51. $D = \{(x,y) \mid 0 \le x \le 1,\ -x+1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0, x+1 \le y \le 1\}$

$\cup \{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le x-1\} \cup \{(x,y) \mid -1 \le x \le 0,\ -1 \le y \le -x-1\}$, all type I.

$$\iint_D x^2\,dA = \int_0^1 \int_{1-x}^1 x^2\,dy\,dx + \int_{-1}^0 \int_{x+1}^1 x^2\,dy\,dx + \int_0^1 \int_{-1}^{x-1} x^2\,dy\,dx + \int_{-1}^0 \int_{-1}^{-x-1} x^2\,dy\,dx$$

$$= 4\int_0^1 \int_{1-x}^1 x^2\,dy\,dx \qquad \text{[by symmetry of the regions and because } f(x,y) = x^2 \ge 0\text{]}$$

$$= 4\int_0^1 x^3\,dx = 4\left[\tfrac{1}{4}x^4\right]_0^1 = 1$$

52. $D = \{(x,y) \mid -1 \le y \le 0,\ -1 \le x \le y - y^3\} \cup \{(x,y) \mid 0 \le y \le 1, \sqrt{y} - 1 \le x \le y - y^3\}$, both type II.

$$\iint_D y\,dA = \int_{-1}^0 \int_{-1}^{y-y^3} y\,dx\,dy + \int_0^1 \int_{\sqrt{y}-1}^{y-y^3} y\,dx\,dy = \int_{-1}^0 [xy]_{x=-1}^{x=y-y^3}\,dy + \int_0^1 [xy]_{x=\sqrt{y}-1}^{x=y-y^3}\,dy$$

$$= \int_{-1}^0 (y^2 - y^4 + y)\,dy + \int_0^1 (y^2 - y^4 - y^{3/2} + y)\,dy$$

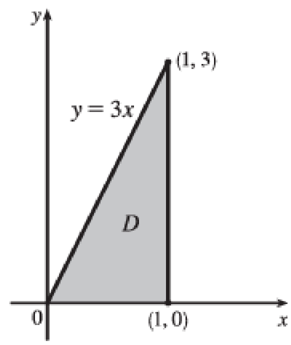
$$= \left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5 + \tfrac{1}{2}y^2\right]_{-1}^0 + \left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5 - \tfrac{2}{5}y^{5/2} + \tfrac{1}{2}y^2\right]_0^1$$

$$= \left(0 - \tfrac{11}{30}\right) + \left(\tfrac{7}{30} - 0\right) = -\tfrac{2}{15}$$

53. Here $Q = \{(x,y) \mid x^2 + y^2 \le \frac{1}{4}, x \ge 0, y \ge 0\}$, and $0 \le (x^2+y^2)^2 \le \left(\frac{1}{4}\right)^2 \Rightarrow -\frac{1}{16} \le -(x^2+y^2)^2 \le 0$ so $e^{-1/16} \le e^{-(x^2+y^2)^2} \le e^0 = 1$ since e^t is an increasing function. We have $A(Q) = \frac{1}{4}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{16}$, so by Property 11, $e^{-1/16} A(Q) \le \iint_Q e^{-(x^2+y^2)^2} dA \le 1 \cdot A(Q) \Rightarrow \frac{\pi}{16} e^{-1/16} \le \iint_Q e^{-(x^2+y^2)^2} dA \le \frac{\pi}{16}$ or we can say $0.1844 < \iint_Q e^{-(x^2+y^2)^2} dA < 0.1964$. (We have rounded the lower bound down and the upper bound up to preserve the inequalities.)

54. T is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ so $A(T) = \frac{1}{2}(1)(2) = 1$. We have $0 \le \sin^4(x+y) \le 1$ for all x, y, and Property 11 gives $0 \cdot A(T) \le \iint_T \sin^4(x+y)dA \le 1 \cdot A(T) \Rightarrow 0 \le \iint_T \sin^4(x+y)dA \le 1$.

55. The average value of a function f of two variables defined on a rectangle R was defined in Section 16.1 [ET 15.1] as $f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)dA$. Extending this definition to general regions D, we have $f_{\text{ave}} = \frac{1}{A(D)} \iint_D f(x,y)dA$. Here $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 3x\}$, so $A(D) = \frac{1}{2}(1)(3) = \frac{3}{2}$ and

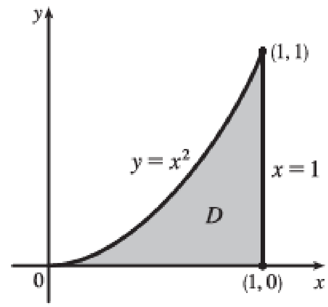


$$f_{\text{ave}} = \frac{1}{A(D)} \iint_D f(x,y)dA = \frac{1}{3/2} \int_0^1 \int_0^{3x} xy\,dy\,dx$$

$$= \tfrac{2}{3} \int_0^1 \left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=3x} dx = \tfrac{1}{3}\int_0^1 9x^3\,dx = \tfrac{3}{4}x^4\Big]_0^1 = \tfrac{3}{4}$$

