1. $\iiint_B xyz^2\, dV = \int_0^1\int_0^3\int_{-1}^2 xyz^2\, dy\, dz\, dx = \int_0^1\int_0^3 \left[\frac{1}{2}xy^2z^2\right]_{y=-1}^{y=2} dz\, dx = \int_0^1\int_0^3 \frac{3}{2}xz^2\, dz\, dx$

$= \int_0^1 \left[\frac{1}{2}xz^3\right]_{z=0}^{z=3} dx = \int_0^1 \frac{27}{2}\, x\, dx = \left.\frac{27}{4}x^2\right]_0^1 = \frac{27}{4}$

2. There are six different possible orders of integration.

$\iiint_E (xz - y^3)\, dV = \int_{-1}^1\int_0^2\int_0^1 (xz-y^3)\, dz\, dy\, dx = \int_{-1}^1\int_0^2 \left[\frac{1}{2}xz^2 - y^3z\right]_{z=0}^{z=1} dy\, dx = \int_{-1}^1\int_0^2 \left(\frac{1}{2}x - y^3\right) dy\, dx$

$= \int_{-1}^1 \left[\frac{1}{2}xy - \frac{1}{4}y^4\right]_{y=0}^{y=2} dx = \int_{-1}^1 (x-4)\, dx = \left[\frac{1}{2}x^2 - 4x\right]_{-1}^1 = -8$

$\iiint_E (xz - y^3)\, dV = \int_0^2\int_{-1}^1\int_0^1 (xz-y^3)\, dz\, dx\, dy = \int_0^2\int_{-1}^1 \left[\frac{1}{2}xz^2 - y^3z\right]_{z=0}^{z=1} dx\, dy$

$= \int_0^2\int_{-1}^1 \left(\frac{1}{2}x - y^3\right) dx\, dy = \int_0^2 \left[\frac{1}{4}x^2 - xy^3\right]_{x=-1}^{x=1} dy = \int_0^2 -2y^3\, dy = \left.-\frac{1}{2}y^4\right]_0^2 = -8$

$\iiint_E (xz - y^3)\, dV = \int_{-1}^1\int_0^1\int_0^2 (xz-y^3)\, dy\, dz\, dx = \int_{-1}^1\int_0^1 \left[xyz - \frac{1}{4}y^4\right]_{y=0}^{y=2} dz\, dx$

$= \int_{-1}^1\int_0^1 (2xz - 4)\, dz\, dx = \int_{-1}^1 \left[xz^2 - 4z\right]_{z=0}^{z=1} dx = \int_{-1}^1 (x-4)\, dx = \left[\frac{1}{2}x^2 - 4x\right]_{-1}^1 = -8$

$\iiint_E (xz - y^3)\, dV = \int_0^1\int_{-1}^1\int_0^2 (xz-y^3)\, dy\, dx\, dz = \int_0^1\int_{-1}^1 \left[xyz - \frac{1}{4}y^4\right]_{y=0}^{y=2} dx\, dz$

$= \int_0^1\int_{-1}^1 (2xz - 4)\, dx\, dz = \int_0^1 \left[x^2z - 4x\right]_{x=-1}^{x=1} dz = \int_0^1 -8\, dz = \left.-8z\right]_0^1 = -8$

$\iiint_E (xz - y^3)\, dV = \int_0^2\int_0^1\int_{-1}^1 (xz-y^3)\, dx\, dz\, dy = \int_0^2\int_0^1 \left[\frac{1}{2}x^2z - xy^3\right]_{x=-1}^{x=1} dz\, dy$

$= \int_0^2\int_0^1 -2y^3\, dz\, dy = \int_0^2 \left[-2y^3z\right]_{z=0}^{z=1} dy = \int_0^2 -2y^3\, dy = \left.-\frac{1}{2}y^4\right]_0^2 = -8$

$\iiint_E (xz - y^3)\, dV = \int_0^1\int_0^2\int_{-1}^1 (xz-y^3)\, dx\, dy\, dz = \int_0^1\int_0^2 \left[\frac{1}{2}x^2z - xy^3\right]_{x=-1}^{x=1} dy\, dz$

$= \int_0^1\int_0^2 -2y^3\, dy\, dz = \int_0^1 \left[-\frac{1}{2}y^4\right]_{y=0}^{y=2} dz = \int_0^1 -8\, dz = \left.-8z\right]_0^1 = -8$

3. $\int_0^1\int_0^z\int_0^{x+z} 6xz\, dy\, dx\, dz = \int_0^1\int_0^z \left[6xyz\right]_{y=0}^{y=x+z} dx\, dz = \int_0^1\int_0^z 6xz(x+z)\, dx\, dz$

$= \int_0^1 \left[2x^3z + 3x^2z^2\right]_{x=0}^{x=z} dz = \int_0^1 (2z^4 + 3z^4)\, dz = \int_0^1 5z^4\, dz = \left.z^5\right]_0^1 = 1$

4. $\int_0^1\int_x^{2x}\int_0^y 2xyz\, dz\, dy\, dx = \int_0^1\int_x^{2x} \left[xyz^2\right]_{z=0}^{z=y} dy\, dx = \int_0^1\int_x^{2x} xy^3\, dy\, dx$

$= \int_0^1 \left[\frac{1}{4}xy^4\right]_{y=x}^{y=2x} dx = \int_0^1 \frac{15}{4}x^5\, dx = \left.\frac{5}{8}x^6\right]_0^1 = \frac{5}{8}$

5. $\int_0^3\int_0^1\int_0^{\sqrt{1-z^2}} ze^y\, dx\, dz\, dy = \int_0^3\int_0^1 \left[xze^y\right]_{x=0}^{x=\sqrt{1-z^2}} dz\, dy = \int_0^3\int_0^1 ze^y\sqrt{1-z^2}\, dz\, dy$

$= \int_0^3 \left[-\frac{1}{3}(1-z^2)^{3/2}e^y\right]_{z=0}^{z=1} dy = \int_0^3 \frac{1}{3}e^y\, dy = \left.\frac{1}{3}e^y\right]_0^3 = \frac{1}{3}(e^3 - 1)$

6. $\int_0^1\int_0^z\int_0^y ze^{-y^2}\, dx\, dy\, dz = \int_0^1\int_0^z \left[xze^{-y^2}\right]_{x=0}^{x=y} dy\, dz = \int_0^1\int_0^z yze^{-y^2}\, dy\, dz = \int_0^1 \left[-\frac{1}{2}ze^{-y^2}\right]_{y=0}^{y=z} dz$

$= \int_0^1 -\frac{1}{2}z\left(e^{-z^2} - 1\right) dz = \frac{1}{2}\int_0^1 \left(z - ze^{-z^2}\right) dz$

$= \frac{1}{2}\left[\frac{1}{2}z^2 + \frac{1}{2}e^{-z^2}\right]_0^1 = \frac{1}{4}(1 + e^{-1} - 0 - 1) = \frac{1}{4e}$

7. $\int_0^{\pi/2}\int_0^y\int_0^x \cos(x+y+z)\,dz\,dx\,dy = \int_0^{\pi/2}\int_0^y \left[\sin(x+y+z)\right]_{z=0}^{z=x}\,dx\,dy$

$= \int_0^{\pi/2}\int_0^y \left[\sin(2x+y)-\sin(x+y)\right]dx\,dy$

$= \int_0^{\pi/2}\left[-\frac{1}{2}\cos(2x+y)+\cos(x+y)\right]_{x=0}^{x=y} dy$

$= \int_0^{\pi/2}\left[-\frac{1}{2}\cos 3y+\cos 2y+\frac{1}{2}\cos y-\cos y\right] dy$

$= \left[-\frac{1}{6}\sin 3y+\frac{1}{2}\sin 2y-\frac{1}{2}\sin y\right]_0^{\pi/2} = \frac{1}{6}-\frac{1}{2} = -\frac{1}{3}$

8. $\int_0^{\sqrt{\pi}}\int_0^x\int_0^{xz} x^2\sin y\,dy\,dz\,dx = \int_0^{\sqrt{\pi}}\int_0^x \left[-x^2\cos y\right]_{y=0}^{y=xz} dz\,dx = \int_0^{\sqrt{\pi}}\int_0^x (x^2-x^2\cos xz)\,dz\,dx$

$= \int_0^{\sqrt{\pi}}\left[x^2z - x\sin xz\right]_{z=0}^{z=x} dx = \int_0^{\sqrt{\pi}}(x^3-x\sin x^2)\,dx$

$= \left[\frac{1}{4}x^4+\frac{1}{2}\cos x^2\right]_0^{\sqrt{\pi}} = \frac{1}{4}\pi^2-\frac{1}{2}-\frac{1}{2} = \frac{1}{4}\pi^2-1$

