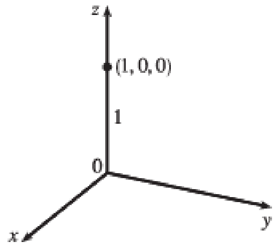
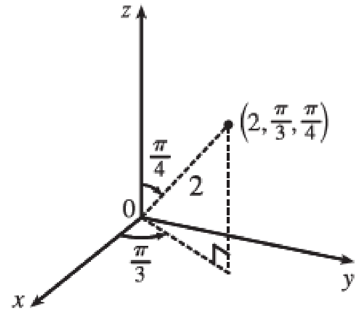
1. (a)



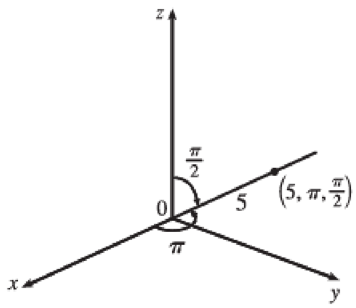
$x = \rho \sin\phi \cos\theta = (1) \sin 0 \cos 0 = 0$,
$y = \rho \sin\phi \sin\theta = (1) \sin 0 \sin 0 = 0$, and
$z = \rho \cos\phi = (1) \cos 0 = 1$ so the point is
$(0, 0, 1)$ in rectangular coordinates.

(b)



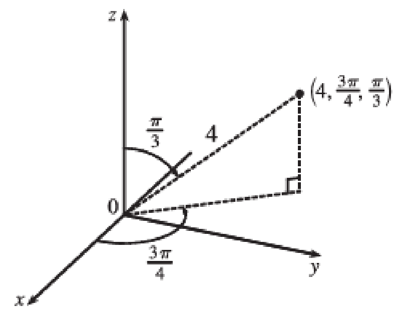
$x = 2 \sin\frac{\pi}{4} \cos\frac{\pi}{3} = \frac{\sqrt{2}}{2}$, $y = 2 \sin\frac{\pi}{4} \sin\frac{\pi}{3} = \frac{\sqrt{6}}{2}$,
$z = 2 \cos\frac{\pi}{4} = \sqrt{2}$ so the point is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}, \sqrt{2}\right)$ in
rectangular coordinates.

2. (a)



$x = 5 \sin\frac{\pi}{2} \cos\pi = -5$, $y = 5 \sin\frac{\pi}{2} \sin\pi = 0$,
$z = 5 \cos\frac{\pi}{2} = 0$ so the point is $(-5, 0, 0)$ in
rectangular coordinates.

(b)



$x = 4 \sin\frac{\pi}{3} \cos\frac{3\pi}{4} = 4\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{6}$,
$y = 4 \sin\frac{\pi}{3} \sin\frac{3\pi}{4} = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{6}$,
$z = 4 \cos\frac{\pi}{3} = 4\left(\frac{1}{2}\right) = 2$ so the point is $\left(-\sqrt{6}, \sqrt{6}, 2\right)$
in rectangular coordinates.

3. (a) $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 3 + 12} = 4$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2} \quad \Rightarrow \quad \phi = \dfrac{\pi}{6}$, and

$\cos\theta = \dfrac{x}{\rho \sin\phi} = \dfrac{1}{4 \sin(\pi/6)} = \dfrac{1}{2} \quad \Rightarrow \quad \theta = \dfrac{\pi}{3}$ [since $y > 0$]. Thus spherical coordinates are $\left(4, \dfrac{\pi}{3}, \dfrac{\pi}{6}\right)$.

(b) $\rho = \sqrt{0 + 1 + 1} = \sqrt{2}$, $\cos\phi = \dfrac{-1}{\sqrt{2}} \quad \Rightarrow \quad \phi = \dfrac{3\pi}{4}$, and $\cos\theta = \dfrac{0}{\sqrt{2}\sin(3\pi/4)} = 0 \quad \Rightarrow \quad \theta = \dfrac{3\pi}{2}$ [since $y < 0$].

Thus spherical coordinates are $\left(\sqrt{2}, \dfrac{3\pi}{2}, \dfrac{3\pi}{4}\right)$.

4. (a) $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0+3+1} = 2$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{1}{2} \quad \Rightarrow \quad \phi = \dfrac{\pi}{3}$, and $\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{0}{2\sin(\pi/3)} = 0 \quad \Rightarrow$

$\theta = \dfrac{\pi}{2}$ [since $y > 0$]. Thus spherical coordinates are $\left(2, \dfrac{\pi}{2}, \dfrac{\pi}{3}\right)$.

