1. The region R is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 0 \le r \le 4, 0 \le \theta \le \frac{3\pi}{2}\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_0^{3\pi/2}\int_0^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

2. The region R is more easily described by rectangular coordinates: $R = \left\{(x,y) \mid -1 \le x \le 1, 0 \le y \le 1 - x^2\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_{-1}^1 \int_0^{1-x^2} f(x,y)\,dy\,dx$.

3. The region R is more easily described by rectangular coordinates: $R = \left\{(x,y) \mid -1 \le x \le 1, 0 \le y \le \frac{1}{2}x + \frac{1}{2}\right\}$.

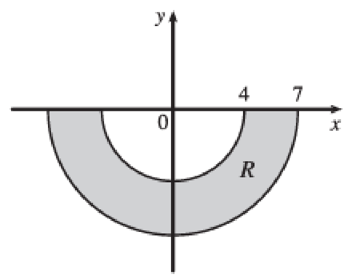
Thus $\iint_R f(x,y)\,dA = \int_{-1}^1 \int_0^{(x+1)/2} f(x,y)\,dy\,dx$.

4. The region R is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 3 \le r \le 6, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_{-\pi/2}^{\pi/2}\int_3^6 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

5. The integral $\int_\pi^{2\pi}\int_4^7 r\,dr\,d\theta$ represents the area of the region

$R = \{(r,\theta) \mid 4 \le r \le 7, \pi \le \theta \le 2\pi\}$, the lower half of a ring.



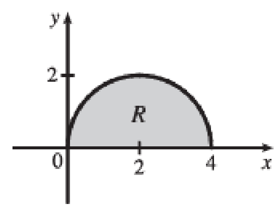
$$\int_\pi^{2\pi}\int_4^7 r\,dr\,d\theta = \left(\int_\pi^{2\pi} d\theta\right)\left(\int_4^7 r\,dr\right)$$
$$= \left[\,\theta\,\right]_\pi^{2\pi}\left[\tfrac{1}{2}r^2\right]_4^7 = \pi \cdot \tfrac{1}{2}(49 - 16) = \tfrac{33\pi}{2}$$

6. The integral $\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta$ represents the area of the region

$R = \{(r,\theta) \mid 0 \le r \le 4\cos\theta, 0 \le \theta \le \pi/2\}$. Since $r = 4\cos\theta \quad \Leftrightarrow$

$r^2 = 4r\cos\theta \quad \Leftrightarrow \quad x^2 + y^2 = 4x \quad \Leftrightarrow \quad (x-2)^2 + y^2 = 4$, R is the

portion in the first quadrant of a circle of radius 2 with center $(2, 0)$.



$$\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=4\cos\theta} d\theta = \int_0^{\pi/2} 8\cos^2\theta\,d\theta = \int_0^{\pi/2} 4(1+\cos 2\theta)\,d\theta = 4\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2\pi$$

7. The disk D can be described in polar coordinates as $D = \{(r,\theta) \mid 0 \le r \le 3, 0 \le \theta \le 2\pi\}$. Then

$$\iint_D xy\,dA = \int_0^{2\pi}\int_0^3 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta = \left(\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\right)\left(\int_0^3 r^3\,dr\right) = \left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^3 = 0.$$

8. $$\iint_R (x+y)\,dA = \int_{\pi/2}^{3\pi/2}\int_1^2 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = \int_{\pi/2}^{3\pi/2}\int_1^2 r^2(\cos\theta + \sin\theta)\,dr\,d\theta$$
$$= \left(\int_{\pi/2}^{3\pi/2}(\cos\theta + \sin\theta)\,d\theta\right)\left(\int_1^2 r^2\,dr\right) = \left[\sin\theta - \cos\theta\right]_{\pi/2}^{3\pi/2}\left[\tfrac{1}{3}r^3\right]_1^2$$
$$= (-1 - 0 - 1 + 0)\left(\tfrac{8}{3} - \tfrac{1}{3}\right) = -\tfrac{14}{3}$$

