1. $Q = \iint_D \sigma(x,y)\,dA = \int_1^3 \int_0^2 (2xy + y^2)\,dy\,dx = \int_1^3 \left[xy^2 + \frac{1}{3}y^3\right]_{y=0}^{y=2} dx$

$= \int_1^3 \left(4x + \frac{8}{3}\right) dx = \left[2x^2 + \frac{8}{3}x\right]_1^3 = 16 + \frac{16}{3} = \frac{64}{3}$ C

2. $Q = \iint_D \sigma(x,y)\,dA = \iint_D (x + y + x^2 + y^2)\,dA = \int_0^{2\pi} \int_0^2 (r\cos\theta + r\sin\theta + r^2)\,r\,dr\,d\theta$

$= \int_0^{2\pi} \int_0^2 [r^2(\cos\theta + \sin\theta) + r^3]\,dr\,d\theta = \int_0^{2\pi} \left[\frac{1}{3}r^3(\cos\theta + \sin\theta) + \frac{1}{4}r^4\right]_{r=0}^{r=2} d\theta$

$= \int_0^{2\pi} \left[\frac{8}{3}(\cos\theta + \sin\theta) + 4\right] d\theta = \left[\frac{8}{3}(\sin\theta - \cos\theta) + 4\theta\right]_0^{2\pi} = 8\pi$ C

3. $m = \iint_D \rho(x,y)\,dA = \int_0^2 \int_{-1}^1 xy^2\,dy\,dx = \int_0^2 x\,dx \int_{-1}^1 y^2\,dy = \left[\frac{1}{2}x^2\right]_0^2 \left[\frac{1}{3}y^3\right]_{-1}^1 = 2 \cdot \frac{2}{3} = \frac{4}{3}$,

$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{3}{4}\int_0^2 \int_{-1}^1 x^2y^2\,dy\,dx = \frac{3}{4}\int_0^2 x^2\,dx \int_{-1}^1 y^2\,dy = \frac{3}{4}\left[\frac{1}{3}x^3\right]_0^2 \left[\frac{1}{3}y^3\right]_{-1}^1 = \frac{3}{4}\cdot\frac{8}{3}\cdot\frac{2}{3} = \frac{4}{3}$,

$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{3}{4}\int_0^2 \int_{-1}^1 xy^3\,dy\,dx = \frac{3}{4}\int_0^2 x\,dx \int_{-1}^1 y^3\,dy = \frac{3}{4}\left[\frac{1}{2}x^2\right]_0^2 \left[\frac{1}{4}y^4\right]_{-1}^1 = \frac{3}{4}\cdot 2 \cdot 0 = 0$.

Hence, $(\overline{x}, \overline{y}) = \left(\frac{4}{3}, 0\right)$.

4. $m = \iint_D \rho(x,y)\,dA = \int_0^a \int_0^b cxy\,dy\,dx = c\int_0^a x\,dx \int_0^b y\,dy = c\left[\frac{1}{2}x^2\right]_0^a \left[\frac{1}{2}y^2\right]_0^b = \frac{1}{4}a^2b^2c$,

$M_y = \iint_D x\rho(x,y)\,dA = \int_0^a \int_0^b cx^2y\,dy\,dx = c\int_0^a x^2\,dx \int_0^b y\,dy = c\left[\frac{1}{3}x^3\right]_0^a \left[\frac{1}{2}y^2\right]_0^b = \frac{1}{6}a^3b^2c$, and

$M_x = \iint_D y\rho(x,y)\,dA = \int_0^a \int_0^b cxy^2\,dy\,dx = c\int_0^a x\,dx \int_0^b y^2\,dy = c\left[\frac{1}{2}x^2\right]_0^a \left[\frac{1}{3}y^3\right]_0^b = \frac{1}{6}a^2b^3c$.

Hence, $(\overline{x}, \overline{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \left(\frac{2}{3}a, \frac{2}{3}b\right)$.

5. $m = \int_0^2 \int_{x/2}^{3-x} (x+y)\,dy\,dx = \int_0^2 \left[xy + \frac{1}{2}y^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left[x\left(3 - \frac{3}{2}x\right) + \frac{1}{2}(3-x)^2 - \frac{1}{8}x^2\right] dx$

$= \int_0^2 \left(-\frac{9}{8}x^2 + \frac{9}{2}\right) dx = \left[-\frac{9}{8}\left(\frac{1}{3}x^3\right) + \frac{9}{2}x\right]_0^2 = 6$,

$M_y = \int_0^2 \int_{x/2}^{3-x} (x^2 + xy)\,dy\,dx = \int_0^2 \left[x^2y + \frac{1}{2}xy^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left(\frac{9}{2}x - \frac{9}{8}x^3\right) dx = \frac{9}{2}$,

$M_x = \int_0^2 \int_{x/2}^{3-y} (xy + y^2)\,dy\,dx = \int_0^2 \left[\frac{1}{2}xy^2 + \frac{1}{3}y^3\right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left(9 - \frac{9}{2}x\right) dx = 9$.

Hence $m = 6$, $(\overline{x}, \overline{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \left(\dfrac{3}{4}, \dfrac{3}{2}\right)$.

6. Here $D = \{(x,y) \mid 0 \le x \le 2, x \le y \le 6 - 2x\}$.

$m = \int_0^2 \int_x^{6-2x} x^2\,dy\,dx = \int_0^2 x^2\,(6 - 2x - x)\,dx = \int_0^2 \left(6x^2 - 3x^3\right) dx = \left[2x^3 - \frac{3}{4}x^4\right]_0^2 = 4$,

$M_y = \int_0^2 \int_x^{6-2x} x \cdot x^2\,dy\,dx = \int_0^2 x^3\,(6 - 2x - x)\,dx = \int_0^2 \left(6x^3 - 3x^4\right) dx = \left[\frac{3}{2}x^4 - \frac{3}{5}x^5\right]_0^2 = \frac{24}{5}$,

$M_x = \int_0^2 \int_x^{6-2x} y \cdot x^2\,dy\,dx = \int_0^2 x^2\left[\frac{1}{2}(6-2x)^2 - \frac{1}{2}x^2\right] dx = \frac{1}{2}\int_0^2 \left(3x^4 - 24x^3 + 36x^2\right) dx$

$= \frac{1}{2}\left[\frac{3}{5}x^5 - 6x^4 + 12x^3\right]_0^2 = \frac{48}{5}$.

Hence $m = 4$, $(\overline{x}, \overline{y}) = \left(\frac{24/5}{4}, \frac{48/5}{4}\right) = \left(\frac{6}{5}, \frac{12}{5}\right)$.