(b) $\rho = \sqrt{1+1+6} = 2\sqrt{2}$, $\cos\phi = \dfrac{\sqrt{6}}{2\sqrt{2}} = \dfrac{\sqrt{3}}{2} \quad \Rightarrow \quad \phi = \dfrac{\pi}{6}$, and $\cos\theta = \dfrac{-1}{2\sqrt{2}\sin(\pi/6)} = -\dfrac{1}{\sqrt{2}} \quad \Rightarrow$

$\theta = \dfrac{3\pi}{4}$ [since $y > 0$]. Thus spherical coordinates are $\left(2\sqrt{2}, \dfrac{3\pi}{4}, \dfrac{\pi}{6}\right)$.

5. Since $\phi = \frac{\pi}{3}$, the surface is the top half of the right circular cone with vertex at the origin and axis the positive z-axis.

6. Since $\rho = 3$, $x^2+y^2+z^2 = 9$ and the surface is a sphere with center the origin and radius 3.

7. $\rho = \sin\theta\sin\phi \quad \Rightarrow \quad \rho^2 = \rho\sin\theta\sin\phi \quad \Leftrightarrow \quad x^2+y^2+z^2 = y \quad \Leftrightarrow \quad x^2+y^2-y+\frac{1}{4}+z^2 = \frac{1}{4} \quad \Leftrightarrow$

$x^2 + (y-\frac{1}{2})^2 + z^2 = \frac{1}{4}$. Therefore, the surface is a sphere of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2}, 0)$.

8. $\rho^2\left(\sin^2\phi\sin^2\theta + \cos^2\phi\right) = 9 \quad \Leftrightarrow \quad (\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi)^2 = 9 \quad \Leftrightarrow \quad y^2+z^2 = 9$. Thus the surface is a circular cylinder of radius 3 with axis the x-axis.

9. (a) $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, so the equation $z^2 = x^2+y^2$ becomes

$(\rho\cos\phi)^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$ or $\rho^2\cos^2\phi = \rho^2\sin^2\phi$. If $\rho \neq 0$, this becomes $\cos^2\phi = \sin^2\phi$. ($\rho = 0$ corresponds to the origin which is included in the surface.) There are many equivalent equations in spherical coordinates, such as $\tan^2\phi = 1$, $2\cos^2\phi = 1$, $\cos 2\phi = 0$, or even $\phi = \frac{\pi}{4}$, $\phi = \frac{3\pi}{4}$.

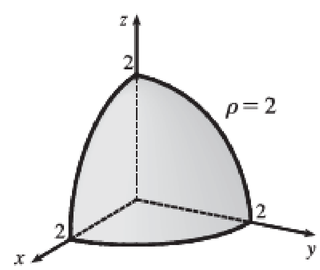
(b) $x^2+z^2 = 9 \quad \Leftrightarrow \quad (\rho\sin\phi\cos\theta)^2 + (\rho\cos\phi)^2 = 9 \quad \Leftrightarrow \quad \rho^2\sin^2\phi\cos^2\theta + \rho^2\cos^2\phi = 9$ or

$\rho^2\left(\sin^2\phi\cos^2\theta + \cos^2\phi\right) = 9$.

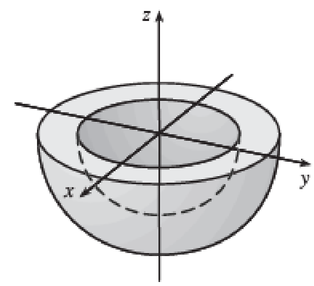
10. (a) $x^2 - 2x + y^2 + z^2 = 0 \quad \Leftrightarrow \quad (x^2+y^2+z^2) - 2x = 0 \quad \Leftrightarrow \quad \rho^2 - 2(\rho\sin\phi\cos\theta) = 0$ or $\rho = 2\sin\phi\cos\theta$.

(b) $x + 2y + 3z = 1 \quad \Leftrightarrow \quad \rho\sin\phi\cos\theta + 2\rho\sin\phi\sin\theta + 3\rho\cos\phi = 1$ or $\rho = 1/\left(\sin\phi\cos\theta + 2\sin\phi\sin\theta + 3\cos\phi\right)$.

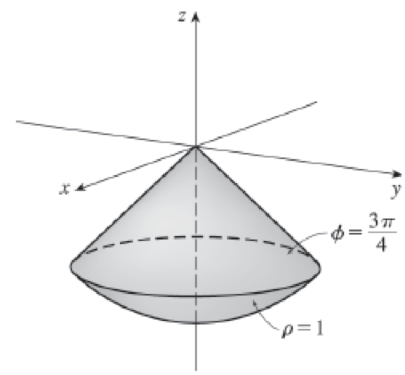
11. $\rho = 2$ represents a sphere of radius 2, centered at the origin, so $\rho \leq 2$ is this sphere and its interior. $0 \leq \phi \leq \frac{\pi}{2}$ restricts the solid to that portion of the region that lies on or above the xy-plane, and $0 \leq \theta \leq \frac{\pi}{2}$ further restricts the solid to the first octant. Thus the solid is the portion in the first octant of the solid ball centered at the origin with radius 2.