9. $$\iint_R \cos(x^2 + y^2)\,dA = \int_0^\pi\int_0^3 \cos(r^2)\,r\,dr\,d\theta = \left(\int_0^\pi d\theta\right)\left(\int_0^3 r\cos(r^2)\,dr\right)$$
$$= \left[\,\theta\,\right]_0^\pi\left[\tfrac{1}{2}\sin(r^2)\right]_0^3 = \pi \cdot \tfrac{1}{2}(\sin 9 - \sin 0) = \tfrac{\pi}{2}\sin 9$$

10. $\iint_R \sqrt{4-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 \sqrt{4-r^2}\,r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 r\sqrt{4-r^2}\,dr\right)$

$$= \left[\theta\right]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{2}\cdot\tfrac{2}{3}(4-r^2)^{3/2}\right]_0^2 = \left(\tfrac{\pi}{2}+\tfrac{\pi}{2}\right)\left(-\tfrac{1}{3}(0-4^{3/2})\right) = \tfrac{8}{3}\pi$$

11. $\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 re^{-r^2}\,dr\right)$

$$= \left[\theta\right]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)(e^{-4}-e^0) = \tfrac{\pi}{2}(1-e^{-4})$$

12. $\iint_R ye^x\,dA = \int_0^{\pi/2}\int_0^5 (r\sin\theta)\,e^{r\cos\theta}\,r\,dr\,d\theta = \int_0^5\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta\,dr$. First we integrate $\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta$:

Let $u = r\cos\theta \quad\Rightarrow\quad du = -r\sin\theta\,d\theta$, and $\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta = \int_{u=r}^{u=0} -r\,e^u\,du = -r[e^0-e^r] = re^r - r$.

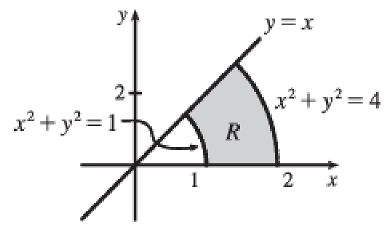
Then $\int_0^5\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta\,dr = \int_0^5 (re^r - r)\,dr = \left[re^r - e^r - \tfrac{1}{2}r^2\right]_0^5 = 4e^5 - \tfrac{23}{2}$, where we integrated by parts in the first term.

13. R is the region shown in the figure, and can be described by $R = \{(r,\theta)\mid 0\le\theta\le\pi/4,\ 1\le r\le 2\}$. Thus

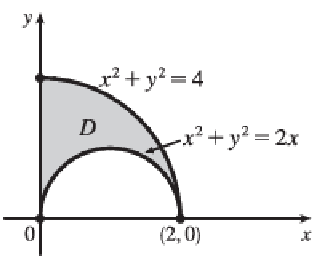
$\iint_R \arctan(y/x)\,dA = \int_0^{\pi/4}\int_1^2 \arctan(\tan\theta)\,r\,dr\,d\theta$ since $y/x = \tan\theta$.

Also, $\arctan(\tan\theta) = \theta$ for $0\le\theta\le\pi/4$, so the integral becomes

$\int_0^{\pi/4}\int_1^2 \theta\,r\,dr\,d\theta = \int_0^{\pi/4}\theta\,d\theta\int_1^2 r\,dr = \left[\tfrac{1}{2}\theta^2\right]_0^{\pi/4}\left[\tfrac{1}{2}r^2\right]_1^2 = \tfrac{\pi^2}{32}\cdot\tfrac{3}{2} = \tfrac{3}{64}\pi^2$.



14.