7. $m = \int_0^1 \int_0^{e^x} y\,dy\,dx = \int_0^1 \left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x} dx = \frac{1}{2}\int_0^1 e^{2x}\,dx = \frac{1}{4}e^{2x}\Big]_0^1 = \frac{1}{4}(e^2 - 1),$

$M_y = \int_0^1 \int_0^{e^x} xy\,dy\,dx = \frac{1}{2}\int_0^1 xe^{2x}\,dx = \frac{1}{2}\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_0^1 = \frac{1}{8}(e^2 + 1),$

$M_x = \int_0^1 \int_0^{e^x} y^2\,dy\,dx = \int_0^1 \left[\frac{1}{3}y^3\right]_{y=0}^{y=e^x} dx = \frac{1}{3}\int_0^1 e^{3x}\,dx = \frac{1}{3}\left[\frac{1}{3}e^{3x}\right]_0^1 = \frac{1}{9}(e^3 - 1).$

Hence $m = \frac{1}{4}(e^2 - 1)$, $(\overline{x}, \overline{y}) = \left(\dfrac{\frac{1}{8}(e^2+1)}{\frac{1}{4}(e^2-1)}, \dfrac{\frac{1}{9}(e^3-1)}{\frac{1}{4}(e^2-1)}\right) = \left(\dfrac{e^2+1}{2(e^2-1)}, \dfrac{4(e^3-1)}{9(e^2-1)}\right).$

8. $m = \int_0^1 \int_0^{\sqrt{x}} x\,dy\,dx = \int_0^1 x\,[y]_{y=0}^{y=\sqrt{x}}\,dx = \int_0^1 x^{3/2}\,dx = \frac{2}{5}x^{5/2}\Big]_0^1 = \frac{2}{5},$

$M_y = \int_0^1 \int_0^{\sqrt{x}} x^2\,dy\,dx = \int_0^1 x\,[y]_{y=0}^{y=\sqrt{x}}\,dx = \int_0^1 x^{5/2}\,dx = \frac{2}{7}x^{7/2}\Big]_0^1 = \frac{2}{7},$

$M_x = \int_0^1 \int_0^{\sqrt{x}} yx\,dy\,dx = \int_0^1 x\left[\frac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}} dx = \frac{1}{2}\int_0^1 x^2\,dx = \frac{1}{2}\left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{6}.$

Hence $m = \frac{2}{5}$, $(\overline{x}, \overline{y}) = \left(\frac{2/7}{2/5}, \frac{1/6}{2/5}\right) = \left(\frac{5}{7}, \frac{5}{12}\right).$

9. Note that $\sin(\pi x/L) \geq 0$ for $0 \leq x \leq L$.

$m = \int_0^L \int_0^{\sin(\pi x/L)} y\,dy\,dx = \int_0^L \frac{1}{2}\sin^2(\pi x/L)\,dx = \frac{1}{2}\left[\frac{1}{2}x - \frac{L}{4\pi}\sin(2\pi x/L)\right]_0^L = \frac{1}{4}L,$

$M_y = \int_0^L \int_0^{\sin(\pi x/L)} x \cdot y\,dy\,dx = \frac{1}{2}\int_0^L x\sin^2(\pi x/L)\,dx \quad \left[\begin{array}{c}\text{integrate by parts with} \\ u = x,\ dv = \sin^2(\pi x/L)\,dx\end{array}\right]$

$= \frac{1}{2} \cdot x\left(\frac{1}{2}x - \frac{L}{4\pi}\sin(2\pi x/L)\right)\Big]_0^L - \frac{1}{2}\int_0^L \left[\frac{1}{2}x - \frac{L}{4\pi}\sin(2\pi x/L)\right] dx$

$= \frac{1}{4}L^2 - \frac{1}{2}\left[\frac{1}{4}x^2 + \frac{L^2}{4\pi^2}\cos(2\pi x/L)\right]_0^L = \frac{1}{4}L^2 - \frac{1}{2}\left(\frac{1}{4}L^2 + \frac{L^2}{4\pi^2} - \frac{L^2}{4\pi^2}\right) = \frac{1}{8}L^2,$

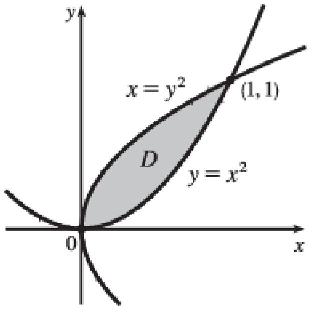
$M_x = \int_0^L \int_0^{\sin(\pi x/L)} y \cdot y\,dy\,dx = \int_0^L \frac{1}{3}\sin^3(\pi x/L)\,dx = \frac{1}{3}\int_0^L \left[1 - \cos^2(\pi x/L)\right]\sin(\pi x/L)\,dx$

[substitute $u = \cos(\pi x/L)$] $\Rightarrow$ $du = -\frac{\pi}{L}\sin(\pi x/L)$]

$= \frac{1}{3}\left(-\frac{L}{\pi}\right)\left[\cos(\pi x/L) - \frac{1}{3}\cos^3(\pi x/L)\right]_0^L = -\frac{L}{3\pi}\left(-1 + \frac{1}{3} - 1 + \frac{1}{3}\right) = \frac{4}{9\pi}L.$

Hence $m = \dfrac{L}{4}$, $(\overline{x}, \overline{y}) = \left(\dfrac{L^2/8}{L/4}, \dfrac{4L/(9\pi)}{L/4}\right) = \left(\dfrac{L}{2}, \dfrac{16}{9\pi}\right).$

10.



$m = \int_0^1 \int_{x^2}^{\sqrt{x}} \sqrt{x}\, dy\, dx = \int_0^1 \sqrt{x}(\sqrt{x} - x^2)\, dx$

$= \int_0^1 (x - x^{5/2})\, dx = \left[\frac{1}{2}x^2 - \frac{2}{7}x^{7/2}\right]_0^1 = \frac{3}{14},$

$M_y = \int_0^1 \int_{x^2}^{\sqrt{x}} x\sqrt{x}\, dy\, dx = \int_0^1 x\sqrt{x}(\sqrt{x} - x^2)dx = \int_0^1 (x^2 - x^{7/2})dx = \left[\frac{1}{3}x^3 - \frac{2}{9}x^{9/2}\right]_0^1 = \frac{1}{9}$

$M_x = \int_0^1 \int_{x^2}^{\sqrt{x}} y\sqrt{x}\, dy\, dx = \int_0^1 \sqrt{x} \cdot \frac{1}{2}(x - x^4)dx = \frac{1}{2}\int_0^1 (x^{3/2} - x^{9/2})dx$

$= \frac{1}{2}\left[\frac{2}{5}x^{5/2} - \frac{2}{11}x^{11/2}\right]_0^1 = \frac{1}{2} \cdot \frac{12}{55} = \frac{6}{55}.$

Hence $m = \frac{3}{14}$, $(\overline{x}, \overline{y}) = \left(\frac{1/9}{3/14}, \frac{6/55}{3/14}\right) = \left(\frac{14}{27}, \frac{28}{55}\right)$.