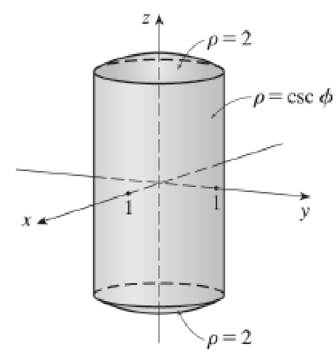
12. $2 \le \rho \le 3$ represents the solid region between and including the spheres of radii 2 and 3, centered at the origin. $\frac{\pi}{2} \le \phi \le \pi$ restricts the solid to that portion on or below the xy-plane.



13. $\rho \le 1$ represents the solid sphere of radius 1 centered at the origin. $\frac{3\pi}{4} \le \phi \le \pi$ restricts the solid to that portion on or below the cone $\phi = \frac{3\pi}{4}$.



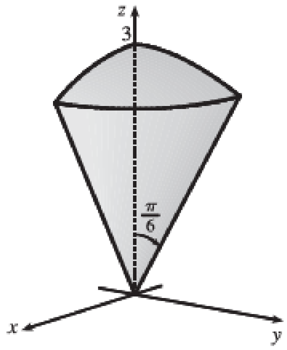
14. $\rho \le 2$ represents the solid sphere of radius 2 centered at the origin. Notice that $x^2 + y^2 = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi$. Then $\rho = \csc\phi \quad \Rightarrow \quad \rho\sin\phi = 1 \quad \Rightarrow \quad \rho^2\sin^2\phi = x^2 + y^2 = 1$, so $\rho \le \csc\phi$ restricts the solid to that portion on or inside the circular cylinder $x^2 + y^2 = 1$.



15. $z \ge \sqrt{x^2 + y^2}$ because the solid lies above the cone. Squaring both sides of this inequality gives $z^2 \ge x^2 + y^2 \quad \Rightarrow$ $2z^2 \ge x^2 + y^2 + z^2 = \rho^2 \quad \Rightarrow \quad z^2 = \rho^2\cos^2\phi \ge \frac{1}{2}\rho^2 \quad \Rightarrow \quad \cos^2\phi \ge \frac{1}{2}$. The cone opens upward so that the inequality is $\cos\phi \ge \frac{1}{\sqrt{2}}$, or equivalently $0 \le \phi \le \frac{\pi}{4}$. In spherical coordinates the sphere $z = x^2 + y^2 + z^2$ is $\rho\cos\phi = \rho^2 \quad \Rightarrow$ $\rho = \cos\phi$. $0 \le \rho \le \cos\phi$ because the solid lies below the sphere. The solid can therefore be described as the region in spherical coordinates satisfying $0 \le \rho \le \cos\phi$, $0 \le \phi \le \frac{\pi}{4}$.

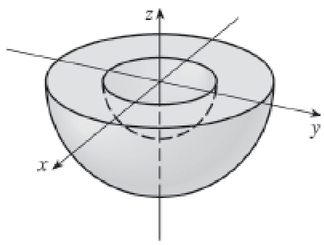
16. (a) The hollow ball is a spherical shell with outer radius 15 cm and inner radius 14.5 cm. If we center the ball at the origin of the coordinate system and use centimeters as the unit of measurement, then spherical coordinates conveniently describe the hollow ball as $14.5 \le \rho \le 15$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

(b) If we position the ball as in part (a), one possibility is to take the half of the ball that is above the xy-plane which is described by $14.5 \le \rho \le 15$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/2$.

17. 

The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 3, 0 \le \theta \le \pi/2, 0 \le \phi \le \pi/6\}$. This represents the solid region in the first octant bounded above by the sphere $\rho = 3$ and below by the cone $\phi = \pi/6$.

$$\begin{aligned}\int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi &= \int_0^{\pi/6} \sin\phi \, d\phi \int_0^{\pi/2} d\theta \int_0^3 \rho^2 \, d\rho \\ &= \left[-\cos\phi\right]_0^{\pi/6} \left[\theta\right]_0^{\pi/2} \left[\tfrac{1}{3}\rho^3\right]_0^3 \\ &= \left(1 - \frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2}\right)(9) = \frac{9\pi}{4}(2 - \sqrt{3})\end{aligned}$$

18. 