9. $\iiint_E 2x\,dV = \int_0^2\int_0^{\sqrt{4-y^2}}\int_0^y 2x\,dz\,dx\,dy = \int_0^2\int_0^{\sqrt{4-y^2}}\left[2xz\right]_{z=0}^{z=y} dx\,dy = \int_0^2\int_0^{\sqrt{4-y^2}} 2xy\,dx\,dy$

$= \int_0^2 \left[x^2y\right]_{x=0}^{x=\sqrt{4-y^2}} dy = \int_0^2 (4-y^2)y\,dy = \left[2y^2-\frac{1}{4}y^4\right]_0^2 = 4$

10. $\iiint_E yz\cos(x^5)\,dV = \int_0^1\int_0^x\int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx = \int_0^1\int_0^x \left[\frac{1}{2}yz^2\cos(x^5)\right]_{z=x}^{z=2x} dy\,dx$

$= \frac{1}{2}\int_0^1\int_0^x 3x^2y\cos(x^5)\,dy\,dx = \frac{1}{2}\int_0^1\left[\frac{3}{2}x^2y^2\cos(x^5)\right]_{y=0}^{y=x} dx$

$= \frac{3}{4}\int_0^1 x^4\cos(x^5)\,dx = \frac{3}{4}\left[\frac{1}{5}\sin(x^5)\right]_0^1 = \frac{3}{20}(\sin 1-\sin 0) = \frac{3}{20}\sin 1$

11. Here $E = \{(x,y,z) \mid 0\le x\le 1, 0\le y\le \sqrt{x}, 0\le z\le 1+x+y\}$, so

$\iiint_E 6xy\,dV = \int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y} 6xy\,dz\,dy\,dx = \int_0^1\int_0^{\sqrt{x}}\left[6xyz\right]_{z=0}^{z=1+x+y} dy\,dx = \int_0^1\int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx$

$= \int_0^1 \left[3xy^2+3x^2y^2+2xy^3\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 (3x^2+3x^3+2x^{5/2})\,dx = \left[x^3+\frac{3}{4}x^4+\frac{4}{7}x^{7/2}\right]_0^1 = \frac{65}{28}$

12. Here E is the region in the first octant that lies below the plane $2x+2y+z=4$ (and above the region in the xy-plane bounded by the lines $x=0$, $y=0$, $x+y=2$). So

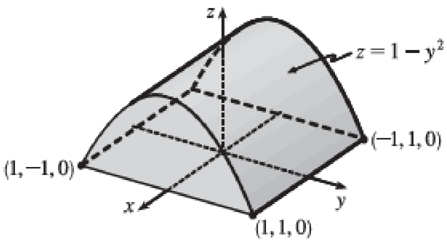
$\iiint_E y\,dV = \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} y\,dz\,dy\,dx = \int_0^2\int_0^{2-x} y(4-2x-2y)\,dy\,dx = \int_0^2\int_0^{2-x}(4y-2xy-2y^2)\,dy\,dx$

$= \int_0^2 \left[2y^2-xy^2-\frac{2}{3}y^3\right]_{y=0}^{y=2-x} dx = \int_0^2 \left[2(2-x)^2-x(2-x)^2-\frac{2}{3}(2-x)^3\right] dx$

$= \int_0^2 \left[(2-x)(2-x)^2-\frac{2}{3}(2-x)^3\right] dx = \frac{1}{3}\int_0^2 (2-x)^3\,dx$

$= \frac{1}{3}\left[-\frac{1}{4}(2-x)^4\right]_0^2 = -\frac{1}{12}(0-16) = \frac{4}{3}$

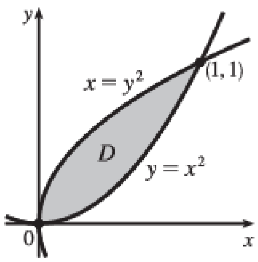
13.



E is the region below the parabolic cylinder $z = 1 - y^2$ and above the square $[-1,1] \times [-1,1]$ in the xy-plane.

$$\begin{aligned}\iiint_E x^2 e^y \, dV &= \int_{-1}^{1} \int_{-1}^{1} \int_0^{1-y^2} x^2 e^y \, dz \, dy \, dx \\ &= \int_{-1}^{1} \int_{-1}^{1} x^2 e^y (1-y^2) \, dy \, dx \\ &= \int_{-1}^{1} x^2 \, dx \ \int_{-1}^{1} (e^y - y^2 e^y) \, dy \\ &= \left[\tfrac{1}{3}x^3\right]_{-1}^{1} \left[e^y - (y^2 - 2y + 2)e^y\right]_{-1}^{1} \qquad \left[\begin{array}{l}\text{integrate by} \\ \text{parts twice}\end{array}\right] \\ &= \tfrac{1}{3}(2)[e - e - e^{-1} + 5e^{-1}] = \tfrac{8}{3e}\end{aligned}$$

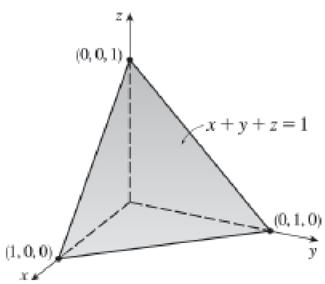
14.



E is the solid above the region shown in the xy-plane and below the plane $z = x + y$. Thus,

$$\begin{aligned}\iiint_E xy \, dV &= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy \, dz \, dy \, dx = \int_0^1 \int_{x^2}^{\sqrt{x}} xy(x+y) \, dy \, dx \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 y + xy^2) \, dy \, dx = \int_0^1 \left[\tfrac{1}{2}x^2 y^2 + \tfrac{1}{3}xy^3\right]_{y=x^2}^{y=\sqrt{x}} \, dx \\ &= \int_0^1 \left(\tfrac{1}{2}x^3 + \tfrac{1}{3}x^{5/2} - \tfrac{1}{2}x^6 - \tfrac{1}{3}x^7\right) dx \\ &= \left[\tfrac{1}{8}x^4 + \tfrac{2}{21}x^{7/2} - \tfrac{1}{14}x^7 - \tfrac{1}{24}x^8\right]_0^1 = \tfrac{1}{8} + \tfrac{2}{21} - \tfrac{1}{14} - \tfrac{1}{24} = \tfrac{3}{28}\end{aligned}$$

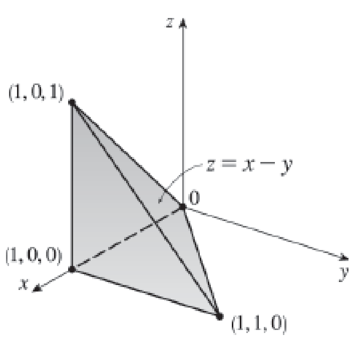
15.



Here $T = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}$, so

$$\begin{aligned}\iiint_T x^2 \, dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^2 \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} x^2(1-x-y) dy \, dx \\ &= \int_0^1 \int_0^{1-x} (x^2 - x^3 - x^2 y) dy \, dx = \int_0^1 \left[x^2 y - x^3 y - \tfrac{1}{2}x^2 y^2\right]_{y=0}^{y=1-x} dx \\ &= \int_0^1 \left[x^2(1-x) - x^3(1-x) - \tfrac{1}{2}x^2(1-x)^2\right] dx \\ &= \int_0^1 \left(\tfrac{1}{2}x^4 - x^3 + \tfrac{1}{2}x^2\right) dx = \left[\tfrac{1}{10}x^5 - \tfrac{1}{4}x^4 + \tfrac{1}{6}x^3\right]_0^1 \\ &= \tfrac{1}{10} - \tfrac{1}{4} + \tfrac{1}{6} = \tfrac{1}{60}\end{aligned}$$

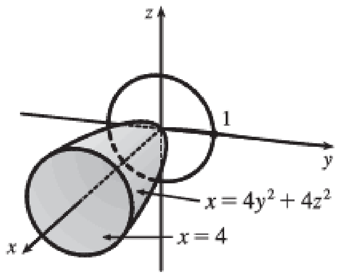
16.



Here $T = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le x - y\}$, so

$$\begin{aligned}\iiint_T xyz \, dV &= \int_0^1 \int_0^x \int_0^{x-y} xyz \, dz \, dy \, dx = \int_0^1 \int_0^x \left[\tfrac{1}{2}xyz^2\right]_{z=0}^{z=x-y} dy \, dx \\ &= \int_0^1 \int_0^x \tfrac{1}{2}xy(x-y)^2 dy \, dx = \tfrac{1}{2}\int_0^1 \int_0^x (x^3 y - 2x^2 y^2 + xy^3) dy \, dx \\ &= \tfrac{1}{2}\int_0^1 \left[\tfrac{1}{2}x^3 y^2 - \tfrac{2}{3}x^2 y^3 + \tfrac{1}{4}xy^4\right]_{y=0}^{y=x} dx \\ &= \tfrac{1}{2}\int_0^1 \left(\tfrac{1}{2}x^5 - \tfrac{2}{3}x^5 + \tfrac{1}{4}x^5\right) dx \\ &= \tfrac{1}{2}\int_0^1 \tfrac{1}{12}x^5 \, dx = \tfrac{1}{144}x^6\Big]_0^1 = \tfrac{1}{144}\end{aligned}$$

17.