11. $\rho(x, y) = ky = kr\sin\theta$, $m = \int_0^{\pi/2}\int_0^1 kr^2 \sin\theta\, dr\, d\theta = \frac{1}{3}k\int_0^{\pi/2} \sin\theta\, d\theta = \frac{1}{3}k\left[-\cos\theta\right]_0^{\pi/2} = \frac{1}{3}k,$

$M_y = \int_0^{\pi/2}\int_0^1 kr^3 \sin\theta\cos\theta\, dr\, d\theta = \frac{1}{4}k\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \frac{1}{8}k\left[-\cos 2\theta\right]_0^{\pi/2} = \frac{1}{8}k,$

$M_x = \int_0^{\pi/2}\int_0^1 kr^3 \sin^2\theta\, dr\, d\theta = \frac{1}{4}k\int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{1}{8}k\left[\theta + \sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{16}k.$

Hence $(\overline{x}, \overline{y}) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)$.

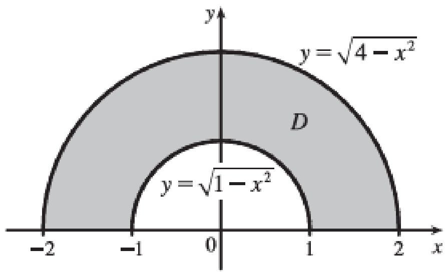
12. $\rho(x, y) = k(x^2 + y^2) = kr^2$, $m = \int_0^{\pi/2}\int_0^1 kr^3\, dr\, d\theta = \frac{\pi}{8}k,$

$M_y = \int_0^{\pi/2}\int_0^1 kr^4 \cos\theta\, dr\, d\theta = \frac{1}{5}k\int_0^{\pi/2} \cos\theta\, d\theta = \frac{1}{5}k\left[\sin\theta\right]_0^{\pi/2} = \frac{1}{5}k,$

$M_x = \int_0^{\pi/2}\int_0^1 kr^4 \sin\theta\, dr\, d\theta = \frac{1}{5}k\int_0^{\pi/2} \sin\theta\, d\theta = \frac{1}{5}k\left[-\cos\theta\right]_0^{\pi/2} = \frac{1}{5}k.$

Hence $(\overline{x}, \overline{y}) = \left(\frac{8}{5\pi}, \frac{8}{5\pi}\right)$.

13.



$$\rho(x,y) = k\sqrt{x^2+y^2} = kr,$$

$$m = \iint_D \rho(x,y)\,dA = \int_0^\pi \int_1^2 kr \cdot r\,dr\,d\theta$$

$$= k\int_0^\pi d\theta \int_1^2 r^2\,dr = k(\pi)\left[\tfrac{1}{3}r^3\right]_1^2 = \tfrac{7}{3}\pi k,$$

$$M_y = \iint_D x\rho(x,y)\,dA = \int_0^\pi \int_1^2 (r\cos\theta)(kr)\,r\,dr\,d\theta = k\int_0^\pi \cos\theta\,d\theta \int_1^2 r^3\,dr$$

$$= k\left[\sin\theta\right]_0^\pi \left[\tfrac{1}{4}r^4\right]_1^2 = k(0)\left(\tfrac{15}{4}\right) = 0$$

[this is to be expected as the region and density function are symmetric about the y-axis]

$$M_x = \iint_D y\rho(x,y)\,dA = \int_0^\pi \int_1^2 (r\sin\theta)(kr)\,r\,dr\,d\theta = k\int_0^\pi \sin\theta\,d\theta \int_1^2 r^3\,dr$$

$$= k\left[-\cos\theta\right]_0^\pi \left[\tfrac{1}{4}r^4\right]_1^2 = k(1+1)\left(\tfrac{15}{4}\right) = \tfrac{15}{2}k.$$

Hence $(\overline{x},\overline{y}) = \left(0, \frac{15k/2}{7\pi k/3}\right) = \left(0, \frac{45}{14\pi}\right)$.

14. Now $\rho(x,y) = k/\sqrt{x^2+y^2} = k/r$, so

$$m = \iint_D \rho(x,y)\,dA = \int_0^\pi \int_1^2 (k/r)\,r\,dr\,d\theta = k\int_0^\pi d\theta \int_1^2 dr = k(\pi)(1) = \pi k,$$

$$M_y = \iint_D x\rho(x,y)\,dA = \int_0^\pi \int_1^2 (r\cos\theta)(k/r)\,r\,dr\,d\theta = k\int_0^\pi \cos\theta\,d\theta \int_1^2 r\,dr$$

$$= k\left[\sin\theta\right]_0^\pi \left[\tfrac{1}{2}r^2\right]_1^2 = k(0)\left(\tfrac{3}{2}\right) = 0,$$

$$M_x = \iint_D y\rho(x,y)\,dA = \int_0^\pi \int_1^2 (r\sin\theta)(k/r)\,r\,dr\,d\theta = k\int_0^\pi \sin\theta\,d\theta \int_1^2 r\,dr$$

$$= k\left[-\cos\theta\right]_0^\pi \left[\tfrac{1}{2}r^2\right]_1^2 = k(1+1)\left(\tfrac{3}{2}\right) = 3k.$$

Hence $(\overline{x},\overline{y}) = \left(0, \frac{3k}{\pi k}\right) = \left(0, \frac{3}{\pi}\right)$.

15. Placing the vertex opposite the hypotenuse at $(0,0)$, $\rho(x,y) = k(x^2+y^2)$. Then

$$m = \int_0^a \int_0^{a-x} k\left(x^2+y^2\right)dy\,dx = k\int_0^a \left[ax^2 - x^3 + \tfrac{1}{3}(a-x)^3\right]dx = k\left[\tfrac{1}{3}ax^3 - \tfrac{1}{4}x^4 - \tfrac{1}{12}(a-x)^4\right]_0^a = \tfrac{1}{6}ka^4.$$

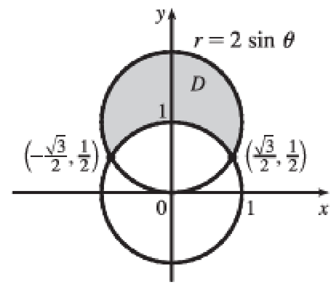
By symmetry,

$$M_y = M_x = \int_0^a \int_0^{a-x} ky(x^2+y^2)\,dy\,dx = k\int_0^a \left[\tfrac{1}{2}(a-x)^2x^2 + \tfrac{1}{4}(a-x)^4\right]dx$$

$$= k\left[\tfrac{1}{6}a^2x^3 - \tfrac{1}{4}ax^4 + \tfrac{1}{10}x^5 - \tfrac{1}{20}(a-x)^5\right]_0^a = \tfrac{1}{15}ka^5$$

Hence $(\overline{x},\overline{y}) = \left(\tfrac{2}{5}a, \tfrac{2}{5}a\right)$.