$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x\ge0,\,y\ge0}} x\,dA - \iint_{\substack{(x-1)^2+y^2\le1\\ y\ge0}} x\,dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\tfrac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\tfrac{1}{3}(8\cos^4\theta)\,d\theta$$

$$= \tfrac{8}{3} - \tfrac{8}{12}\left[\cos^3\theta\sin\theta + \tfrac{3}{2}(\theta+\sin\theta\cos\theta)\right]_0^{\pi/2}$$

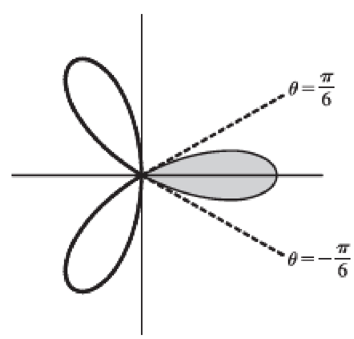
$$= \tfrac{8}{3} - \tfrac{2}{3}\left[0+\tfrac{3}{2}\left(\tfrac{\pi}{2}\right)\right] = \tfrac{16-3\pi}{6}$$

15. One loop is given by the region $D = \{(r,\theta)\mid -\pi/6\le\theta\le\pi/6,\ 0\le r\le\cos3\theta\}$, so the area is

$$\iint_D dA = \int_{-\pi/6}^{\pi/6}\int_0^{\cos3\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos3\theta} d\theta$$

$$= \int_{-\pi/6}^{\pi/6}\frac{1}{2}\cos^2 3\theta\,d\theta = 2\int_0^{\pi/6}\frac{1}{2}\left(\frac{1+\cos6\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{6}\sin6\theta\right]_0^{\pi/6} = \frac{\pi}{12}$$



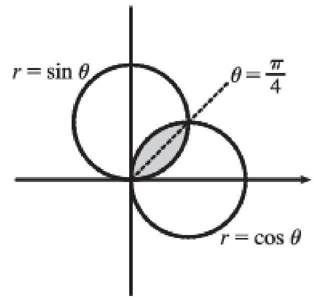
16. $D = \{(r, \theta) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4 + 3\cos\theta\}$, so

$$A(D) = \iint_D dA = \int_0^{2\pi} \int_0^{4+3\cos\theta} r\,dr\,d\theta = \int_0^{2\pi} \left[\tfrac{1}{2}r^2\right]_{r=0}^{r=4+3\cos\theta} d\theta = \tfrac{1}{2}\int_0^{2\pi} (4 + 3\cos\theta)^2\,d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} (16 + 24\cos\theta + 9\cos^2\theta)\,d\theta = \tfrac{1}{2}\int_0^{2\pi} \left(16 + 24\cos\theta + 9 \cdot \tfrac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \tfrac{1}{2}\left[16\theta + 24\sin\theta + \tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{41}{2}\pi$$

17. By symmetry,



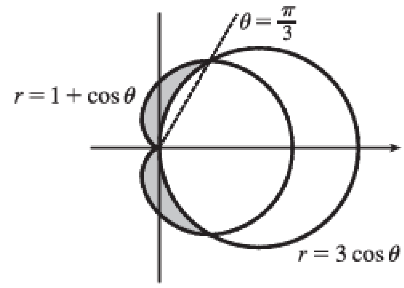
$$A = 2\int_0^{\pi/4} \int_0^{\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/4} \left[\tfrac{1}{2}r^2\right]_{r=0}^{r=\sin\theta} d\theta$$

$$= \int_0^{\pi/4} \sin^2\theta\,d\theta = \int_0^{\pi/4} \tfrac{1}{2}(1 - \cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= \tfrac{1}{2}\left[\tfrac{\pi}{4} - \tfrac{1}{2}\sin\tfrac{\pi}{2} - 0 + \tfrac{1}{2}\sin 0\right] = \tfrac{1}{8}(\pi - 2)$$

18. The region lies between the two polar curves in quadrants I and IV, but in quadrants II and III the region is enclosed by the cardiod. In the first quadrant, $1 + \cos\theta = 3\cos\theta$ when $\cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$, so the area of the region inside the cardiod and outside the circle is



$$A_1 = \int_{\pi/3}^{\pi/2} \int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta = \int_{\pi/3}^{\pi/2} \left[\tfrac{1}{2}r^2\right]_{r=3\cos\theta}^{r=1+\cos\theta} d\theta$$

$$= \tfrac{1}{2}\int_{\pi/3}^{\pi/2} (1 + 2\cos\theta - 8\cos^2\theta)d\theta = \tfrac{1}{2}\left[\theta + 2\sin\theta - 8\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right)\right]_{\pi/3}^{\pi/2}$$

$$= \left[-\tfrac{3}{2}\theta + \sin\theta - \sin 2\theta\right]_{\pi/3}^{\pi/2} = \left(-\tfrac{3\pi}{4} + 1 - 0\right) - \left(-\tfrac{\pi}{2} + \tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{2}\right) = 1 - \tfrac{\pi}{4}.$$