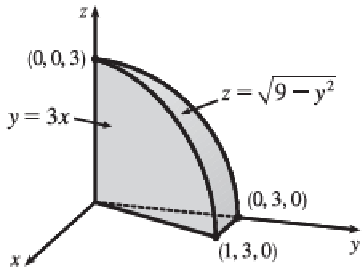
The projection E on the yz-plane is the disk $y^2 + z^2 \leq 1$. Using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we get

$$\iiint_E x\, dV = \iint_D \left[\int_{4y^2+4z^2}^{4} x\, dx\right] dA = \tfrac{1}{2}\iint_D \left[4^2 - (4y^2+4z^2)^2\right] dA$$

$$= 8\int_0^{2\pi}\int_0^1 (1-r^4)\, r\, dr\, d\theta = 8\int_0^{2\pi} d\theta \int_0^1 (r - r^5)\, dr$$

$$= 8(2\pi)\left[\tfrac{1}{2}r^2 - \tfrac{1}{6}r^6\right]_0^1 = \tfrac{16\pi}{3}$$

18.



$$\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} z\, dz\, dy\, dx = \int_0^1\int_{3x}^3 \tfrac{1}{2}(9-y^2)\, dy\, dx$$

$$= \int_0^1 \left[\tfrac{9}{2}y - \tfrac{1}{6}y^3\right]_{y=3x}^{y=3} dx$$

$$= \int_0^1 \left[9 - \tfrac{27}{2}x + \tfrac{9}{2}x^3\right] dx$$

$$= \left[9x - \tfrac{27}{4}x^2 + \tfrac{9}{8}x^4\right]_0^1 = \tfrac{27}{8}$$

19. The plane $2x + y + z = 4$ intersects the xy-plane when

$2x + y + 0 = 4 \quad\Rightarrow\quad y = 4 - 2x$, so

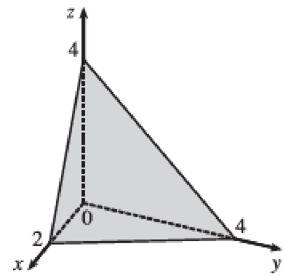
$E = \{(x, y, z) \mid 0 \leq x \leq 2, 0 \leq y \leq 4 - 2x, 0 \leq z \leq 4 - 2x - y\}$ and

$$V = \int_0^2\int_0^{4-2x}\int_0^{4-2x-y} dz\, dy\, dx = \int_0^2\int_0^{4-2x} (4 - 2x - y)\, dy\, dx$$

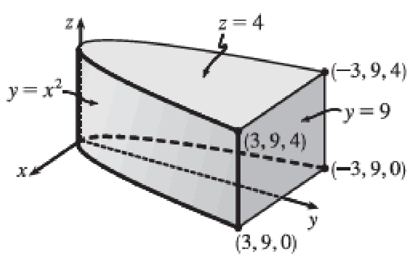
$$= \int_0^2 \left[4y - 2xy - \tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x} dx$$

$$= \int_0^2 \left[4(4-2x) - 2x(4-2x) - \tfrac{1}{2}(4-2x)^2\right] dx$$

$$= \int_0^2 (2x^2 - 8x + 8)\, dx = \left[\tfrac{2}{3}x^3 - 4x^2 + 8x\right]_0^2 = \tfrac{16}{3}$$



20.



$$V = \iiint_E dV = \int_{-3}^3\int_{x^2}^9\int_0^4 dz\, dy\, dx$$

$$= 4\int_{-3}^3\int_{x^2}^9 dy\, dx = 4\int_{-3}^3 (9 - x^2)\, dx$$

$$= 4\left[9x - \tfrac{1}{3}x^3\right]_{-3}^3 = 4(27 - 9 + 27 - 9) = 144$$

21. $V = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} dz\,dy\,dx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (5-y-1)\,dy\,dx = \int_{-3}^{3}\left[4y - \frac{1}{2}y^2\right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} dx$

$= \int_{-3}^{3} 8\sqrt{9-x^2}\,dx = 8\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_{-3}^{3}$ [using trigonometric substitution or Formula 30 in the Table of Integrals]

$= 8\left[\frac{9}{2}\sin^{-1}(1) - \frac{9}{2}\sin^{-1}(-1)\right] = 36\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 36\pi$

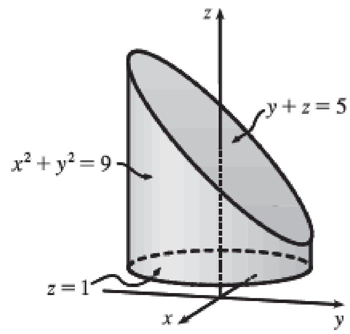
Alternatively, use polar coordinates to evaluate the double integral:

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-y)\,dy\,dx = \int_{0}^{2\pi}\int_{0}^{3} (4 - r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_{0}^{2\pi}\left[2r^2 - \frac{1}{3}r^3\sin\theta\right]_{r=0}^{r=3} d\theta$$

$$= \int_{0}^{2\pi}(18 - 9\sin\theta)\,d\theta$$

$$= 18\theta + 9\cos\theta\Big]_{0}^{2\pi} = 36\pi$$



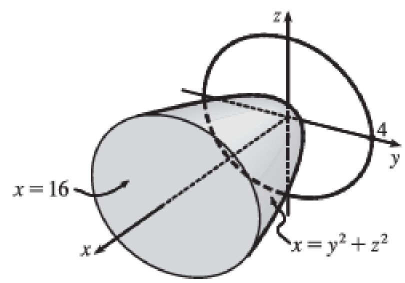
22. The paraboloid $x = y^2 + z^2$ intersects the plane $x = 16$ in the circle $y^2 + z^2 = 16$, $x = 16$. Thus, $E = \{(x,y,z) \mid y^2 + z^2 \le x \le 16,\, y^2 + z^2 \le 16\}$.

Let $D = \{(y,z) \mid y^2 + z^2 \le 16\}$. Then using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we have

$V = \iint_D \left(\int_{y^2+z^2}^{16} dx\right) dA = \iint_D (16 - (y^2+z^2))\,dA$

$= \int_{0}^{2\pi}\int_{0}^{4}(16 - r^2)\,r\,dr\,d\theta = \int_{0}^{2\pi} d\theta \int_{0}^{4}(16r - r^3)\,dr$

$= \left[\theta\right]_{0}^{2\pi}\left[8r^2 - \frac{1}{4}r^4\right]_{0}^{4} = 2\pi(128 - 64) = 128\pi$



23. (a) The wedge can be described as the region

$D = \{(x,y,z) \mid y^2 + z^2 \le 1,\, 0 \le x \le 1,\, 0 \le y \le x\}$

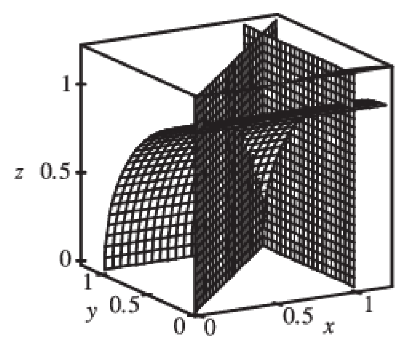
$= \left\{(x,y,z) \mid 0 \le x \le 1,\, 0 \le y \le x,\, 0 \le z \le \sqrt{1-y^2}\right\}$

So the integral expressing the volume of the wedge is

$\iiint_D dV = \int_0^1\int_0^x\int_0^{\sqrt{1-y^2}} dz\,dy\,dx$.

(b) A CAS gives $\int_0^1\int_0^x\int_0^{\sqrt{1-y^2}} dz\,dy\,dx = \frac{\pi}{4} - \frac{1}{3}$.

(Or use Formulas 30 and 87 from the Table of Integrals.)



24. (a) Divide B into 8 cubes of size $\Delta V = 8$. With $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, the Midpoint Rule gives

$$\begin{aligned}\iiint_B \sqrt{x^2+y^2+z^2}\,dV &\approx \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f(\overline{x}_i,\overline{y}_j,\overline{z}_k)\,\Delta V \\ &= 8[f(1,1,1)+f(1,1,3)+f(1,3,1)+f(1,3,3)+f(3,1,1) \\ &\qquad + f(3,1,3)+f(3,3,1)+f(3,3,3)] \\ &\approx 239.64\end{aligned}$$

(b) Using a CAS we have $\iiint_B \sqrt{x^2+y^2+z^2}\,dV = \int_0^4\int_0^4\int_0^4 \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx \approx 245.91$. This differs from the estimate in part (a) by about 2.5%.