16. $\rho(x, y) = k/\sqrt{x^2 + y^2} = k/r$.



$$m = \int_{\pi/6}^{5\pi/6} \int_{1}^{2\sin\theta} \frac{k}{r}\, r\, dr\, d\theta = k \int_{\pi/6}^{5\pi/6} [(2\sin\theta) - 1]\, d\theta$$
$$= k\big[-2\cos\theta - \theta\big]_{\pi/6}^{5\pi/6} = 2k\big(\sqrt{3} - \tfrac{\pi}{3}\big)$$

By symmetry of D and $f(x) = x$, $M_y = 0$, and

$$M_x = \int_{\pi/6}^{5\pi/6} \int_{1}^{2\sin\theta} kr\sin\theta\, dr\, d\theta = \tfrac{1}{2}k \int_{\pi/6}^{5\pi/6} (4\sin^3\theta - \sin\theta)\, d\theta$$
$$= \tfrac{1}{2}k\big[-3\cos\theta + \tfrac{4}{3}\cos^3\theta\big]_{\pi/6}^{5\pi/6} = \sqrt{3}\, k$$

Hence $(\overline{x}, \overline{y}) = \left(0, \frac{3\sqrt{3}}{2(3\sqrt{3} - \pi)}\right)$.

17. $I_x = \iint_D y^2 \rho(x, y) dA = \int_0^1 \int_0^{e^x} y^2 \cdot y\; dy\, dx = \int_0^1 \left[\frac{1}{4}y^4\right]_{y=0}^{y=e^x} dx = \frac{1}{4}\int_0^1 e^{4x}\, dx = \frac{1}{4}\left[\frac{1}{4}e^{4x}\right]_0^1 = \frac{1}{16}(e^4 - 1)$,

$I_y = \iint_D x^2 \rho(x, y)\, dA = \int_0^1 \int_0^{e^x} x^2 y\, dy\, dx = \int_0^1 x^2 \left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x} dx = \frac{1}{2}\int_0^1 x^2 e^{2x}\, dx$

$= \frac{1}{2}\left[\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right)e^{2x}\right]_0^1$ [integrate by parts twice] $= \frac{1}{8}(e^2 - 1)$,

and $I_0 = I_x + I_y = \frac{1}{16}(e^4 - 1) + \frac{1}{8}(e^2 - 1) = \frac{1}{16}(e^4 + 2e^2 - 3)$.

18. $I_x = \int_0^{\pi/2} \int_0^1 (r^2 \sin^2\theta)(kr^2)\, r\, dr\, d\theta = \frac{1}{6}k \int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{1}{6}k\left[\frac{1}{4}(2\theta - \sin 2\theta)\right]_0^{\pi/2} = \frac{\pi}{24}k$,

$I_y = \int_0^{\pi/2} \int_0^1 (r^2 \cos^2\theta)(kr^2)\, r\, dr\, d\theta = \frac{1}{6}k \int_0^{\pi/6} \cos^2\theta\, d\theta = \frac{1}{6}k\left[\frac{1}{4}(2\theta + \sin 2\theta)\right]_0^{\pi/2} = \frac{\pi}{24}k$,

and $I_0 = I_x + I_y = \frac{\pi}{12}k$.

19. As in Exercise 15, we place the vertex opposite the hypotenuse at $(0, 0)$ and the equal sides along the positive axes.

$I_x = \int_0^a \int_0^{a-x} y^2 k(x^2 + y^2)\, dy\, dx = k\int_0^a \int_0^{a-x} (x^2y^2 + y^4)\, dy\, dx = k\int_0^a \left[\frac{1}{3}x^2y^3 + \frac{1}{5}y^5\right]_{y=0}^{y=a-x} dx$

$= k\int_0^a \left[\frac{1}{3}x^2(a-x)^3 + \frac{1}{5}(a-x)^5\right] dx = k\left[\frac{1}{3}\left(\frac{1}{3}a^3x^3 - \frac{3}{4}a^2x^4 + \frac{3}{5}ax^5 - \frac{1}{6}x^6\right) - \frac{1}{30}(a-x)^6\right]_0^a = \frac{7}{180}ka^6$,

$I_y = \int_0^a \int_0^{a-x} x^2 k(x^2 + y^2)\, dy\, dx = k\int_0^a \int_0^{a-x} (x^4 + x^2y^2)\, dy\, dx = k\int_0^a \left[x^4y + \frac{1}{3}x^2y^3\right]_{y=0}^{y=a-x} dx$

$= k\int_0^a \left[x^4(a-x) + \frac{1}{3}x^2(a-x)^3\right] dx = k\left[\frac{1}{5}ax^5 - \frac{1}{6}x^6 + \frac{1}{3}\left(\frac{1}{3}a^3x^3 - \frac{3}{4}a^2x^4 + \frac{3}{5}ax^5 - \frac{1}{6}x^6\right)\right]_0^a = \frac{7}{180}ka^6$,

and $I_0 = I_x + I_y = \frac{7}{90}ka^6$.

20. If we find the moments of inertia about the x- and y-axes, we can determine in which direction rotation will be more difficult. (See the explanation following Example 4.) The moment of inertia about the x-axis is given by

$$I_x = \iint_D y^2 \rho(x,y)\,dA = \int_0^2 \int_0^2 y^2(1+0.1x)\,dy\,dx = \int_0^2 (1+0.1x)\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=2} dx$$
$$= \tfrac{8}{3}\int_0^2 (1+0.1x)\,dx = \tfrac{8}{3}\left[x + 0.1\cdot\tfrac{1}{2}x^2\right]_0^2 = \tfrac{8}{3}(2.2) \approx 5.87$$

Similarly, the moment of inertia about the y-axis is given by

$$I_y = \iint_D x^2 \rho(x,y)\,dA = \int_0^2 \int_0^2 x^2(1+0.1x)\,dy\,dx = \int_0^2 x^2(1+0.1x)\left[y\right]_{y=0}^{y=2} dx$$
$$= 2\int_0^2 (x^2+0.1x^3)\,dx = 2\left[\tfrac{1}{3}x^3 + 0.1\cdot\tfrac{1}{4}x^4\right]_0^2 = 2\left(\tfrac{8}{3}+0.4\right) \approx 6.13$$

Since $I_y > I_x$, more force is required to rotate the fan blade about the y-axis.

21. Using a CAS, we find $m = \iint_D \rho(x,y)\,dA = \int_0^\pi \int_0^{\sin x} xy\,dy\,dx = \dfrac{\pi^2}{8}$. Then

$\overline{x} = \dfrac{1}{m}\displaystyle\iint_D x\rho(x,y)\,dA = \dfrac{8}{\pi^2}\int_0^\pi \int_0^{\sin x} x^2 y\,dy\,dx = \dfrac{2\pi}{3} - \dfrac{1}{\pi}$ and

$\overline{y} = \dfrac{1}{m}\displaystyle\iint_D y\rho(x,y)\,dA = \dfrac{8}{\pi^2}\int_0^\pi \int_0^{\sin x} xy^2\,dy\,dx = \dfrac{16}{9\pi}$, so $(\overline{x},\overline{y}) = \left(\dfrac{2\pi}{3} - \dfrac{1}{\pi}, \dfrac{16}{9\pi}\right)$.