The area of the region in the second quadrant is

$$A_2 = \int_{\pi/2}^{\pi} \int_0^{1+\cos\theta} r\,dr\,d\theta = \int_{\pi/2}^{\pi} \left[\tfrac{1}{2}r^2\right]_{r=0}^{r=1+\cos\theta} d\theta = \tfrac{1}{2}\int_{\pi/2}^{\pi} (1 + 2\cos\theta + \cos^2\theta)d\theta$$

$$= \tfrac{1}{2}\left[\theta + 2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi} = \tfrac{1}{2}\left(\tfrac{3\pi}{4} - 2\right) = \tfrac{3\pi}{8} - 1.$$

By symmetry, the total area is $A = 2(A_1 + A_2) = 2\left(1 - \frac{\pi}{4} + \frac{3\pi}{8} - 1\right) = \frac{\pi}{4}$.

19. $V = \iint_{x^2+y^2 \le 4} \sqrt{x^2 + y^2}\,dA = \int_0^{2\pi} \int_0^2 \sqrt{r^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 r^2\,dr = \left[\theta\right]_0^{2\pi} \left[\frac{1}{3}r^3\right]_0^2 = 2\pi\left(\frac{8}{3}\right) = \frac{16}{3}\pi$

20. The paraboloid $z = 18 - 2x^2 - 2y^2$ intersects the xy-plane in the circle $x^2 + y^2 = 9$, so

$$V = \iint\limits_{x^2+y^2 \le 9} (18 - 2x^2 - 2y^2)\,dA = \iint\limits_{x^2+y^2 \le 9} \left[18 - 2\left(x^2 + y^2\right)\right] dA = \int_0^{2\pi} \int_0^3 \left(18 - 2r^2\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^3 (18r - 2r^3)\,dr = \left[\theta\right]_0^{2\pi} \left[9r^2 - \tfrac{1}{2}r^4\right]_0^3 = (2\pi)\left(81 - \tfrac{81}{2}\right) = 81\pi$$

21. The hyperboloid of two sheets $-x^2 - y^2 + z^2 = 1$ intersects the plane $z = 2$ when $-x^2 - y^2 + 4 = 1$ or $x^2 + y^2 = 3$. So the solid region lies above the surface $z = \sqrt{1 + x^2 + y^2}$ and below the plane $z = 2$ for $x^2 + y^2 \leq 3$, and its volume is

$$V = \iint_{x^2+y^2 \leq 3} \left(2 - \sqrt{1 + x^2 + y^2}\right) dA = \int_0^{2\pi} \int_0^{\sqrt{3}} (2 - \sqrt{1 + r^2})\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} (2r - r\sqrt{1 + r^2}) dr = \left[\theta\right]_0^{2\pi} \left[r^2 - \tfrac{1}{3}(1 + r^2)^{3/2}\right]_0^{\sqrt{3}}$$

$$= 2\pi\left(3 - \tfrac{8}{3} - 0 + \tfrac{1}{3}\right) = \tfrac{4}{3}\pi$$

22. The sphere $x^2 + y^2 + z^2 = 16$ intersects the xy-plane in the circle $x^2 + y^2 = 16$, so

$$V = 2 \iint_{4 \leq x^2+y^2 \leq 16} \sqrt{16 - x^2 - y^2}\, dA \quad \text{[by symmetry]} \quad = 2\int_0^{2\pi} \int_2^4 \sqrt{16 - r^2}\, r\, dr\, d\theta = 2\int_0^{2\pi} d\theta \int_2^4 r(16 - r^2)^{1/2} dr$$

$$= 2\left[\theta\right]_0^{2\pi} \left[-\tfrac{1}{3}(16 - r^2)^{3/2}\right]_2^4 = -\tfrac{2}{3}(2\pi)(0 - 12^{3/2}) = \tfrac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\,\pi$$