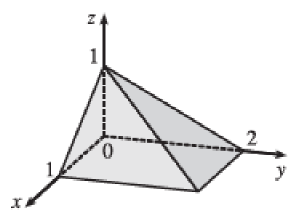
25. Here $f(x,y,z) = \dfrac{1}{\ln(1+x+y+z)}$ and $\Delta V = 2\cdot 4\cdot 2 = 16$, so the Midpoint Rule gives

$$\begin{aligned}\iiint_B f(x,y,z)\,dV &\approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\overline{x}_i,\overline{y}_j,\overline{z}_k)\,\Delta V \\ &= 16[f(1,2,1)+f(1,2,3)+f(1,6,1)+f(1,6,3) \\ &\qquad + f(3,2,1)+f(3,2,3)+f(3,6,1)+f(3,6,3)] \\ &= 16\left[\tfrac{1}{\ln 5}+\tfrac{1}{\ln 7}+\tfrac{1}{\ln 9}+\tfrac{1}{\ln 11}+\tfrac{1}{\ln 7}+\tfrac{1}{\ln 9}+\tfrac{1}{\ln 11}+\tfrac{1}{\ln 13}\right] \approx 60.533\end{aligned}$$

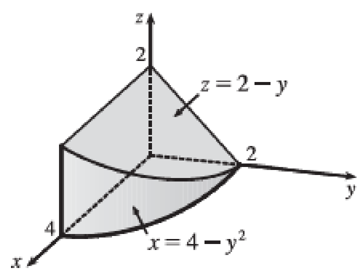
26. Here $f(x,y,z) = \sin(xy^2z^3)$ and $\Delta V = 2\cdot 1\cdot \frac{1}{2} = 1$, so the Midpoint Rule gives

$$\begin{aligned}\iiint_B f(x,y,z)\,dV &\approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\overline{x}_i,\overline{y}_j,\overline{z}_k)\,\Delta V \\ &= 1\left[f\left(1,\tfrac12,\tfrac14\right)+f\left(1,\tfrac12,\tfrac34\right)+f\left(1,\tfrac32,\tfrac14\right)+f\left(1,\tfrac32,\tfrac34\right)\right. \\ &\qquad \left.+ f\left(3,\tfrac12,\tfrac14\right)+f\left(3,\tfrac12,\tfrac34\right)+f\left(3,\tfrac32,\tfrac14\right)+f\left(3,\tfrac32,\tfrac34\right)\right] \\ &= \sin\tfrac{1}{256}+\sin\tfrac{27}{256}+\sin\tfrac{9}{256}+\sin\tfrac{243}{256}+\sin\tfrac{3}{256}+\sin\tfrac{81}{256}+\sin\tfrac{27}{256}+\sin\tfrac{729}{256} \approx 1.675\end{aligned}$$

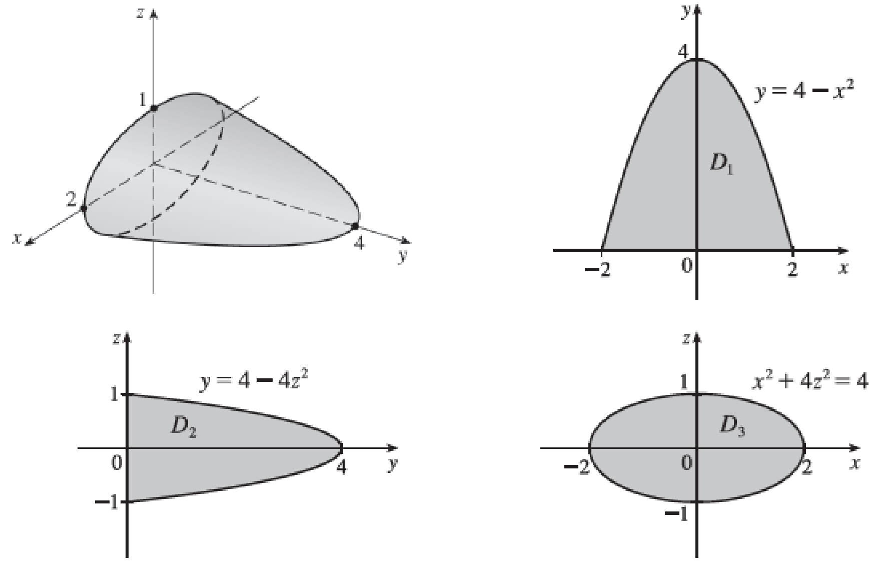
27. $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le 1-x, 0 \le y \le 2-2z\}$, the solid bounded by the three coordinate planes and the planes $z = 1-x$, $y = 2-2z$.



28. $E = \{(x,y,z) \mid 0 \le y \le 2, 0 \le z \le 2-y, 0 \le x \le 4-y^2\}$, the solid bounded by the three coordinate planes, the plane $z = 2-y$, and the cylindrical surface $x = 4-y^2$.



29.



If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \left\{(x, y) \mid -2 \le x \le 2, 0 \le y \le 4 - x^2\right\} = \left\{(x, y) \mid 0 \le y \le 4,\ -\sqrt{4-y} \le x \le \sqrt{4-y}\,\right\}$$

$$D_2 = \left\{(y, z) \mid 0 \le y \le 4,\ -\tfrac{1}{2}\sqrt{4-y} \le z \le \tfrac{1}{2}\sqrt{4-y}\right\} = \left\{(y, z) \mid -1 \le z \le 1, 0 \le y \le 4 - 4z^2\right\}$$

$$D_3 = \left\{(x, z) \mid x^2 + 4z^2 \le 4\right\}$$

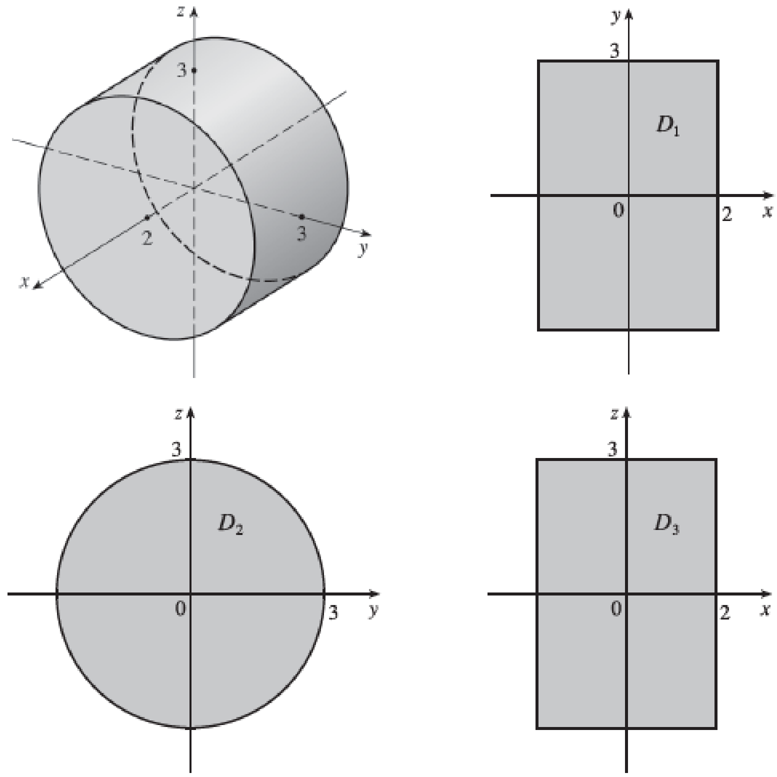
Therefore

$$\begin{aligned}
E &= \left\{(x, y, z) \mid -2 \le x \le 2, 0 \le y \le 4 - x^2,\ -\tfrac{1}{2}\sqrt{4 - x^2 - y} \le z \le \tfrac{1}{2}\sqrt{4 - x^2 - y}\,\right\} \\
&= \left\{(x, y, z) \mid 0 \le y \le 4,\ -\sqrt{4-y} \le x \le \sqrt{4-y},\ -\tfrac{1}{2}\sqrt{4 - x^2 - y} \le z \le \tfrac{1}{2}\sqrt{4 - x^2 - y}\,\right\} \\
&= \left\{(x, y, z) \mid -1 \le z \le 1, 0 \le y \le 4 - 4z^2,\ -\sqrt{4 - y - 4z^2} \le x \le \sqrt{4 - y - 4z^2}\,\right\} \\
&= \left\{(x, y, z) \mid 0 \le y \le 4,\ -\tfrac{1}{2}\sqrt{4-y} \le z \le \tfrac{1}{2}\sqrt{4-y},\ -\sqrt{4 - y - 4z^2} \le x \le \sqrt{4 - y - 4z^2}\,\right\} \\
&= \left\{(x, y, z) \mid -2 \le x \le 2,\ -\tfrac{1}{2}\sqrt{4 - x^2} \le z \le \tfrac{1}{2}\sqrt{4 - x^2}, 0 \le y \le 4 - x^2 - 4z^2\right\} \\
&= \left\{(x, y, z) \mid -1 \le z \le 1,\ -\sqrt{4 - 4z^2} \le x \le \sqrt{4 - 4z^2}, 0 \le y \le 4 - x^2 - 4z^2\right\}
\end{aligned}$$

Then

$$\begin{aligned}
\iiint_E f(x, y, z)\, dV &= \int_{-2}^{2} \int_0^{4-x^2} \int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\, dz\, dy\, dx = \int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\, dz\, dx\, dy \\
&= \int_{-1}^{1} \int_0^{4-4z^2} \int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\, dx\, dy\, dz = \int_0^4 \int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2} \int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\, dx\, dz\, dy \\
&= \int_{-2}^{2} \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} \int_0^{4-x^2-4z^2} f(x, y, z)\, dy\, dz\, dx = \int_{-1}^{1} \int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}} \int_0^{4-x^2-4z^2} f(x, y, z)\, dy\, dx\, dz
\end{aligned}$$

30.