The moments of inertia are $I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^\pi \int_0^{\sin x} xy^3\,dy\,dx = \dfrac{3\pi^2}{64}$,

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^\pi \int_0^{\sin x} x^3 y\,dy\,dx = \dfrac{\pi^2}{16}(\pi^2-3)$, and $I_0 = I_x + I_y = \dfrac{\pi^2}{64}(4\pi^2-9)$.

22. Using a CAS, we find $m = \iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r^2\,dr\,d\theta = \frac{5}{3}\pi$,

$\overline{x} = \frac{1}{m}\iint_D x\sqrt{x^2+y^2}\,dA = \frac{3}{5\pi}\int_0^{2\pi}\int_0^{1+\cos\theta} r^3\cos\theta\,dr\,d\theta = \frac{21}{20}$ and

$\overline{y} = \frac{1}{m}\iint_D y\sqrt{x^2+y^2}\,dA = \frac{3}{5\pi}\int_0^{2\pi}\int_0^{1+\cos\theta} r^3\sin\theta\,dr\,d\theta = 0$, so $(\overline{x},\overline{y}) = \left(\frac{21}{20}, 0\right)$.

The moments of inertia are $I_x = \iint_D y^2\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r^4\sin^2\theta\,dr\,d\theta = \frac{33}{40}\pi$,

$I_y = \iint_D x^2\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r^4\cos^2\theta\,dr\,d\theta = \frac{93}{40}\pi$, and $I_0 = I_x + I_y = \frac{63}{20}\pi$.

23. $I_x = \iint_D y^2\rho(x,y)dA = \int_0^h\int_0^b \rho y^2\,dx\,dy = \rho\int_0^b dx\int_0^h y^2\,dy = \rho\left[x\right]_0^b\left[\frac{1}{3}y^3\right]_0^h = \rho b\left(\frac{1}{3}h^3\right) = \frac{1}{3}\rho bh^3$,

$I_y = \iint_D x^2\rho(x,y)dA = \int_0^h\int_0^b \rho x^2\,dx\,dy = \rho\int_0^b x^2\,dx\int_0^h dy = \rho\left[\frac{1}{3}x^3\right]_0^b\left[y\right]_0^h = \frac{1}{3}\rho b^3 h$,

and $m = \rho\,(\text{area of rectangle}) = \rho bh$ since the lamina is homogeneous. Hence $\overline{\overline{x}}^2 = \dfrac{I_y}{m} = \dfrac{\frac{1}{3}\rho b^3 h}{\rho bh} = \dfrac{b^2}{3} \quad\Rightarrow\quad \overline{\overline{x}} = \dfrac{b}{\sqrt{3}}$

and $\overline{\overline{y}}^2 = \dfrac{I_x}{m} = \dfrac{\frac{1}{3}\rho bh^3}{\rho bh} = \dfrac{h^2}{3} \quad\Rightarrow\quad \overline{\overline{y}} = \dfrac{h}{\sqrt{3}}$.

24. Here we assume $b > 0$, $h > 0$ but note that we arrive at the same results if $b < 0$ or $h < 0$. We have

$D = \left\{(x, y) \mid 0 \leq x \leq b, 0 \leq y \leq h - \frac{h}{b}x\right\}$, so

$$I_x = \int_0^b \int_0^{h-hx/b} y^2 \rho \, dy \, dx = \rho \int_0^b \left[\tfrac{1}{3}y^3\right]_{y=0}^{y=h-hx/b} dx = \tfrac{1}{3}\rho \int_0^b \left(h - \tfrac{h}{b}x\right)^3 dx$$

$$= \tfrac{1}{3}\rho \left[-\tfrac{b}{h}\left(\tfrac{1}{4}\right)\left(h - \tfrac{h}{b}x\right)^4\right]_0^b = -\tfrac{b}{12h}\rho(0 - h^4) = \tfrac{1}{12}\rho b h^3,$$

$$I_y = \int_0^b \int_0^{h-hx/b} x^2 \rho \, dy \, dx = \rho \int_0^b x^2 \left(h - \tfrac{h}{b}x\right) dx = \rho \int_0^b \left(hx^2 - \tfrac{h}{b}x^3\right) dx$$

$$= \rho \left[\tfrac{h}{3}x^3 - \tfrac{h}{4b}x^4\right]_0^b = \rho\left(\tfrac{hb^3}{3} - \tfrac{hb^3}{4}\right) = \tfrac{1}{12}\rho b^3 h,$$

and $m = \int_0^b \int_0^{h-hx/b} \rho \, dy \, dx = \rho \int_0^b \left(h - \frac{h}{b}x\right) dx = \rho \left[hx - \frac{h}{2b}x^2\right]_0^b = \frac{1}{2}\rho b h$. Hence $\bar{\bar{x}}^2 = \dfrac{I_y}{m} = \dfrac{\frac{1}{12}\rho b^3 h}{\frac{1}{2}\rho b h} = \dfrac{b^2}{6} \quad \Rightarrow$

$\bar{\bar{x}} = \dfrac{b}{\sqrt{6}}$ and $\bar{\bar{y}}^2 = \dfrac{I_x}{m} = \dfrac{\frac{1}{12}\rho b h^3}{\frac{1}{2}\rho b h} = \dfrac{h^2}{6} \quad \Rightarrow \quad \bar{\bar{y}} = \dfrac{h}{\sqrt{6}}$.

25. In polar coordinates, the region is $D = \left\{(r, \theta) \mid 0 \leq r \leq a, 0 \leq \theta \leq \frac{\pi}{2}\right\}$, so

$$I_x = \iint_D y^2 \rho \, dA = \int_0^{\pi/2} \int_0^a \rho(r \sin\theta)^2 \, r \, dr \, d\theta = \rho \int_0^{\pi/2} \sin^2 d\theta \int_0^a r^3 \, dr$$

$$= \rho\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left[\tfrac{1}{4}r^4\right]_0^a = \rho\left(\tfrac{\pi}{4}\right)\left(\tfrac{1}{4}a^4\right) = \tfrac{1}{16}\rho a^4 \pi,$$

$$I_y = \iint_D x^2 \rho \, dA = \int_0^{\pi/2} \int_0^a \rho(r \cos\theta)^2 \, r \, dr \, d\theta = \rho \int_0^{\pi/2} \cos^2 d\theta \int_0^a r^3 \, dr$$

$$= \rho\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left[\tfrac{1}{4}r^4\right]_0^a = \rho\left(\tfrac{\pi}{4}\right)\left(\tfrac{1}{4}a^4\right) = \tfrac{1}{16}\rho a^4 \pi,$$

and $m = \rho \cdot A(D) = \rho \cdot \frac{1}{4}\pi a^2$ since the lamina is homogeneous. Hence $\bar{\bar{x}}^2 = \bar{\bar{y}}^2 = \dfrac{\frac{1}{16}\rho a^4 \pi}{\frac{1}{4}\rho a^2 \pi} = \dfrac{a^2}{4} \quad \Rightarrow \quad \bar{\bar{x}} = \bar{\bar{y}} = \dfrac{a}{2}$.