The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 1 \le \rho \le 2, 0 \le \theta \le 2\pi, \pi/2 \le \phi \le \pi\}$. This represents the solid region between the spheres $\rho = 1$ and $\rho = 2$ and below the xy-plane.

$$\begin{aligned}\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta &= \int_0^{2\pi} d\theta \int_{\pi/2}^{\pi} \sin\phi \, d\phi \int_1^2 \rho^2 \, d\rho \\ &= \left[\theta\right]_0^{2\pi} \left[-\cos\phi\right]_{\pi/2}^{\pi} \left[\tfrac{1}{3}\rho^3\right]_1^2 \\ &= 2\pi(1)\left(\tfrac{7}{3}\right) = \tfrac{14\pi}{3}\end{aligned}$$

19. The solid E is most conveniently described if we use cylindrical coordinates:

$E = \left\{(r, \theta, z) \mid 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 3, 0 \le z \le 2\right\}$. Then

$\iiint_E f(x, y, z) \, dV = \int_0^{\pi/2} \int_0^3 \int_0^2 f(r\cos\theta, r\sin\theta, z) \, r \, dz \, dr \, d\theta$.

20. The solid E is most conveniently described if we use spherical coordinates:

$E = \left\{(\rho, \theta, \phi) \mid 1 \le \rho \le 2, \frac{\pi}{2} \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}\right\}$. Then

$\iiint_E f(x, y, z) \, dV = \int_0^{\pi/2} \int_{\pi/2}^{2\pi} \int_1^2 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$.

21. In spherical coordinates, B is represented by $\{(\rho, \theta, \phi) \mid 0 \le \rho \le 5, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$. Thus

$$\begin{aligned}\iiint_B (x^2 + y^2 + z^2)^2 \, dV &= \int_0^{\pi} \int_0^{2\pi} \int_0^5 (\rho^2)^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^5 \rho^6 \, d\rho \\ &= \left[-\cos\phi\right]_0^{\pi} \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{7}\rho^7\right]_0^5 = (2)(2\pi)\left(\tfrac{78{,}125}{7}\right) \\ &= \tfrac{312{,}500}{7}\pi \approx 140{,}249.7\end{aligned}$$

22. In spherical coordinates, H is represented by $\{(\rho,\theta,\phi)\mid 0\le\rho\le 3, 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{2}\}$. Thus

$$\begin{aligned}\iiint_H (9-x^2-y^2)\,dV &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^3 \left[9-(\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta)\right]\ \rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^3 (9-\rho^2\sin^2\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \int_0^{\pi/2}\int_0^{2\pi}\left[3\rho^3-\tfrac{1}{5}\rho^5\sin^2\phi\right]_{\rho=0}^{\rho=3}\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \int_0^{\pi/2}\int_0^{2\pi}\left(81\sin\phi-\tfrac{243}{5}\sin^3\phi\right)d\theta\,d\phi \\ &= \int_0^{2\pi}d\theta\int_0^{\pi/2}\left[81\sin\phi-\tfrac{243}{5}(1-\cos^2\phi)\sin\phi\right]d\phi \\ &= 2\pi\left[-81\cos\phi-\tfrac{243}{5}\left(\tfrac{1}{3}\cos^3\phi-\cos\phi\right)\right]_0^{\pi/2} \\ &= 2\pi\left[0+81+\tfrac{243}{5}\left(-\tfrac{2}{3}\right)\right]=\tfrac{486}{5}\pi\end{aligned}$$

23. In spherical coordinates, E is represented by $\{(\rho,\theta,\phi)\mid 1\le\rho\le 2, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}\}$. Thus

$$\iiint_E z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\int_0^{\pi/2}d\theta\int_1^2\rho^3\,d\rho$$

$$= \left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi/2}\left[\theta\right]_0^{\pi/2}\left[\tfrac{1}{4}\rho^4\right]_1^2 = \left(\tfrac{1}{2}\right)\left(\tfrac{\pi}{2}\right)\left(\tfrac{15}{4}\right)=\tfrac{15\pi}{16}$$

24. $$\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 e^{\sqrt{\rho^2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3\rho^2e^\rho\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi/2}d\theta\int_0^{\pi/2}\sin\phi\,d\phi\int_0^3\rho^2e^\rho\,d\rho = \left[\theta\right]_0^{\pi/2}\left[-\cos\phi\right]_0^{\pi/2}\left[(\rho^2-2\rho+2)e^\rho\right]_0^3 \quad \text{[integrate by parts twice]}$$

$$= \tfrac{\pi}{2}(0+1)(5e^3-2)=\tfrac{\pi}{2}(5e^3-2)$$

25. $$\iiint_E x^2\,dV = \int_0^\pi\int_0^\pi\int_3^4(\rho\sin\phi\cos\theta)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^\pi\cos^2\theta\,d\theta\int_0^\pi\sin^3\phi\,d\phi\int_3^4\rho^4\,d\rho$$

$$= \left[\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta\right]_0^\pi\left[-\tfrac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^\pi\left[\tfrac{1}{5}\rho^5\right]_3^4 = \left(\tfrac{\pi}{2}\right)\left(\tfrac{2}{3}+\tfrac{2}{3}\right)\tfrac{1}{5}(4^5-3^5)=\tfrac{1562}{15}\pi$$