If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x, y) \mid -2 \le x \le 2,\ -3 \le y \le 3\}$$

$$D_2 = \left\{(y, z) \mid y^2 + z^2 \le 9\right\}$$

$$D_3 = \{(x, z) \mid -2 \le x \le 2,\ -3 \le z \le 3\}$$

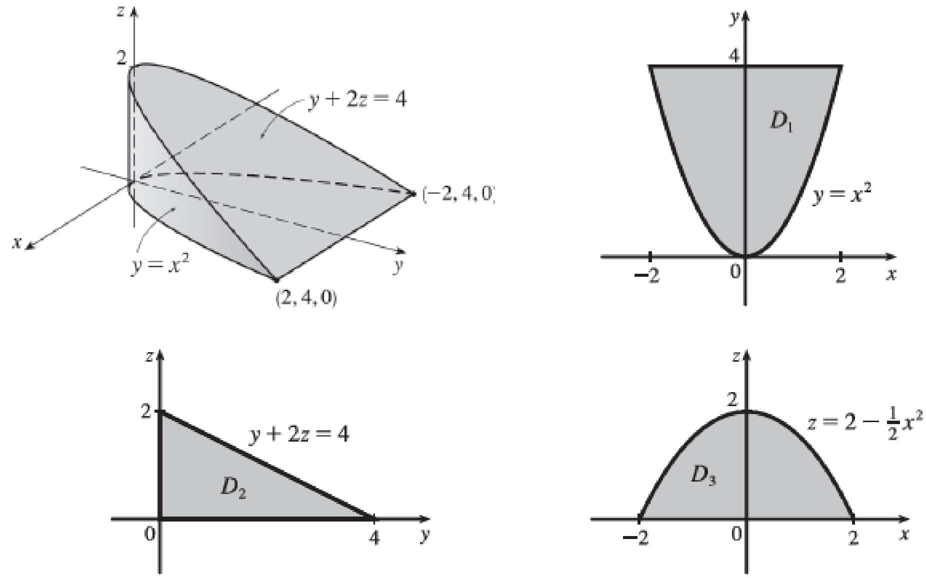
Therefore

$$\begin{aligned} E &= \left\{(x, y, z) \mid -2 \le x \le 2,\ -3 \le y \le 3,\ -\sqrt{9 - y^2} \le z \le \sqrt{9 - y^2}\right\} \\ &= \left\{(x, y, z) \mid -3 \le y \le 3,\ -\sqrt{9 - y^2} \le z \le \sqrt{9 - y^2},\ -2 \le x \le 2\right\} \\ &= \left\{(x, y, z) \mid -3 \le z \le 3,\ -\sqrt{9 - z^2} \le y \le \sqrt{9 - z^2},\ -2 \le x \le 2\right\} \\ &= \left\{(x, y, z) \mid -2 \le x \le 2,\ -3 \le z \le 3,\ -\sqrt{9 - z^2} \le y \le \sqrt{9 - z^2}\right\} \end{aligned}$$

and

$$\begin{aligned} \iiint_E f(x, y, z)\, dV &= \int_{-2}^{2} \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y, z)\, dz\, dy\, dx = \int_{-3}^{3} \int_{-2}^{2} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y, z)\, dz\, dx\, dy \\ &= \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{-2}^{2} f(x, y, z)\, dx\, dz\, dy = \int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \int_{-2}^{2} f(x, y, z)\, dx\, dy\, dz \\ &= \int_{-2}^{2} \int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x, y, z)\, dy\, dz\, dx = \int_{-3}^{3} \int_{-2}^{2} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x, y, z)\, dy\, dx\, dz \end{aligned}$$

31.



If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \left\{(x,y) \mid -2 \le x \le 2, x^2 \le y \le 4\right\} = \left\{(x,y) \mid 0 \le y \le 4, -\sqrt{y} \le x \le \sqrt{y}\right\},$$

$$D_2 = \left\{(y,z) \mid 0 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y\right\} = \left\{(y,z) \mid 0 \le z \le 2, 0 \le y \le 4 - 2z\right\}, \text{ and}$$

$$D_3 = \left\{(x,z) \mid -2 \le x \le 2, 0 \le z \le 2 - \tfrac{1}{2}x^2\right\} = \left\{(x,z) \mid 0 \le z \le 2, -\sqrt{4-2z} \le x \le \sqrt{4-2z}\right\}$$

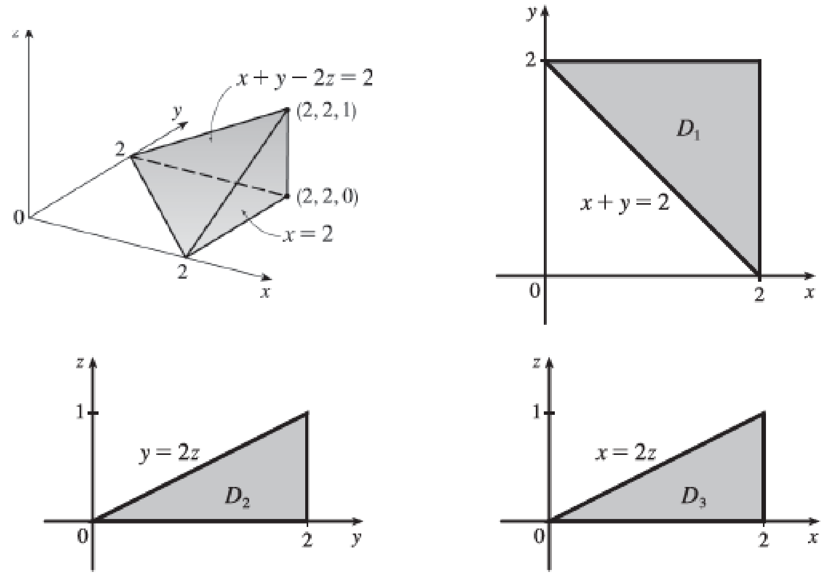
Therefore

$$\begin{aligned} E &= \left\{(x,y,z) \mid -2 \le x \le 2, x^2 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 4, -\sqrt{y} \le x \le \sqrt{y}, 0 \le z \le 2 - \tfrac{1}{2}y\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y, -\sqrt{y} \le x \le \sqrt{y}\right\} \\ &= \left\{(x,y,z) \mid 0 \le z \le 2, 0 \le y \le 4 - 2z, -\sqrt{y} \le x \le \sqrt{y}\right\} \\ &= \left\{(x,y,z) \mid -2 \le x \le 2, 0 \le z \le 2 - \tfrac{1}{2}x^2, x^2 \le y \le 4 - 2z\right\} \\ &= \left\{(x,y,z) \mid 0 \le z \le 2, -\sqrt{4-2z} \le x \le \sqrt{4-2z}, x^2 \le y \le 4 - 2z\right\} \end{aligned}$$

Then

$$\begin{aligned} \iiint_E f(x,y,z)\,dV &= \int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{2-y/2} f(x,y,z)\,dz\,dy\,dx = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{2-y/2} f(x,y,z)\,dz\,dx\,dy \\ &= \int_0^4\int_0^{2-y/2}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dz\,dy = \int_0^2\int_0^{4-2z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz \\ &= \int_{-2}^{2}\int_0^{2-x^2/2}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dz\,dx = \int_0^2\int_{-\sqrt{4-2z}}^{\sqrt{4-2z}}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

32.