26. $m = \int_0^\pi \int_0^{\sin x} \rho \, dy \, dx = \rho \int_0^\pi \sin x \, dx = \rho\left[-\cos x\right]_0^\pi = 2\rho,$

$$I_x = \int_0^\pi \int_0^{\sin x} \rho y^2 \, dy \, dx = \tfrac{1}{3}\rho \int_0^\pi \sin^3 x \, dx = \tfrac{1}{3}\rho \int_0^\pi (1 - \cos^2 x) \sin x \, dx = \tfrac{1}{3}\rho\left[-\cos\theta + \tfrac{1}{3}\cos^3\theta\right]_0^\pi = \tfrac{4}{9}\rho,$$

$$I_y = \int_0^\pi \int_0^{\sin x} \rho x^2 \, dy \, dx = \rho \int_0^\pi x^2 \sin x \, dx = \rho\left[-x^2 \cos x + 2x \sin x + 2\cos x\right]_0^\pi \quad \text{[by integrating by parts twice]}$$

$$= \rho(\pi^2 - 4).$$

Then $\bar{\bar{y}}^2 = \dfrac{I_x}{m} = \dfrac{2}{9}$, so $\bar{\bar{y}} = \dfrac{\sqrt{2}}{3}$ and $\bar{\bar{x}}^2 = \dfrac{I_y}{m} = \dfrac{\pi^2 - 4}{2}$, so $\bar{\bar{x}} = \sqrt{\dfrac{\pi^2 - 4}{2}}$.

27. (a) $f(x,y)$ is a joint density function, so we know $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,1] \times [0,2]$, we can say

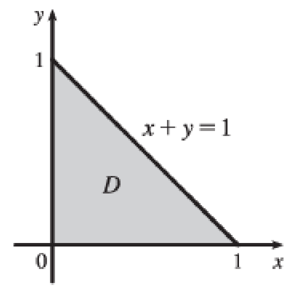
$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^1\int_0^2 Cx(1+y)\,dy\,dx$$
$$= C\int_0^1 x\left[y + \tfrac{1}{2}y^2\right]_{y=0}^{y=2} dx = C\int_0^1 4x\,dx = C\left[2x^2\right]_0^1 = 2C$$

Then $2C = 1 \;\Rightarrow\; C = \frac{1}{2}$.

(b) $P(X \le 1, Y \le 1) = \int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y)\,dy\,dx = \int_0^1\int_0^1 \frac{1}{2}x(1+y)\,dy\,dx$

$$= \int_0^1 \tfrac{1}{2}x\left[y + \tfrac{1}{2}y^2\right]_{y=0}^{y=1} dx = \int_0^1 \tfrac{1}{2}x\left(\tfrac{3}{2}\right) dx = \tfrac{3}{4}\left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{3}{8} \text{ or } 0.375$$

(c) $P(X + Y \le 1) = P((X,Y) \in D)$ where D is the triangular region shown in the figure. Thus



$$P(X+Y \le 1) = \iint_D f(x,y)\,dA = \int_0^1\int_0^{1-x} \tfrac{1}{2}x(1+y)\,dy\,dx$$
$$= \int_0^1 \tfrac{1}{2}x\left[y + \tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \int_0^1 \tfrac{1}{2}x\left(\tfrac{1}{2}x^2 - 2x + \tfrac{3}{2}\right) dx$$
$$= \tfrac{1}{4}\int_0^1 \left(x^3 - 4x^2 + 3x\right) dx = \tfrac{1}{4}\left[\tfrac{x^4}{4} - 4\tfrac{x^3}{3} + 3\tfrac{x^2}{2}\right]_0^1$$
$$= \tfrac{5}{48} \approx 0.1042$$

28. (a) $f(x,y) \ge 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Here, $f(x,y) = 0$ outside the square $[0,1] \times [0,1]$, so $\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_0^1\int_0^1 4xy\,dy\,dx = \int_0^1 \left[2xy^2\right]_{y=0}^{y=1} dx = \int_0^1 2x\,dx = x^2\Big]_0^1 = 1$.

Thus, $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on Y, so

$$P\left(X \ge \tfrac{1}{2}\right) = \int_{1/2}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_{1/2}^{1}\int_0^1 4xy\,dy\,dx = \int_{1/2}^{1}\left[2xy^2\right]_{y=0}^{y=1} dx = \int_{1/2}^{1} 2x\,dx = x^2\Big]_{1/2}^{1} = \tfrac{3}{4}.$$

(ii) $P\left(X \ge \frac{1}{2}, Y \le \frac{1}{2}\right) = \int_{1/2}^{\infty}\int_{-\infty}^{1/2} f(x,y)\,dy\,dx = \int_{1/2}^{1}\int_0^{1/2} 4xy\,dy\,dx$

$$= \int_{1/2}^{1}\left[2xy^2\right]_{y=0}^{y=1/2} dx = \int_{1/2}^{1} \tfrac{1}{2}x\,dx = \tfrac{1}{2}\cdot\tfrac{1}{2}x^2\Big]_{1/2}^{1} = \tfrac{3}{16}$$

(c) The expected value of X is given by

$$\mu_1 = \iint_{\mathbb{R}^2} x\,f(x,y)\,dA = \int_0^1\int_0^1 x(4xy)\,dy\,dx = \int_0^1 2x^2\left[y^2\right]_{y=0}^{y=1} dx = 2\int_0^1 x^2\,dx = 2\left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{2}{3}$$

The expected value of Y is

$$\mu_2 = \iint_{\mathbb{R}^2} y\,f(x,y)\,dA = \int_0^1\int_0^1 y(4xy)\,dy\,dx = \int_0^1 4x\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=1} dx = \tfrac{4}{3}\int_0^1 x\,dx = \tfrac{4}{3}\left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{2}{3}$$

29. (a) $f(x,y) \geq 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Here, $f(x,y) = 0$ outside the first quadrant, so

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_0^\infty \int_0^\infty 0.1e^{-(0.5x+0.2y)}\,dy\,dx = 0.1\int_0^\infty \int_0^\infty e^{-0.5x}e^{-0.2y}\,dy\,dx = 0.1\int_0^\infty e^{-0.5x}\,dx \int_0^\infty e^{-0.2y}\,dy$$

$$= 0.1 \lim_{t\to\infty} \int_0^t e^{-0.5x}\,dx \lim_{t\to\infty} \int_0^t e^{-0.2y}\,dy = 0.1 \lim_{t\to\infty} \left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty} \left[-5e^{-0.2y}\right]_0^t$$

$$= 0.1 \lim_{t\to\infty} \left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty} \left[-5(e^{-0.2t}-1)\right] = (0.1)\cdot(-2)(0-1)\cdot(-5)(0-1) = 1$$