26. $$\iiint_E xyz\,dV = \int_0^{\pi/3}\int_0^{2\pi}\int_2^4(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi/3}\sin^3\phi\cos\phi\,d\phi\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\int_2^4\rho^5\,d\rho = \left[\tfrac{1}{4}\sin^4\phi\right]_0^{\pi/3}\left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi}\left[\tfrac{1}{6}\rho^6\right]_2^4=0$$

27. The solid region is given by $E=\{(\rho,\theta,\phi)\mid 0\le\rho\le a, 0\le\theta\le 2\pi, \frac{\pi}{6}\le\phi\le\frac{\pi}{3}\}$ and its volume is

$$V=\iiint_E dV = \int_{\pi/6}^{\pi/3}\int_0^{2\pi}\int_0^a\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/6}^{\pi/3}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^a\rho^2\,d\rho$$

$$= \left[-\cos\phi\right]_{\pi/6}^{\pi/3}\left[\theta\right]_0^{2\pi}\left[\tfrac{1}{3}\rho^3\right]_0^a = \left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\right)(2\pi)\left(\tfrac{1}{3}a^3\right)=\tfrac{\sqrt{3}-1}{3}\pi a^3$$

28. If we center the ball at the origin, then the ball is given by $B=\{(\rho,\theta,\phi)\mid 0\le\rho\le a, 0\le\theta\le 2\pi, 0\le\phi\le\pi\}$ and the distance from any point (x,y,z) in the ball to the center $(0,0,0)$ is $\sqrt{x^2+y^2+z^2}=\rho$. Thus the average distance is

$$\frac{1}{V(B)}\iiint_B\rho\,dV = \frac{1}{\frac{4}{3}\pi a^3}\int_0^\pi\int_0^{2\pi}\int_0^a\rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{3}{4\pi a^3}\int_0^\pi\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^a\rho^3\,d\rho$$

$$= \frac{3}{4\pi a^3}\left[-\cos\phi\right]_0^\pi\left[\theta\right]_0^{2\pi}\left[\tfrac{1}{4}\rho^4\right]_0^a = \frac{3}{4\pi a^3}(2)(2\pi)\left(\tfrac{1}{4}a^4\right)=\tfrac{3}{4}a$$

29. (a) Since $\rho = 4\cos\phi$ implies $\rho^2 = 4\rho\cos\phi$, the equation is that of a sphere of radius 2 with center at $(0,0,2)$. Thus

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left[\tfrac{1}{3}\rho^3\right]_{\rho=0}^{\rho=4\cos\phi} \sin\phi \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left(\tfrac{64}{3}\cos^3\phi\right)\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\tfrac{16}{3}\cos^4\phi\right]_{\phi=0}^{\phi=\pi/3} d\theta = \int_0^{2\pi} -\tfrac{16}{3}\left(\tfrac{1}{16}-1\right) d\theta = 5\theta\Big]_0^{2\pi} = 10\pi$$

(b) By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^3 \cos\phi\sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \cos\phi\sin\phi\left(64\cos^4\phi\right) d\phi \, d\theta$$

$$= \int_0^{2\pi} 64\left[-\tfrac{1}{6}\cos^6\phi\right]_{\phi=0}^{\phi=\pi/3} d\theta = \int_0^{2\pi} \tfrac{21}{2} \, d\theta = 21\pi$$

Hence $(\overline{x}, \overline{y}, \overline{z}) = (0, 0, 2.1)$.

30. In spherical coordinates, the sphere $x^2+y^2+z^2=4$ is equivalent to $\rho = 2$ and the cone $z = \sqrt{x^2+y^2}$ is represented by $\phi = \frac{\pi}{4}$. Thus, the solid is given by $\left\{(\rho,\theta,\phi) \,\middle|\, 0 \le \rho \le 2, 0 \le \theta \le 2\pi, \frac{\pi}{4} \le \phi \le \frac{\pi}{2}\right\}$ and

$$V = \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2 \rho^2\sin\phi \, d\rho \, d\theta \, d\phi = \int_{\pi/4}^{\pi/2}\sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^2 \rho^2 \, d\rho$$

$$= \left[-\cos\phi\right]_{\pi/4}^{\pi/2} \left[\,\theta\,\right]_0^{2\pi} \left[\tfrac{1}{3}\rho^3\right]_0^2 = \left(\tfrac{\sqrt{2}}{2}\right)(2\pi)\left(\tfrac{8}{3}\right) = \tfrac{8\sqrt{2}\pi}{3}$$