If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x, y) \mid 0 \le x \le 2, 2 - x \le y \le 2\} = \{(x, y) \mid 0 \le y \le 2, 2 - y \le x \le 2\},$$

$$D_2 = \left\{(y, z) \mid 0 \le y \le 2, 0 \le z \le \tfrac{1}{2}y\right\} = \{(y, z) \mid 0 \le z \le 1, 2z \le y \le 2\}, \text{ and}$$

$$D_3 = \left\{(x, z) \mid 0 \le x \le 2, 0 \le z \le \tfrac{1}{2}x\right\} = \{(x, z) \mid 0 \le z \le 1, 2z \le x \le 2\}$$

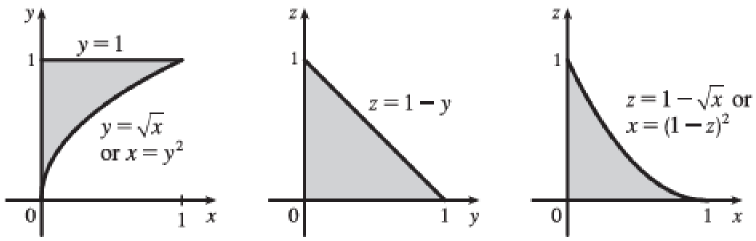
Therefore

$$\begin{aligned} E &= \left\{(x, y, z) \mid 0 \le x \le 2, 2 - x \le y \le 2, 0 \le z \le \tfrac{1}{2}(x + y - 2)\right\} \\ &= \left\{(x, y, z) \mid 0 \le y \le 2, 2 - y \le x \le 2, 0 \le z \le \tfrac{1}{2}(x + y - 2)\right\} \\ &= \left\{(x, y, z) \mid 0 \le y \le 2, 0 \le z \le \tfrac{1}{2}y, 2 - y + 2z \le x \le 2\right\} \\ &= \{(x, y, z) \mid 0 \le z \le 1, 2z \le y \le 2, 2 - y + 2z \le x \le 2\} \\ &= \left\{(x, y, z) \mid 0 \le x \le 2, 0 \le z \le \tfrac{1}{2}x, 2 - x + 2z \le y \le 2\right\} \\ &= \{(x, y, z) \mid 0 \le z \le 1, 2z \le x \le 2, 2 - x + 2z \le y \le 2\} \end{aligned}$$

Then

$$\begin{aligned} \iiint_E f(x, y, z)\, dV &= \int_0^2 \int_{2-x}^2 \int_0^{(x+y-2)/2} f(x, y, z)\, dz\, dy\, dx = \int_0^2 \int_{2-y}^2 \int_0^{(x+y-2)/2} f(x, y, z)\, dz\, dx\, dy \\ &= \int_0^2 \int_0^{y/2} \int_{2-y+2z}^2 f(x, y, z)\, dx\, dz\, dy = \int_0^1 \int_{2z}^2 \int_{2-y+2z}^2 f(x, y, z)\, dx\, dy\, dz \\ &= \int_0^2 \int_0^{x/2} \int_{2-x+2z}^2 f(x, y, z)\, dy\, dz\, dx = \int_0^1 \int_{2z}^2 \int_{2-x+2z}^2 f(x, y, z)\, dy\, dx\, dz \end{aligned}$$

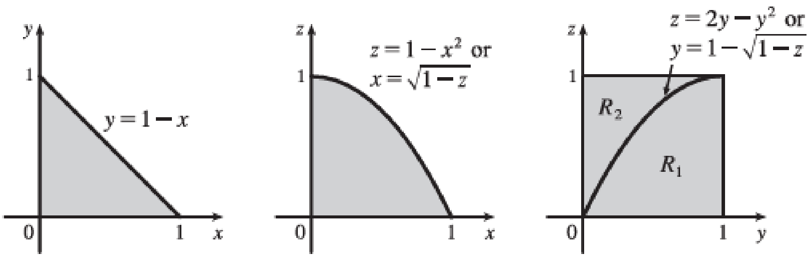
33.



The diagrams show the projections of E on the xy-, yz-, and xz-planes. Therefore

$$\begin{aligned}\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx &= \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz \\ &= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy = \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz\end{aligned}$$

34.



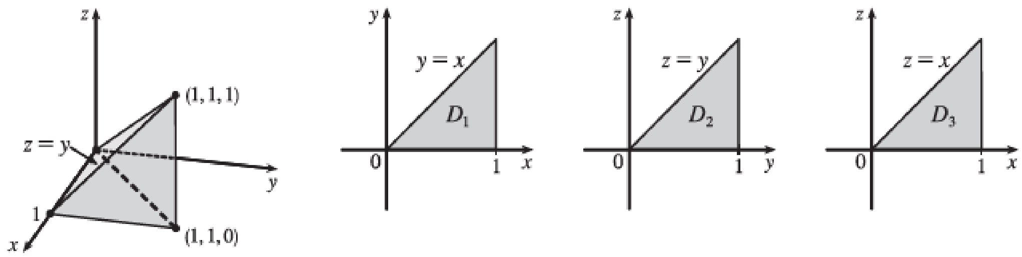
The projections of E onto the xy- and xz-planes are as in the first two diagrams and so

$$\begin{aligned}\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,dz\,dx &= \int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\,dy\,dx\,dz \\ &= \int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx\end{aligned}$$

Now the surface $z = 1 - x^2$ intersects the plane $y = 1 - x$ in a curve whose projection in the yz-plane is $z = 1 - (1 - y)^2$ or $z = 2y - y^2$. So we must split up the projection of E on the yz-plane into two regions as in the third diagram. For (y, z) in R_1, $0 \le x \le 1 - y$ and for (y, z) in R_2, $0 \le x \le \sqrt{1-z}$, and so the given integral is also equal to

$$\begin{aligned}&\int_0^1\int_0^{1-\sqrt{1-z}}\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz + \int_0^1\int_{1-\sqrt{1-z}}^1\int_0^{1-y} f(x,y,z)\,dx\,dy\,dz \\ &\qquad = \int_0^1\int_0^{2y-y^2}\int_0^{1-y} f(x,y,z)\,dx\,dz\,dy + \int_0^1\int_{2y-y^2}^1\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy.\end{aligned}$$

35.



$\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le z \le y, y \le x \le 1, 0 \le y \le 1\}$.

If D_1, D_2, and D_3 are the projections of E on the xy-, yz- and xz-planes then

$$D_1 = \{(x,y) \mid 0 \le y \le 1, y \le x \le 1\} = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\},$$

$$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\}, \text{ and}$$

$$D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le x\} = \{(x,z) \mid 0 \le z \le 1, z \le x \le 1\}.$$

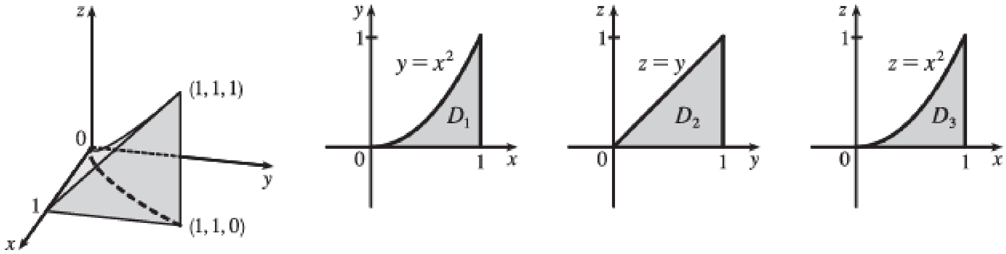
Thus we also have

$$\begin{aligned} E &= \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le y\} = \{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, y \le x \le 1\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, z \le y \le 1, y \le x \le 1\} = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le x, z \le y \le x\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, z \le x \le 1, z \le y \le x\}. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy &= \int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_0^y \int_y^1 f(x,y,z)\,dx\,dz\,dy \\ &= \int_0^1 \int_z^1 \int_y^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_0^x \int_z^x f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1 \int_z^1 \int_z^x f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

36.



$\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x^2, 0 \le z \le y\}$.

If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x^2\} = \{(x,y) \mid 0 \le y \le 1, \sqrt{y} \le x \le 1\},$$
$$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\},$$
$$D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le x^2\} = \{(x,z) \mid 0 \le z \le 1, \sqrt{z} \le x \le 1\}.$$

Thus we also have

$$\begin{aligned} E &= \{(x,y,z) \mid 0 \le y \le 1, \sqrt{y} \le x \le 1, 0 \le z \le y\} = \{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, \sqrt{y} \le x \le 1\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, z \le y \le 1, \sqrt{y} \le x \le 1\} = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le x^2, z \le y \le x^2\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, \sqrt{z} \le x \le 1, z \le y \le x^2\} \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx &= \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x,y,z)\,dz\,dx\,dy = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy \\ &= \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_0^{x^2} \int_z^{x^2} f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