Thus $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on X, so

$$P(Y \geq 1) = \int_{-\infty}^\infty \int_1^\infty f(x,y)\,dy\,dx = \int_0^\infty \int_1^\infty 0.1e^{-(0.5x+0.2y)}\,dy\,dx$$

$$= 0.1\int_0^\infty e^{-0.5x}\,dx \int_1^\infty e^{-0.2y}\,dy = 0.1 \lim_{t\to\infty} \int_0^t e^{-0.5x}\,dx \lim_{t\to\infty} \int_1^t e^{-0.2y}\,dy$$

$$= 0.1 \lim_{t\to\infty} \left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty} \left[-5e^{-0.2y}\right]_1^t = 0.1 \lim_{t\to\infty} \left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty} \left[-5(e^{-0.2t}-e^{-0.2})\right]$$

$$(0.1)\cdot(-2)(0-1)\cdot(-5)(0-e^{-0.2}) = e^{-0.2} \approx 0.8187$$

(ii) $P(X \leq 2, Y \leq 4) = \int_{-\infty}^2 \int_{-\infty}^4 f(x,y)\,dy\,dx = \int_0^2 \int_0^4 0.1e^{-(0.5x+0.2y)}\,dy\,dx$

$$= 0.1\int_0^2 e^{-0.5x}\,dx \int_0^4 e^{-0.2y}\,dy = 0.1\left[-2e^{-0.5x}\right]_0^2 \left[-5e^{-0.2y}\right]_0^4$$

$$= (0.1)\cdot(-2)(e^{-1}-1)\cdot(-5)(e^{-0.8}-1)$$

$$= (e^{-1}-1)(e^{-0.8}-1) = 1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481$$

(c) The expected value of X is given by

$$\mu_1 = \iint_{\mathbb{R}^2} x\, f(x,y)\, dA = \int_0^\infty \int_0^\infty x\left[0.1e^{-(0.5x+0.2y)}\right]\, dy\, dx$$
$$= 0.1 \int_0^\infty xe^{-0.5x}\, dx \int_0^\infty e^{-0.2y}\, dy = 0.1 \lim_{t\to\infty} \int_0^t xe^{-0.5x}\, dx \lim_{t\to\infty} \int_0^t e^{-0.2y}\, dy$$

To evaluate the first integral, we integrate by parts with $u = x$ and $dv = e^{-0.5x}\, dx$ (or we can use Formula 96 in the Table of Integrals): $\int xe^{-0.5x}\, dx = -2xe^{-0.5x} - \int -2e^{-0.5x}\, dx = -2xe^{-0.5x} - 4e^{-0.5x} = -2(x+2)e^{-0.5x}$. Thus

$$\mu_1 = 0.1 \lim_{t\to\infty} \left[-2(x+2)e^{-0.5x}\right]_0^t \lim_{t\to\infty} \left[-5e^{-0.2y}\right]_0^t$$
$$= 0.1 \lim_{t\to\infty} (-2)\left[(t+2)e^{-0.5t} - 2\right] \lim_{t\to\infty} (-5)\left[e^{-0.2t} - 1\right]$$
$$= 0.1(-2)\left(\lim_{t\to\infty} \frac{t+2}{e^{0.5t}} - 2\right)(-5)(-1) = 2 \qquad \text{[by l'Hospital's Rule]}$$

The expected value of Y is given by

$$\mu_2 = \iint_{\mathbb{R}^2} y\, f(x,y)\, dA = \int_0^\infty \int_0^\infty y\left[0.1e^{-(0.5+0.2y)}\right]\, dy\, dx$$
$$= 0.1 \int_0^\infty e^{-0.5x}\, dx \int_0^\infty ye^{-0.2y}\, dy = 0.1 \lim_{t\to\infty} \int_0^t e^{-0.5x}\, dx \lim_{t\to\infty} \int_0^t ye^{-0.2y}\, dy$$

To evaluate the second integral, we integrate by parts with $u = y$ and $dv = e^{-0.2y}\, dy$ (or again we can use Formula 96 in the Table of Integrals) which gives $\int ye^{-0.2y}\, dy = -5ye^{-0.2y} + \int 5e^{-0.2y}\, dy = -5(y+5)e^{-0.2y}$. Then

$$\mu_2 = 0.1 \lim_{t\to\infty} \left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty} \left[-5(y+5)e^{-0.2y}\right]_0^t$$
$$= 0.1 \lim_{t\to\infty} \left[-2(e^{-0.5t} - 1)\right] \lim_{t\to\infty} \left(-5\left[(t+5)e^{-0.2t} - 5\right]\right)$$
$$= 0.1(-2)(-1)\cdot(-5)\left(\lim_{t\to\infty} \frac{t+5}{e^{0.2t}} - 5\right) = 5 \qquad \text{[by l'Hospital's Rule]}$$

30. (a) The lifetime of each bulb has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000} e^{-t/1000} & \text{if } t \geq 0 \end{cases}$$

If X and Y are the lifetimes of the individual bulbs, then X and Y are independent, so the joint density function is the product of the individual density functions:

$$f(x,y) = \begin{cases} 10^{-6} e^{-(x+y)/1000} & \text{if } x \geq 0,\, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

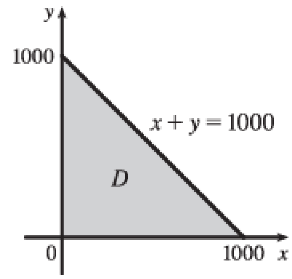
$$P(X \leq 1000, Y \leq 1000) = \int_{-\infty}^{1000} \int_{-\infty}^{1000} f(x,y)\, dy\, dx = \int_0^{1000} \int_0^{1000} 10^{-6} e^{-(x+y)/1000}\, dy\, dx$$

$$= 10^{-6} \int_0^{1000} e^{-x/1000}\, dx \int_0^{1000} e^{-y/1000}\, dy$$

$$= 10^{-6} \left[-1000 e^{-x/1000}\right]_0^{1000} \left[-1000 e^{-y/1000}\right]_0^{1000}$$

$$= \left(e^{-1} - 1\right)^2 \approx 0.3996$$

(b) Now we are asked for the probability that the combined lifetimes of both bulbs is 1000 hours or less. Thus we want to find $P(X + Y \leq 1000)$, or equivalently $P((X,Y) \in D)$ where D is the triangular region shown in the figure. Then



$$P(X + Y \leq 1000) = \iint_D f(x,y)\, dA$$

$$= \int_0^{1000} \int_0^{1000-x} 10^{-6} e^{-(x+y)/1000}\, dy\, dx$$

$$= 10^{-6} \int_0^{1000} \left[-1000 e^{-(x+y)/1000}\right]_{y=0}^{y=1000-x} dx = -10^{-3} \int_0^{1000} \left(e^{-1} - e^{-x/1000}\right) dx$$

$$= -10^{-3} \left[e^{-1} x + 1000 e^{-x/1000}\right]_0^{1000} = 1 - 2e^{-1} \approx 0.2642$$

31. (a) The random variables X and Y are normally distributed with $\mu_1 = 45$, $\mu_2 = 20$, $\sigma_1 = 0.5$, and $\sigma_2 = 0.1$.