23. By symmetry,

$$V = 2\iint_{x^2+y^2 \leq a^2} \sqrt{a^2 - x^2 - y^2}\, dA = 2\int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2}\, r\, dr\, d\theta = 2\int_0^{2\pi} d\theta \int_0^a r\sqrt{a^2 - r^2}\, dr$$

$$= 2\left[\theta\right]_0^{2\pi} \left[-\tfrac{1}{3}(a^2 - r^2)^{3/2}\right]_0^a = 2(2\pi)\left(0 + \tfrac{1}{3}a^3\right) = \tfrac{4\pi}{3}a^3$$

24. The paraboloid $z = 1 + 2x^2 + 2y^2$ intersects the plane $z = 7$ when $7 = 1 + 2x^2 + 2y^2$ or $x^2 + y^2 = 3$ and we are restricted to the first octant, so

$$V = \iint_{\substack{x^2+y^2 \leq 3,\\ x \geq 0, y \geq 0}} \left[7 - (1 + 2x^2 + 2y^2)\right] dA = \int_0^{\pi/2} \int_0^{\sqrt{3}} \left[7 - (1 + 2r^2)\right] r\, dr\, d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{\sqrt{3}} (6r - 2r^3)\, dr = \left[\theta\right]_0^{\pi/2} \left[3r^2 - \tfrac{1}{2}r^4\right]_0^{\sqrt{3}} = \tfrac{\pi}{2} \cdot \tfrac{9}{2} = \tfrac{9}{4}\pi$$

25. The cone $z = \sqrt{x^2 + y^2}$ intersects the sphere $x^2 + y^2 + z^2 = 1$ when $x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2 = 1$ or $x^2 + y^2 = \frac{1}{2}$. So

$$V = \iint_{x^2+y^2 \leq 1/2} \left(\sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2}\right) dA = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \left(\sqrt{1 - r^2} - r\right) r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{1/\sqrt{2}} \left(r\sqrt{1 - r^2} - r^2\right) dr = \left[\theta\right]_0^{2\pi} \left[-\tfrac{1}{3}(1 - r^2)^{3/2} - \tfrac{1}{3}r^3\right]_0^{1/\sqrt{2}} = 2\pi\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{\sqrt{2}} - 1\right) = \tfrac{\pi}{3}\left(2 - \sqrt{2}\right)$$

26. The two paraboloids intersect when $3x^2 + 3y^2 = 4 - x^2 - y^2$ or $x^2 + y^2 = 1$. So

$$V = \iint_{x^2+y^2 \leq 1} \left[(4 - x^2 - y^2) - 3(x^2 + y^2)\right] dA = \int_0^{2\pi} \int_0^1 4(1 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (4r - 4r^3)\, dr = \left[\theta\right]_0^{2\pi} \left[2r^2 - r^4\right]_0^1 = 2\pi$$

27. The given solid is the region inside the cylinder $x^2 + y^2 = 4$ between the surfaces $z = \sqrt{64 - 4x^2 - 4y^2}$ and $z = -\sqrt{64 - 4x^2 - 4y^2}$. So

$$V = \iint\limits_{x^2+y^2 \le 4} \left[\sqrt{64-4x^2-4y^2} - \left(-\sqrt{64-4x^2-4y^2}\right)\right] dA = \iint\limits_{x^2+y^2\le 4} 2\sqrt{64-4x^2-4y^2}\, dA$$
$$= 4\int_0^{2\pi}\int_0^2 \sqrt{16-r^2}\, r\, dr\, d\theta = 4\int_0^{2\pi} d\theta \int_0^2 r\sqrt{16-r^2}\, dr = 4\left[\,\theta\,\right]_0^{2\pi}\left[-\tfrac{1}{3}(16-r^2)^{3/2}\right]_0^2$$
$$= 8\pi\left(-\tfrac{1}{3}\right)\left(12^{3/2} - 16^{2/3}\right) = \tfrac{8\pi}{3}\left(64 - 24\sqrt{3}\right)$$