31. By the symmetry of the region, $M_{xy} = 0$ and $M_{yz} = 0$. Assuming constant density K,

$$m = \iiint_E KV = K\int_0^{\pi}\int_0^{\pi}\int_3^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = K\int_0^{\pi} d\theta \int_0^{\pi}\sin\phi \, d\phi \int_3^4 \rho^2 \, d\rho$$

$$= K\pi\left[-\cos\phi\right]_0^{\pi}\left[\tfrac{1}{3}\rho^3\right]_3^4 = 2K\pi \cdot \tfrac{37}{3} = \tfrac{74}{3}\pi K$$

and $$M_{xz} = \iiint_E y\,K\,dV = K\int_0^{\pi}\int_0^{\pi}\int_3^4 (\rho\sin\phi\sin\theta)\,\rho^2\sin\phi \, d\rho \, d\phi \, d\theta = K\int_0^{\pi}\sin\theta \, d\theta \int_0^{\pi}\sin^2\phi \, d\phi \int_3^4 \rho^3 \, d\rho$$

$$= K\left[-\cos\theta\right]_0^{\pi}\left[\tfrac{1}{2}\phi - \tfrac{1}{4}\sin 2\phi\right]_0^{\pi}\left[\tfrac{1}{4}\rho^4\right]_3^4 = K(2)\left(\tfrac{\pi}{2}\right)\tfrac{1}{4}(256-81) = \tfrac{175}{4}\pi K$$

Thus the centroid is $(\overline{x}, \overline{y}, \overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(0, \dfrac{175\pi K/4}{74\pi K/3}, 0\right) = \left(0, \frac{525}{296}, 0\right)$.

32. (a) Placing the center of the base at $(0,0,0)$, $\rho(x,y,z) = K\sqrt{x^2+y^2+z^2}$ is the density function. So

$$m = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^3\sin\phi \, d\rho \, d\phi \, d\theta = K\int_0^{2\pi} d\theta \int_0^{\pi/2}\sin\phi \, d\phi \int_0^a \rho^3 \, d\rho$$

$$= K\left[\,\theta\,\right]_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/2}\left[\tfrac{1}{4}\rho^4\right]_0^a = K(2\pi)(1)\left(\tfrac{1}{4}a^4\right) = \tfrac{1}{2}\pi K a^4$$

(b) By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^4\sin\phi\cos\phi \, d\rho \, d\phi \, d\theta = K\int_0^{2\pi} d\theta \int_0^{\pi/2}\sin\phi\cos\phi \, d\phi \int_0^a \rho^4 \, d\rho$$

$$= K\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi/2}\left[\tfrac{1}{5}\rho^5\right]_0^a = K(2\pi)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{5}a^5\right) = \tfrac{1}{5}\pi K a^5$$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{2}{5}a\right)$.

(c) $$I_z = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (K\rho^3\sin\phi)(\rho^2\sin^2\phi) \, d\rho \, d\phi \, d\theta = K\int_0^{2\pi} d\theta \int_0^{\pi/2}\sin^3\phi \, d\phi \int_0^a \rho^5 \, d\rho$$

$$= K\left[\,\theta\,\right]_0^{2\pi}\left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi/2}\left[\tfrac{1}{6}\rho^6\right]_0^a = K(2\pi)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{6}a^6\right) = \tfrac{2}{9}\pi K a^6$$

33. (a) The density function is $\rho(x,y,z) = K$, a constant, and by the symmetry of the problem $M_{xz} = M_{yz} = 0$. Then

$M_{xy} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a K\rho^3 \sin\phi\,\cos\phi\,d\rho\,d\phi\,d\theta = \frac{1}{2}\pi K a^4 \int_0^{\pi/2} \sin\phi\,\cos\phi\,d\phi = \frac{1}{8}\pi K a^4$. But the mass is K(volume of the hemisphere) $= \frac{2}{3}\pi K a^3$, so the centroid is $\left(0, 0, \frac{3}{8}a\right)$.

(b) Place the center of the base at $(0,0,0)$; the density function is $\rho(x,y,z) = K$. By symmetry, the moments of inertia about any two such diameters will be equal, so we just need to find I_x:

$$
\begin{aligned}
I_x &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a (K\rho^2 \sin\phi)\,\rho^2\,(\sin^2\phi\,\sin^2\theta + \cos^2\phi)\,d\rho\,d\phi\,d\theta \\
&= K \int_0^{2\pi} \int_0^{\pi/2} (\sin^3\phi\,\sin^2\theta + \sin\phi\,\cos^2\phi)\left(\tfrac{1}{5}a^5\right) d\phi\,d\theta \\
&= \tfrac{1}{5}Ka^5 \int_0^{2\pi} \left[\sin^2\theta\left(-\cos\phi + \tfrac{1}{3}\cos^3\phi\right) + \left(-\tfrac{1}{3}\cos^3\phi\right)\right]_{\phi=0}^{\phi=\pi/2} d\theta = \tfrac{1}{5}Ka^5 \int_0^{2\pi} \left[\tfrac{2}{3}\sin^2\theta + \tfrac{1}{3}\right] d\theta \\
&= \tfrac{1}{5}Ka^5 \left[\tfrac{2}{3}\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right) + \tfrac{1}{3}\theta\right]_0^{2\pi} = \tfrac{1}{5}Ka^5\left[\tfrac{2}{3}(\pi - 0) + \tfrac{1}{3}(2\pi - 0)\right] = \tfrac{4}{15}Ka^5\pi
\end{aligned}
$$

34. Place the center of the base at $(0,0,0)$, then the density is $\rho(x,y,z) = Kz$, K a constant. Then

$m = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a (K\rho\cos\phi)\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 2\pi K \int_0^{\pi/2} \cos\phi\sin\phi \cdot \frac{1}{4}a^4\,d\phi = \frac{1}{2}\pi K a^4 \left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \frac{\pi}{4}Ka^4$.

By the symmetry of the problem $M_{xz} = M_{yz} = 0$, and

$M_{xy} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a K\rho^4 \cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2}{5}\pi K a^5 \int_0^{\pi/2} \cos^2\phi\sin\phi\,d\phi = \frac{2}{5}\pi K a^5 \left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \frac{2}{15}\pi K a^5$.

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{8}{15}a\right)$.

35. In spherical coordinates $z = \sqrt{x^2 + y^2}$ becomes $\cos\phi = \sin\phi$ or $\phi = \frac{\pi}{4}$. Then

$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi\,d\phi \int_0^1 \rho^2\,d\rho = \frac{1}{3}\pi\left(2 - \sqrt{2}\right)$,

$M_{xy} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^3 \sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = 2\pi\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/4}\left(\frac{1}{4}\right) = \frac{\pi}{8}$ and by symmetry $M_{yz} = M_{xz} = 0$.

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \dfrac{3}{8\left(2 - \sqrt{2}\right)}\right)$.

36. Place the center of the sphere at $(0,0,0)$, let the diameter of intersection be along the z-axis, one of the planes be the xz-plane and the other be the plane whose angle with the xz-plane is $\theta = \frac{\pi}{6}$. Then in spherical coordinates the volume is given by

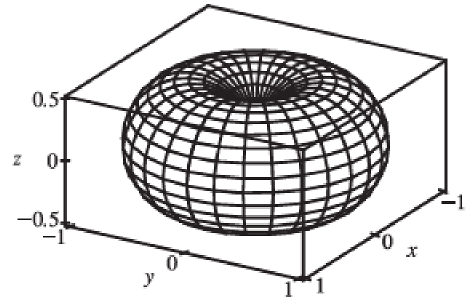
$V = \int_0^{\pi/6} \int_0^{\pi} \int_0^a \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/6} d\theta \int_0^{\pi} \sin\phi\,d\phi \int_0^a \rho^2\,d\rho = \frac{\pi}{6}(2)\left(\frac{1}{3}a^3\right) = \frac{1}{9}\pi a^3$.

37. In cylindrical coordinates the paraboloid is given by $z = r^2$ and the plane by $z = 2r\sin\theta$ and they intersect in the circle $r = 2\sin\theta$. Then $\iiint_E z\,dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_{r^2}^{2r\sin\theta} rz\,dz\,dr\,d\theta = \frac{5\pi}{6}$ [using a CAS].

38. (a) The region enclosed by the torus is $\{(\rho,\theta,\phi) \mid 0 \le \theta \le 2\pi, 0 \le \phi \le \pi, 0 \le \rho \le \sin\phi\}$, so its volume is

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\phi} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = 2\pi \int_0^{\pi} \tfrac{1}{3}\sin^4\phi\, d\phi = \tfrac{2}{3}\pi\left[\tfrac{3}{8}\phi - \tfrac{1}{4}\sin 2\phi + \tfrac{1}{16}\sin 4\phi\right]_0^{\pi} = \tfrac{1}{4}\pi^2.$$

(b) In Maple, we can plot the torus using the `coords=spherical` option in a regular `plot3d` command. In Mathematica, use `ParametricPlot3D`.