The individual density functions for X and Y, then, are $f_1(x) = \dfrac{1}{0.5\sqrt{2\pi}}\, e^{-(x-45)^2/0.5}$ and

$f_2(y) = \dfrac{1}{0.1\sqrt{2\pi}}\, e^{-(y-20)^2/0.02}$. Since X and Y are independent, the joint density function is the product

$f(x,y) = f_1(x) f_2(y) = \dfrac{1}{0.5\sqrt{2\pi}}\, e^{-(x-45)^2/0.5} \dfrac{1}{0.1\sqrt{2\pi}}\, e^{-(y-20)^2/0.02} = \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2}$.

Then $P(40 \le X \le 50, 20 \le Y \le 25) = \int_{40}^{50} \int_{20}^{25} f(x,y)\, dy\, dx = \frac{10}{\pi} \int_{40}^{50} \int_{20}^{25} e^{-2(x-45)^2 - 50(y-20)^2}\, dy\, dx$.

Using a CAS or calculator to evaluate the integral, we get $P(40 \le X \le 50, 20 \le Y \le 25) \approx 0.500$.

(b) $P(4(X-45)^2 + 100(Y-20)^2 \le 2) = \iint_D \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2}\, dA$, where D is the region enclosed by the ellipse $4(x-45)^2 + 100(y-20)^2 = 2$. Solving for y gives $y = 20 \pm \frac{1}{10}\sqrt{2 - 4(x-45)^2}$, the upper and lower halves of the ellipse, and these two halves meet where $y = 20$ [since the ellipse is centered at $(45, 20)$] $\Rightarrow$ $4(x-45)^2 = 2$ $\Rightarrow$ $x = 45 \pm \frac{1}{\sqrt{2}}$. Thus

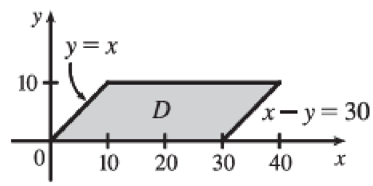
$$\iint_D \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2}\, dA = \frac{10}{\pi} \int_{45-1/\sqrt{2}}^{45+1/\sqrt{2}} \int_{20-\frac{1}{10}\sqrt{2-4(x-45)^2}}^{20+\frac{1}{10}\sqrt{2-4(x-45)^2}} e^{-2(x-45)^2 - 50(y-20)^2}\, dy\, dx.$$

Using a CAS or calculator to evaluate the integral, we get $P(4(X-45)^2 + 100(Y-20)^2 \le 2) \approx 0.632$.

32. Because X and Y are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x,y) = f_1(x) f_2(y) = \begin{cases} \frac{1}{50} e^{-x} y & \text{if } x \ge 0, 0 \le y \le 10 \\ 0 & \text{otherwise} \end{cases}$$

Since Xavier won't wait for Yolanda, they won't meet unless $X \ge Y$. Additionally, Yolanda will wait up to half an hour but no longer, so they won't meet unless $X - Y \le 30$. Thus the probability that they meet is $P((X,Y) \in D)$ where D is the parallelogram shown in the figure. The integral is simpler to evaluate if we consider D as a type II region, so



$$P((X,Y) \in D) = \iint_D f(x,y)\, dx\, dy = \int_0^{10} \int_y^{y+30} \frac{1}{50} e^{-x} y\, dx\, dy = \frac{1}{50} \int_0^{10} y \left[-e^{-x}\right]_{x=y}^{x=y+30} dy$$
$$= \frac{1}{50} \int_0^{10} y(-e^{-(y+30)} + e^{-y})\, dy = \frac{1}{50}(1 - e^{-30}) \int_0^{10} y e^{-y}\, dy$$

By integration by parts (or Formula 96 in the Table of Integrals), this is

$\frac{1}{50}(1 - e^{-30})\left[-(y+1)e^{-y}\right]_0^{10} = \frac{1}{50}(1 - e^{-30})(1 - 11e^{-10}) \approx 0.020$. Thus there is only about a 2% chance they will meet. Such is student life!

33. (a) If $f(P,A)$ is the probability that an individual at A will be infected by an individual at P, and $k\,dA$ is the number of infected individuals in an element of area dA, then $f(P,A)k\,dA$ is the number of infections that should result from exposure of the individual at A to infected people in the element of area dA. Integration over D gives the number of infections of the person at A due to all the infected people in D. In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

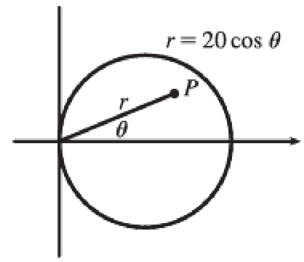
$$E = \iint_D kf(P,A)\,dA = k\iint_D \frac{20 - d(P,A)}{20}\,dA = k\iint_D \left[1 - \frac{\sqrt{(x-x_0)^2 + (y-y_0)^2}}{20}\right] dx\,dy$$

(b) If $A = (0,0)$, then

$$E = k\iint_D \left[1 - \frac{1}{20}\sqrt{x^2+y^2}\right] dx\,dy$$

$$= k\int_0^{2\pi}\int_0^{10}\left(1 - \frac{r}{20}\right) r\,dr\,d\theta = 2\pi k\left[\frac{r^2}{2} - \frac{r^3}{60}\right]_0^{10}$$

$$= 2\pi k\left(50 - \tfrac{50}{3}\right) = \tfrac{200}{3}\pi k \approx 209k$$



For A at the edge of the city, it is convenient to use a polar coordinate system centered at A. Then the polar equation for the circular boundary of the city becomes $r = 20\cos\theta$ instead of $r = 10$, and the distance from A to a point P in the city is again r (see the figure). So

$$E = k\int_{-\pi/2}^{\pi/2}\int_0^{20\cos\theta}\left(1 - \frac{r}{20}\right) r\,dr\,d\theta = k\int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2} - \frac{r^3}{60}\right]_{r=0}^{r=20\cos\theta} d\theta$$

$$= k\int_{-\pi/2}^{\pi/2}\left(200\cos^2\theta - \tfrac{400}{3}\cos^3\theta\right) d\theta = 200k\int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta - \tfrac{2}{3}\left(1 - \sin^2\theta\right)\cos\theta\right] d\theta$$

$$= 200k\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta - \tfrac{2}{3}\sin\theta + \tfrac{2}{3}\cdot\tfrac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2} = 200k\left[\tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9} + \tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9}\right]$$

$$= 200k\left(\tfrac{\pi}{2} - \tfrac{8}{9}\right) \approx 136k$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.