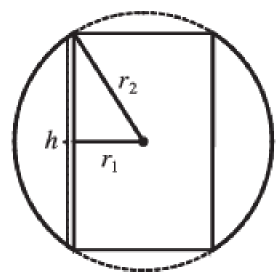
28. (a) Here the region in the xy-plane is the annular region $r_1^2 \le x^2 + y^2 \le r_2^2$ and the desired volume is twice that above the xy-plane. Hence

$$V = 2\iint\limits_{r_1^2 \le x^2+y^2 \le r_2^2} \sqrt{r_2^2 - x^2 - y^2}\, dA = 2\int_0^{2\pi}\int_{r_1}^{r_2} \sqrt{r_2^2 - r^2}\, r\, dr\, d\theta = 2\int_0^{2\pi} d\theta \int_{r_1}^{r_2} \sqrt{r_2^2 - r^2}\, r\, dr$$
$$= \tfrac{4\pi}{3}\left[-(r_2^2 - r^2)^{3/2}\right]_{r_1}^{r_2} = \tfrac{4\pi}{3}(r_2^2 - r_1^2)^{3/2}$$

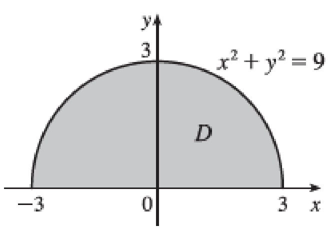
(b) A cross-sectional cut is shown in the figure.

So $r_2^2 = \left(\tfrac{1}{2}h\right)^2 + r_1^2$ or $\tfrac{1}{4}h^2 = r_2^2 - r_1^2$.

Thus the volume in terms of h is $V = \tfrac{4\pi}{3}\left(\tfrac{1}{4}h^2\right)^{3/2} = \tfrac{\pi}{6}h^3$.

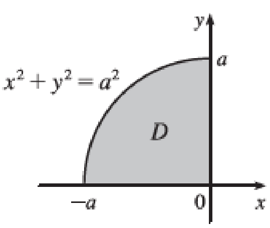


29.



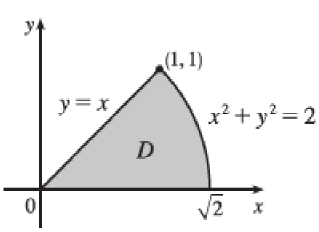
$$\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\, dy\, dx = \int_0^{\pi}\int_0^3 \sin\left(r^2\right) r\, dr\, d\theta$$
$$= \int_0^{\pi} d\theta \int_0^3 r\sin\left(r^2\right)\, dr = [\theta]_0^{\pi}\left[-\tfrac{1}{2}\cos\left(r^2\right)\right]_0^3$$
$$= \pi\left(-\tfrac{1}{2}\right)(\cos 9 - 1) = \tfrac{\pi}{2}(1 - \cos 9)$$

30.

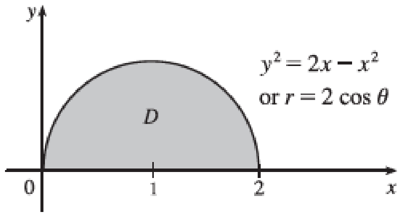


$$\int_{\pi/2}^{\pi}\int_0^a (r\cos\theta)^2 (r\sin\theta)\, r\, dr\, d\theta = \int_{\pi/2}^{\pi}\int_0^a r^4\cos^2\theta\sin\theta\, dr\, d\theta$$
$$= \int_{\pi/2}^{\pi}\cos^2\theta\sin\theta\, d\theta \int_0^a r^4\, dr$$
$$= \left[-\tfrac{1}{3}\cos^3\theta\right]_{\pi/2}^{\pi}\left[\tfrac{1}{5}r^5\right]_0^a$$
$$= -\tfrac{1}{3}\left(\cos^3\pi - \cos^3\tfrac{\pi}{2}\right)\left(\tfrac{1}{5}a^5\right) = \tfrac{1}{15}a^5$$