37. $m = \iiint_E \rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} 2(1+x+y)\,dy\,dx$

$$= \int_0^1 \left[2y + 2xy + y^2\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \left(2\sqrt{x} + 2x^{3/2} + x\right) dx = \left[\tfrac{4}{3}x^{3/2} + \tfrac{4}{5}x^{5/2} + \tfrac{1}{2}x^2\right]_0^1 = \tfrac{79}{30}$$

$$M_{yz} = \iiint_E x\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2x\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} 2x(1+x+y)\,dy\,dx$$
$$= \int_0^1 \left[2xy + 2x^2y + xy^2\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 (2x^{3/2} + 2x^{5/2} + x^2)\,dx = \left[\tfrac{4}{5}x^{5/2} + \tfrac{4}{7}x^{7/2} + \tfrac{1}{3}x^3\right]_0^1 = \tfrac{179}{105}$$

$$M_{xz} = \iiint_E y\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2y\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} 2y(1+x+y)\,dy\,dx$$
$$= \int_0^1 \left[y^2 + xy^2 + \tfrac{2}{3}y^3\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \left(x + x^2 + \tfrac{2}{3}x^{3/2}\right) dx = \left[\tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{4}{15}x^{5/2}\right]_0^1 = \tfrac{11}{10}$$

$$M_{xy} = \iiint_E z\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2z\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} \left[z^2\right]_{z=0}^{z=1+x+y} dy\,dx = \int_0^1 \int_0^{\sqrt{x}} (1+x+y)^2\,dy\,dx$$
$$= \int_0^1 \int_0^{\sqrt{x}} (1 + 2x + 2y + 2xy + x^2 + y^2)\,dy\,dx = \int_0^1 \left[y + 2xy + y^2 + xy^2 + x^2y + \tfrac{1}{3}y^3\right]_{y=0}^{y=\sqrt{x}} dx$$
$$= \int_0^1 \left(\sqrt{x} + \tfrac{7}{3}x^{3/2} + x + x^2 + x^{5/2}\right) dx = \left[\tfrac{2}{3}x^{3/2} + \tfrac{14}{15}x^{5/2} + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{2}{7}x^{7/2}\right]_0^1 = \tfrac{571}{210}$$

Thus the mass is $\frac{79}{30}$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(\dfrac{358}{553}, \dfrac{33}{79}, \dfrac{571}{553}\right)$.

38. $m = \int_{-1}^{1}\int_0^{1-y^2}\int_0^{1-z} 4\,dx\,dz\,dy = 4\int_{-1}^{1}\int_0^{1-y^2}(1-z)\,dz\,dy = 4\int_{-1}^{1}\left[z-\frac{1}{2}z^2\right]_{z=0}^{z=1-y^2}dy = 2\int_{-1}^{1}(1-y^4)\,dy = \frac{16}{5}$,

$$M_{yz} = \int_{-1}^{1}\int_0^{1-y^2}\int_0^{1-z} 4x\,dx\,dz\,dy = 2\int_{-1}^{1}\int_0^{1-y^2}(1-z)^2\,dz\,dy = 2\int_{-1}^{1}\left[-\frac{1}{3}(1-z)^3\right]_{z=0}^{z=1-y^2}dy$$

$$= \frac{2}{3}\int_{-1}^{1}(1-y^6)\,dy = \left(\frac{4}{3}\right)\left(\frac{6}{7}\right) = \frac{24}{21}$$

$$M_{xz} = \int_{-1}^{1}\int_0^{1-y^2}\int_0^{1-z} 4y\,dx\,dz\,dy = \int_{-1}^{1}\int_0^{1-y^2}4y(1-z)\,dz\,dy$$

$$= \int_{-1}^{1}\left[4y(1-y^2)-2y(1-y^2)^2\right]dy = \int_{-1}^{1}(2y-2y^5)\,dy = 0 \quad \text{[the integrand is odd]}$$

$$M_{xy} = \int_{-1}^{1}\int_0^{1-y^2}\int_0^{1-z} 4z\,dx\,dz\,dy = \int_{-1}^{1}\int_0^{1-y^2}(4z-4z^2)\,dz\,dy = 2\int_{-1}^{1}\left[(1-y^2)^2-\frac{2}{3}(1-y^2)^3\right]dy$$

$$= 2\int_{-1}^{1}\left[\frac{1}{3}-y^4+\frac{2}{3}y^6\right]dy = \left[\frac{4}{3}y-\frac{4}{5}y^5+\frac{8}{21}y^7\right]_0^1 = \frac{96}{105} = \frac{32}{35}$$

Thus, $(\overline{x},\overline{y},\overline{z}) = \left(\frac{5}{14}, 0, \frac{2}{7}\right)$

39. $m = \int_0^a\int_0^a\int_0^a(x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^a\int_0^a\left[\frac{1}{3}x^3+xy^2+xz^2\right]_{x=0}^{x=a}dy\,dz = \int_0^a\int_0^a\left(\frac{1}{3}a^3+ay^2+az^2\right)dy\,dz$

$$= \int_0^a\left[\frac{1}{3}a^3y+\frac{1}{3}ay^3+ayz^2\right]_{y=0}^{y=a}dz = \int_0^a\left(\frac{2}{3}a^4+a^2z^2\right)dz = \left[\frac{2}{3}a^4z+\frac{1}{3}a^2z^3\right]_0^a = \frac{2}{3}a^5+\frac{1}{3}a^5 = a^5$$

$$M_{yz} = \int_0^a\int_0^a\int_0^a\left[x^3+x(y^2+z^2)\right]dx\,dy\,dz = \int_0^a\int_0^a\left[\frac{1}{4}a^4+\frac{1}{2}a^2(y^2+z^2)\right]dy\,dz$$

$= \int_0^a\left(\frac{1}{4}a^5+\frac{1}{6}a^5+\frac{1}{2}a^3z^2\right)dz = \frac{1}{4}a^6+\frac{1}{3}a^6 = \frac{7}{12}a^6 = M_{xz} = M_{xy}$ by symmetry of E and $\rho(x,y,z)$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(\frac{7}{12}a, \frac{7}{12}a, \frac{7}{12}a\right)$.

40. $m = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y-y^2\right]dy\,dx$

$$= \int_0^1\left[\frac{1}{2}(1-x)^3-\frac{1}{3}(1-x)^3\right]dx = \frac{1}{6}\int_0^1(1-x)^3\,dx = \frac{1}{24}$$

$$M_{yz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xy\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(x-x^2)y-xy^2\right]dy\,dx$$

$$= \int_0^1\left[\frac{1}{2}x(1-x)^3-\frac{1}{3}x(1-x)^3\right]dx = \frac{1}{6}\int_0^1\left(x-3x^2+3x^3-x^4\right)dx = \frac{1}{6}\left(\frac{1}{2}-1+\frac{3}{4}-\frac{1}{5}\right) = \frac{1}{120}$$

$$M_{xz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y^2\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y^2-y^3\right]dy\,dx$$

$$= \int_0^1\left[\frac{1}{3}(1-x)^4-\frac{1}{4}(1-x)^4\right]dx = \frac{1}{12}\left[-\frac{1}{5}(1-x)^5\right]_0^1 = \frac{1}{60}$$

$$M_{xy} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} yz\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[\frac{1}{2}y(1-x-y)^2\right]dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^{1-x}\left[(1-x)^2y-2(1-x)y^2+y^3\right]dy\,dx = \frac{1}{2}\int_0^1\left[\frac{1}{2}(1-x)^4-\frac{2}{3}(1-x)^4+\frac{1}{4}(1-x)^4\right]dx$$

$$= \frac{1}{24}\int_0^1(1-x)^4\,dx = -\frac{1}{24}\left[\frac{1}{5}(1-x)^5\right]_0^1 = \frac{1}{120}$$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)$.

41. $I_x = \int_0^L\int_0^L\int_0^L k(y^2+z^2)\,dz\,dy\,dx = k\int_0^L\int_0^L\left(Ly^2+\frac{1}{3}L^3\right)dy\,dx = k\int_0^L\frac{2}{3}L^4\,dx = \frac{2}{3}kL^5$.

By symmetry, $I_x = I_y = I_z = \frac{2}{3}kL^5$.

42. Let k be the density. Then

$$I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} k(y^2+z^2)\,dx\,dy\,dz = ka\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}(y^2+z^2)\,dy\,dz$$

$$= ak\int_{-c/2}^{c/2}\left[\tfrac{1}{3}y^3+z^2y\right]_{y=-b/2}^{y=b/2}dz = ak\int_{-c/2}^{c/2}\left(\tfrac{1}{12}b^3+bz^2\right)dz = ak\left[\tfrac{1}{12}b^3z+\tfrac{1}{3}bz^3\right]_{-c/2}^{c/2}$$

$$= ak\left(\tfrac{1}{12}b^3c+\tfrac{1}{12}bc^3\right) = \tfrac{1}{12}kabc(b^2+c^2)$$

By symmetry, $I_y = \frac{1}{12}kabc(a^2+c^2)$ and $I_z = \frac{1}{12}kabc(a^2+b^2)$.

