i}{\sqrt{\sum a_j^2}} \quad\Rightarrow\quad \sum x_i^2=\dfrac{\sum a_i^2}{\sum a_j^2}=1$, and $y_i=\dfrac{b_i}{\sqrt{\sum b_j^2}} \quad\Rightarrow\quad \sum y_i^2=\dfrac{\sum b_i^2}{\sum b_j^2}=1$. Therefore, from part (a),

$\sum x_iy_i=\sum\dfrac{a_ib_i}{\sqrt{\sum a_j^2}\sqrt{\sum b_j^2}}\le 1 \quad\Leftrightarrow\quad \sum a_ib_i\le\sqrt{\sum a_j^2}\sqrt{\sum b_j^2}$.