31.



$$\int_0^{\pi/4}\int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta = \int_0^{\pi/4}(\cos\theta + \sin\theta)\, d\theta \int_0^{\sqrt{2}} r^2\, dr$$
$$= [\sin\theta - \cos\theta]_0^{\pi/4}\left[\tfrac{1}{3}r^3\right]_0^{\sqrt{2}}$$
$$= \left[\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} - 0 + 1\right]\cdot\tfrac{1}{3}\left(2\sqrt{2} - 0\right) = \tfrac{2\sqrt{2}}{3}$$

32. 

$$\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_0^{\pi/2}\left[\tfrac{1}{3}r^3\right]_{r=0}^{r=2\cos\theta} d\theta = \int_0^{\pi/2}\left(\tfrac{8}{3}\cos^3\theta\right)d\theta$$
$$= \tfrac{8}{3}\int_0^{\pi/2}\left(1-\sin^2\theta\right)\cos\theta\,d\theta$$
$$= \tfrac{8}{3}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right]_0^{\pi/2} = \tfrac{16}{9}$$

33. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x,y)$ to be the depth of the water at (x,y), then the volume of water in the pool is the volume of the solid that lies above $D = \left\{(x,y) \mid x^2+y^2 \le 400\right\}$ and below the graph of $f(x,y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0,-20) = 2$ to $f(0,20) = 7$. The trace in the yz-plane is a line segment from $(0,-20,2)$ to $(0,20,7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z - 7 = \frac{1}{8}(y-20) \quad\Rightarrow\quad z = \frac{1}{8}y + \frac{9}{2}$. Since $f(x,y)$ is independent of x, $f(x,y) = \frac{1}{8}y + \frac{9}{2}$. Thus the volume is given by $\iint_D f(x,y)\,dA$, which is most conveniently evaluated using polar coordinates. Then $D = \{(r,\theta) \mid 0 \le r \le 20, 0 \le \theta \le 2\pi\}$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\int_0^{2\pi}\int_0^{20}\left(\tfrac{1}{8}r\sin\theta + \tfrac{9}{2}\right)r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{24}r^3\sin\theta + \tfrac{9}{4}r^2\right]_{r=0}^{r=20} d\theta = \int_0^{2\pi}\left(\tfrac{1000}{3}\sin\theta + 900\right)d\theta$$
$$= \left[-\tfrac{1000}{3}\cos\theta + 900\theta\right]_0^{2\pi} = 1800\pi$$

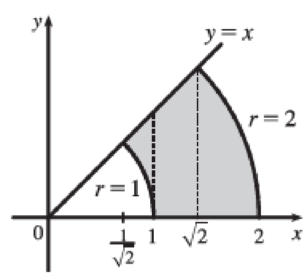
Thus the pool contains $1800\pi \approx 5655$ ft^3 of water.

34. (a) If $R \le 100$, the total amount of water supplied each hour to the region within R feet of the sprinkler is

$$V = \int_0^{2\pi}\int_0^R e^{-r}r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^R re^{-r}\,dr = \left[\theta\right]_0^{2\pi}\left[-re^{-r} - e^{-r}\right]_0^R$$
$$= 2\pi\left[-Re^{-R} - e^{-R} + 0 + 1\right] = 2\pi\left(1 - Re^{-R} - e^{-R}\right)\text{ ft}^3$$

(b) The average amount of water per hour per square foot supplied to the region within R feet of the sprinkler is $\frac{V}{\text{area of region}} = \frac{V}{\pi R^2} = \frac{2\left(1 - Re^{-R} - e^{-R}\right)}{R^2}$ ft^3 (per hour per square foot). See the definition of the average value of a function on page 992 [ET 956].

