51. $V(E) = L^3 \quad \Rightarrow \quad f_{\text{ave}} = \dfrac{1}{L^3}\displaystyle\int_0^L\int_0^L\int_0^L xyz\,dx\,dy\,dz = \dfrac{1}{L^3}\int_0^L x\,dx \int_0^L y\,dy \int_0^L z\,dz$

$$= \frac{1}{L^3}\left[\frac{x^2}{2}\right]_0^L \left[\frac{y^2}{2}\right]_0^L \left[\frac{z^2}{2}\right]_0^L = \frac{1}{L^3}\,\frac{L^2}{2}\,\frac{L^2}{2}\,\frac{L^2}{2} = \frac{L^3}{8}$$

52. $V(E) = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} dz\,dy\,dx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy\,dx$

$$= \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 (r-r^3)\,dr = 2\pi\left(\tfrac{r^2}{2} - \tfrac{r^4}{4}\right)\Big]_0^1 = \tfrac{\pi}{2}.$$

Then $\quad f_{\text{ave}} = \frac{1}{\pi/2}\iiint_E (x^2 z + y^2 z)\,dV = \frac{2}{\pi}\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} (x^2+y^2)\,z\,dz\,dy\,dx$

$$= \tfrac{2}{\pi}\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)\cdot\tfrac{1}{2}(1-x^2-y^2)^2\,dy\,dx = \tfrac{1}{\pi}\int_0^{2\pi}\int_0^1 r^2(1-r^2)^2\,r\,dr\,d\theta$$

$$= \tfrac{1}{\pi}\int_0^{2\pi} d\theta \int_0^1 (r^3 - 2r^5 + r^7)\,dr = \tfrac{1}{\pi}(2\pi)\left[\tfrac{1}{4}r^4 - \tfrac{1}{3}r^6 + \tfrac{1}{8}r^8\right]_0^1 = 2\left(\tfrac{1}{24}\right) = \tfrac{1}{12}$$

53. The triple integral will attain its maximum when the integrand $1 - x^2 - 2y^2 - 3z^2$ is positive in the region E and negative everywhere else. For if E contains some region F where the integrand is negative, the integral could be increased by excluding F from E, and if E fails to contain some part G of the region where the integrand is positive, the integral could be increased by including G in E. So we require that $x^2 + 2y^2 + 3z^2 \le 1$. This describes the region bounded by the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$.