43. $$I_z = \iiint_E (x^2+y^2)\,\rho(x,y,z)\,dV = \iint_{x^2+y^2\le a^2}\left[\int_0^h k(x^2+y^2)\,dz\right]dA = \iint_{x^2+y^2\le a^2} k(x^2+y^2)h\,dA$$

$$= kh\int_0^{2\pi}\int_0^a (r^2)\,r\,dr\,d\theta = kh\int_0^{2\pi}d\theta\int_0^a r^3\,dr = kh(2\pi)\left[\tfrac{1}{4}r^4\right]_0^a = 2\pi kh\cdot\tfrac{1}{4}a^4 = \tfrac{1}{2}\pi kha^4$$

44. $$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV = \iint_{x^2+y^2\le h^2}\left[\int_{\sqrt{x^2+y^2}}^h k(x^2+y^2)\,dz\right]dA$$

$$= \iint_{x^2+y^2\le h^2} k(x^2+y^2)\left(h-\sqrt{x^2+y^2}\right)dA = k\int_0^{2\pi}\int_0^h r^2(h-r)\,r\,dr\,d\theta$$

$$= k\int_0^{2\pi}d\theta\int_0^h\left(r^3h-r^4\right)dr = k(2\pi)\left[\tfrac{1}{4}r^4h-\tfrac{1}{5}r^5\right]_0^h = 2\pi k\left(\tfrac{1}{4}h^5-\tfrac{1}{5}h^5\right) = \tfrac{1}{10}\pi kh^5$$

45. (a) $m = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-y}\sqrt{x^2+y^2}\,dz\,dy\,dx$

(b) $(\overline{x},\overline{y},\overline{z}) = \left(\dfrac{M_{yz}}{m},\dfrac{M_{xz}}{m},\dfrac{M_{xy}}{m}\right)$ where

$M_{yz} = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-y} x\sqrt{x^2+y^2}\,dz\,dy\,dx$, $M_{xz} = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-y} y\sqrt{x^2+y^2}\,dz\,dy\,dx$, and

$M_{xy} = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-y} z\sqrt{x^2+y^2}\,dz\,dy\,dx$.

(c) $I_z = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-y}(x^2+y^2)\sqrt{x^2+y^2}\,dz\,dy\,dx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-y}(x^2+y^2)^{3/2}\,dz\,dy\,dx$

46. (a) $m = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$

(b) $(\overline{x},\overline{y},\overline{z})$ where $\overline{x} = m^{-1}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} x\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$,

$\overline{y} = m^{-1}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$,

$\overline{z} = m^{-1}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$

(c) $I_z = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}}(x^2+y^2)(1+x+y+z)\,dz\,dx\,dy$

47. (a) $m = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y (1 + x + y + z)\, dz\, dy\, dx = \frac{3\pi}{32} + \frac{11}{24}$

(b) $$(\overline{x}, \overline{y}, \overline{z}) = \left(m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y x(1 + x + y + z)\, dz\, dy\, dx,\; m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y y(1 + x + y + z)\, dz\, dy\, dx,\; m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y z(1 + x + y + z)\, dz\, dy\, dx \right)$$

$$= \left(\frac{28}{9\pi + 44}, \frac{30\pi + 128}{45\pi + 220}, \frac{45\pi + 208}{135\pi + 660} \right)$$

(c) $I_z = \displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y (x^2 + y^2)(1 + x + y + z)\, dz\, dy\, dx = \frac{68 + 15\pi}{240}$

48. (a) $m = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} (x^2 + y^2)\, dz\, dy\, dx = \frac{56}{5} = 11.2$

(b) $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} x(x^2 + y^2)\, dz\, dy\, dx \approx 0.375$,

$\overline{y} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} y(x^2 + y^2)\, dz\, dy\, dx = \frac{45\pi}{64} \approx 2.209$,

$\overline{z} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z(x^2 + y^2)\, dz\, dy\, dx = \frac{15}{16} = 0.9375$.

(c) $I_z = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} (x^2 + y^2)^2\, dz\, dy\, dx = \frac{10{,}464}{175} \approx 59.79$

49. (a) $f(x, y, z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x, y, z)\, dV = 1$. Here we have

$$\begin{aligned}\iiint_{\mathbb{R}^3} f(x, y, z)\, dV &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dz\, dy\, dx = \int_0^2 \int_0^2 \int_0^2 Cxyz\, dz\, dy\, dx \\ &= C \int_0^2 x\, dx \int_0^2 y\, dy \int_0^2 z\, dz = C\left[\tfrac{1}{2}x^2\right]_0^2 \left[\tfrac{1}{2}y^2\right]_0^2 \left[\tfrac{1}{2}z^2\right]_0^2 = 8C\end{aligned}$$

Then we must have $8C = 1 \Rightarrow C = \frac{1}{8}$.

(b) $$\begin{aligned}P(X \le 1, Y \le 1, Z \le 1) &= \int_{-\infty}^1 \int_{-\infty}^1 \int_{-\infty}^1 f(x, y, z)\, dz\, dy\, dx = \int_0^1 \int_0^1 \int_0^1 \tfrac{1}{8}xyz\, dz\, dy\, dx \\ &= \tfrac{1}{8} \int_0^1 x\, dx \int_0^1 y\, dy \int_0^1 z\, dz = \tfrac{1}{8}\left[\tfrac{1}{2}x^2\right]_0^1 \left[\tfrac{1}{2}y^2\right]_0^1 \left[\tfrac{1}{2}z^2\right]_0^1 = \tfrac{1}{8}\left(\tfrac{1}{2}\right)^3 = \tfrac{1}{64}\end{aligned}$$

(c) $P(X + Y + Z \le 1) = P((X, Y, Z) \in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x + y + z = 1$. The plane $x + y + z = 1$ meets the xy-plane in the line $x + y = 1$, so we have

$$\begin{aligned}P(X + Y + Z \le 1) &= \iiint_E f(x, y, z)\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \tfrac{1}{8}xyz\, dz\, dy\, dx \\ &= \tfrac{1}{8} \int_0^1 \int_0^{1-x} xy\left[\tfrac{1}{2}z^2\right]_{z=0}^{z=1-x-y} dy\, dx = \tfrac{1}{16} \int_0^1 \int_0^{1-x} xy(1 - x - y)^2\, dy\, dx \\ &= \tfrac{1}{16} \int_0^1 \int_0^{1-x} [(x^3 - 2x^2 + x)y + (2x^2 - 2x)y^2 + xy^3]\, dy\, dx \\ &= \tfrac{1}{16} \int_0^1 \left[(x^3 - 2x^2 + x)\tfrac{1}{2}y^2 + (2x^2 - 2x)\tfrac{1}{3}y^3 + x\left(\tfrac{1}{4}y^4\right)\right]_{y=0}^{y=1-x} dx \\ &= \tfrac{1}{192} \int_0^1 (x - 4x^2 + 6x^3 - 4x^4 + x^5)\, dx = \tfrac{1}{192}\left(\tfrac{1}{30}\right) = \tfrac{1}{5760}\end{aligned}$$

50. (a) $f(x,y,z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x,y,z)\,dV = 1$. Here we have

$$\begin{aligned}\iiint_{\mathbb{R}^3} f(x,y,z)\,dV &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} Ce^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx \\ &= C\int_0^{\infty} e^{-0.5x}\,dx \int_0^{\infty} e^{-0.2y}\,dy \int_0^{\infty} e^{-0.1z}\,dz \\ &= C\lim_{t\to\infty}\int_0^t e^{-0.5x}\,dx \lim_{t\to\infty}\int_0^t e^{-0.2y}\,dy \lim_{t\to\infty}\int_0^t e^{-0.1z}\,dz \\ &= C\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t \lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t \\ &= C\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right] \lim_{t\to\infty}\left[-10(e^{-0.1t}-1)\right] \\ &= C\cdot(-2)(0-1)\cdot(-5)(0-1)\cdot(-10)(0-1) = 100C\end{aligned}$$

So we must have $100C = 1 \quad\Rightarrow\quad C = \frac{1}{100}$.

(b) We have no restriction on Z, so

$$\begin{aligned}P(X\le 1, Y\le 1) &= \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^{\infty} \tfrac{1}{100}e^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx \\ &= \tfrac{1}{100}\int_0^1 e^{-0.5x}\,dx \int_0^1 e^{-0.2y}\,dy \int_0^{\infty} e^{-0.1z}\,dz \\ &= \tfrac{1}{100}\left[-2e^{-0.5x}\right]_0^1 \left[-5e^{-0.2y}\right]_0^1 \lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t \qquad \text{[by part (a)]} \\ &= \tfrac{1}{100}(2-2e^{-0.5})(5-5e^{-0.2})(10) = (1-e^{-0.5})(1-e^{-0.2}) \approx 0.07132\end{aligned}$$

(c)
$$\begin{aligned}P(X\le 1, Y\le 1, Z\le 1) &= \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^1 \tfrac{1}{100}e^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx \\ &= \tfrac{1}{100}\int_0^1 e^{-0.5x}\,dx \int_0^1 e^{-0.2y}\,dy \int_0^1 e^{-0.1z}\,dz \\ &= \tfrac{1}{100}\left[-2e^{-0.5x}\right]_0^1 \left[-5e^{-0.2y}\right]_0^1 \left[-10e^{-0.1z}\right]_0^1 \\ &= (1-e^{-0.5})(1-e^{-0.2})(1-e^{-0.1}) \approx 0.006787\end{aligned}$$