35.
$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_1^{\sqrt{2}}\int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} xy\,dy\,dx$$
$$= \int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=1}^{r=2} d\theta$$
$$= \frac{15}{4}\int_0^{\pi/4}\sin\theta\cos\theta\,d\theta = \frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{15}{16}$$



36. (a) $\iint_{D_a} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^a re^{-r^2}\,dr\,d\theta = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^a = \pi\left(1-e^{-a^2}\right)$ for each a. Then $\lim_{a\to\infty}\pi\left(1-e^{-a^2}\right)=\pi$ since $e^{-a^2}\to 0$ as $a\to\infty$. Hence $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA = \pi$.

(b) $\iint_{S_a} e^{-(x^2+y^2)}\,dA = \int_{-a}^{a}\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dx\,dy = \left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)$ for each a.

Then, from (a), $\pi = \iint_{\mathbb{R}^2} -(x^2+y^2)\,dA$, so

$\pi = \lim_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right).$

To evaluate $\lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)$, we are using the fact that these integrals are bounded. This is true since on $[-1,1]$, $0<e^{-x^2}\le 1$ while on $(-\infty,-1)$, $0<e^{-x^2}\le e^{x}$ and on $(1,\infty)$, $0<e^{-x^2}<e^{-x}$. Hence

$0\le\int_{-\infty}^{\infty} e^{-x^2}\,dx \le \int_{-\infty}^{-1} e^{x}\,dx + \int_{-1}^{1}\,dx + \int_{1}^{\infty} e^{-x}\,dx = 2(e^{-1}+1).$

(c) Since $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)=\pi$ and y can be replaced by x, $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2=\pi$ implies that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \pm\sqrt{\pi}$. But $e^{-x^2}\ge 0$ for all x, so $\int_{-\infty}^{\infty} e^{-x^2}\,dx=\sqrt{\pi}$.

(d) Letting $t=\sqrt{2}\,x$, $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2}}\left(e^{-t^2/2}\right)dt$, so that $\sqrt{\pi}=\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-t^2/2}\,dt$ or $\int_{-\infty}^{\infty} e^{-t^2/2}\,dt=\sqrt{2\pi}$.

37. (a) We integrate by parts with $u=x$ and $dv = xe^{-x^2}\,dx$. Then $du=dx$ and $v=-\frac{1}{2}e^{-x^2}$, so

$$\begin{aligned}
\int_0^{\infty} x^2e^{-x^2}\,dx &= \lim_{t\to\infty}\int_0^t x^2e^{-x^2}\,dx = \lim_{t\to\infty}\left(\left[-\tfrac{1}{2}xe^{-x^2}\right]_0^t + \int_0^t \tfrac{1}{2}e^{-x^2}\,dx\right)\\
&= \lim_{t\to\infty}\left(-\tfrac{1}{2}te^{-t^2}\right) + \tfrac{1}{2}\int_0^{\infty} e^{-x^2}\,dx = 0 + \tfrac{1}{2}\int_0^{\infty} e^{-x^2}\,dx \qquad \text{[by l'Hospital's Rule]}\\
&= \tfrac{1}{4}\int_{-\infty}^{\infty} e^{-x^2}\,dx \qquad \text{[since } e^{-x^2} \text{ is an even function]}\\
&= \tfrac{1}{4}\sqrt{\pi} \qquad \text{[by Exercise 36(c)]}
\end{aligned}$$

(b) Let $u=\sqrt{x}$. Then $u^2=x \;\Rightarrow\; dx = 2u\,du \;\Rightarrow$

$\int_0^{\infty}\sqrt{x}e^{-x}\,dx = \lim_{t\to\infty}\int_0^t\sqrt{x}\,e^{-x}\,dx = \lim_{t\to\infty}\int_0^{\sqrt{t}} ue^{-u^2}2u\,du = 2\int_0^{\infty} u^2e^{-u^2}\,du = 2\left(\frac{1}{4}\sqrt{\pi}\right)$ [by part(a)] $=\frac{1}{2}\sqrt{\pi}$.