56. Here $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x^2\}$, so

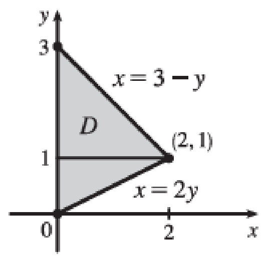
$A(D) = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big]_0^1 = \frac{1}{3}$ and



$$\begin{aligned} f_{\text{ave}} &= \frac{1}{A(D)}\iint_D f(x,y)dA = \frac{1}{1/3}\int_0^1\int_0^{x^2} x\sin y\,dy\,dx \\ &= 3\int_0^1 \left[-x\cos y\right]_{y=0}^{y=x^2}\,dx \\ &= 3\int_0^1 \left[x - x\cos(x^2)\right]dx = 3\left[\tfrac{1}{2}x^2 - \tfrac{1}{2}\sin(x^2)\right]_0^1 \\ &= 3\left(\tfrac{1}{2} - \tfrac{1}{2}\sin 1 - 0\right) = \tfrac{3}{2}(1 - \sin 1) \end{aligned}$$

57. Since $m \le f(x,y) \le M$, $\iint_D m\,dA \le \iint_D f(x,y)\,dA \le \iint_D M\,dA$ by (8) $\Rightarrow$ $m\iint_D 1\,dA \le \iint_D f(x,y)\,dA \le M\iint_D 1\,dA$ by (7) $\Rightarrow$ $mA(D) \le \iint_D f(x,y)\,dA \le MA(D)$ by (10).

58.



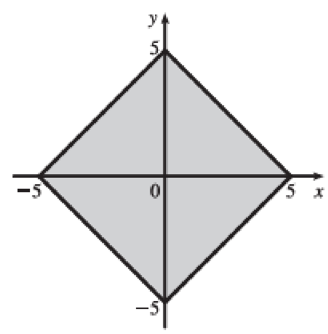
$$\begin{aligned} \iint_D f(x,y)\,dA &= \int_0^1\int_0^{2y} f(x,y)\,dx\,dy + \int_1^3\int_0^{3-y} f(x,y)\,dx\,dy \\ &= \int_0^2\int_{x/2}^{3-x} f(x,y)\,dy\,dx \end{aligned}$$

59. $\iint_D (x^2\tan x + y^3 + 4)\,dA = \iint_D x^2\tan x\,dA + \iint_D y^3\,dA + \iint_D 4\,dA$. But $x^2\tan x$ is an odd function of x and D is symmetric with respect to the y-axis, so $\iint_D x^2\tan x\,dA = 0$. Similarly, y^3 is an odd function of y and D is symmetric with respect to the x-axis, so $\iint_D y^3\,dA = 0$. Thus

$$\iint_D (x^2\tan x + y^3 + 4)\,dA = 4\iint_D\,dA = 4(\text{area of } D) = 4\cdot\pi\left(\sqrt{2}\right)^2 = 8\pi$$

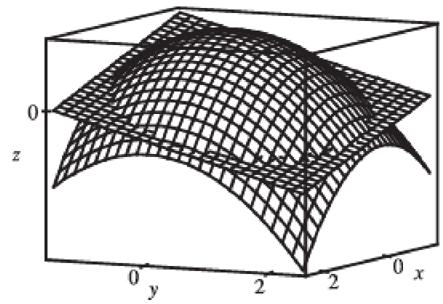
60. First, $\iint_D (2 - 3x + 4y)\,dA = \iint_D 2\,dA - \iint_D 3x\,dA + \iint_D 4y\,dA$. The region D, shown in the figure, is symmetric with respect to the y-axis and $3x$ is an odd function of x, so $\iint_D 3x\,dA = 0$. Similarly, $4y$ is an odd function of y and D is symmetric with respect to the x-axis, so $\iint_D 4y\,dA = 0$. Then

$$\begin{aligned} \iint_D (2 - 3x + 4y)\,dA &= \iint_D 2\,dA = 2\iint_D\,dA \\ &= 2(\text{area of } D) = 2(50) = 100 \end{aligned}$$



61. Since $\sqrt{1 - x^2 - y^2} \ge 0$, we can interpret $\iint_D \sqrt{1 - x^2 - y^2}\,dA$ as the volume of the solid that lies below the graph of $z = \sqrt{1 - x^2 - y^2}$ and above the region D in the xy-plane. $z = \sqrt{1 - x^2 - y^2}$ is equivalent to $x^2 + y^2 + z^2 = 1$, $z \ge 0$ which meets the xy-plane in the circle $x^2 + y^2 = 1$, the boundary of D. Thus, the solid is an upper hemisphere of radius 1 which has volume $\frac{1}{2}\left[\frac{4}{3}\pi(1)^3\right] = \frac{2}{3}\pi$.

62. To find the equations of the boundary curves, we require that the z-values of the two surfaces be the same. In Maple, we use the command `solve(4-x^2-y^2=1-x-y,y);` and in Mathematica, we use `Solve[4-x^2-y^2==1-x-y,y]`. We find that the curves have equations $y = \dfrac{1 \pm \sqrt{13 + 4x - 4x^2}}{2}$. To find the two points of intersection of these curves, we use the CAS to solve $13 + 4x - 4x^2 = 0$, finding that $x = \frac{1 \pm \sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is



$$V = \int_{(1-\sqrt{14})/2}^{(1+\sqrt{14})/2} \int_{\left(1-\sqrt{13+4x-4x^2}\right)/2}^{\left(1+\sqrt{13+4x-4x^2}\right)/2} \left[(4 - x^2 - y^2) - (1 - x - y)\right] dy\, dx = \frac{49\pi}{8}$$