39. The region E of integration is the region above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 2$ in the first octant. Because E is in the first octant we have $0 \le \theta \le \frac{\pi}{2}$. The cone has equation $\phi = \frac{\pi}{4}$ (as in Example 4), so $0 \le \phi \le \frac{\pi}{4}$, and $0 \le \rho \le \sqrt{2}$. So the integral becomes

$$\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} (\rho\sin\phi\cos\theta)\,(\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/4}\sin^3\phi\, d\phi \int_0^{\pi/2}\sin\theta\cos\theta\, d\theta \int_0^{\sqrt{2}} \rho^4\, d\rho = \left(\int_0^{\pi/4}(1-\cos^2\phi)\sin\phi\, d\phi\right)\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2}\left[\tfrac{1}{5}\rho^5\right]_0^{\sqrt{2}}$$

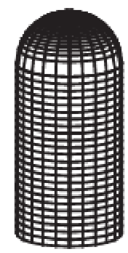
$$= \left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi/4}\cdot\tfrac{1}{2}\cdot\tfrac{1}{5}\left(\sqrt{2}\right)^5 = \left[\tfrac{\sqrt{2}}{12} - \tfrac{\sqrt{2}}{2} - \left(\tfrac{1}{3}-1\right)\right]\cdot\tfrac{2\sqrt{2}}{5} = \tfrac{4\sqrt{2}-5}{15}$$

40. The region of integration is the solid sphere $x^2+y^2+z^2 \le a^2$, so $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, and $0 \le \rho \le a$. Also $x^2z + y^2z + z^3 = (x^2+y^2+z^2)z = \rho^2 z = \rho^3\cos\phi$, so the integral becomes

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{a}\left(\rho^3\cos\phi\right)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi = \int_0^{\pi}\sin\phi\cos\phi\, d\phi\int_0^{2\pi} d\theta \int_0^{a}\rho^5\, d\rho = \left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi}\left[\theta\right]_0^{2\pi}\left[\tfrac{1}{6}\rho^6\right]_0^{a} = 0$$

41. In cylindrical coordinates, the equation of the cylinder is $r = 3$, $0 \le z \le 10$.

The hemisphere is the upper part of the sphere radius 3, center $(0,0,10)$, equation $r^2 + (z-10)^2 = 3^2$, $z \ge 10$. In Maple, we can use the `coords=cylindrical` option in a regular `plot3d` command. In Mathematica, we can use `ParametricPlot3D`.



42. We begin by finding the positions of Los Angeles and Montréal in spherical coordinates, using the method described in the exercise:

Montréal	Los Angeles
$\rho = 3960$ mi	$\rho = 3960$ mi
$\theta = 360° - 73.60° = 286.40°$	$\theta = 360° - 118.25° = 241.75°$
$\phi = 90° - 45.50° = 44.50°$	$\phi = 90° - 34.06° = 55.94°$

Now we change the above to Cartesian coordinates using $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$ to get two position vectors of length 3960 mi (since both cities must lie on the surface of the Earth). In particular:

$$\text{Montréal: } \langle 783.67, -2662.67, 2824.47\rangle \qquad \text{Los Angeles: } \langle -1552.80, -2889.91, 2217.84\rangle$$

To find the angle α between these two vectors we use the dot product:

$\langle 783.67, -2662.67, 2824.47\rangle \cdot \langle -1552.80, -2889.91, 2217.84\rangle = (3960)^2\cos\alpha \quad\Rightarrow\quad \cos\alpha \approx 0.8126 \quad\Rightarrow$
$\alpha \approx 0.6223$ rad. The great circle distance between the cities is $s = \rho\theta \approx 3960(0.6223) \approx 2464$ mi.

43. If E is the solid enclosed by the surface $\rho = 1 + \frac{1}{5}\sin 6\theta\,\sin 5\phi$, it can be described in spherical coordinates as $E = \left\{(\rho,\theta,\phi) \mid 0 \le \rho \le 1 + \frac{1}{5}\sin 6\theta\sin 5\phi, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\right\}$. Its volume is given by
$V(E) = \iiint_E dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{1+(\sin 6\theta\,\sin 5\phi)/5} \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{136\pi}{99}$ [using a CAS].

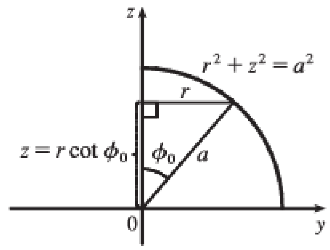
44. The given integral is equal to $\lim\limits_{R\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho e^{-\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \lim\limits_{R\to\infty}\left(\int_0^{2\pi}d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^{R}\rho^3 e^{-\rho^2}\,d\rho\right)$.

Now use integration by parts with $u = \rho^2$, $dv = \rho e^{-\rho^2}\,d\rho$ to get

$$\lim_{R\to\infty} 2\pi(2)\left(\rho^2\left(-\tfrac{1}{2}\right)e^{-\rho^2}\Big]_0^R - \int_0^R 2\rho\left(-\tfrac{1}{2}\right)e^{-\rho^2}\,d\rho\right) = \lim_{R\to\infty} 4\pi\left(-\tfrac{1}{2}R^2e^{-R^2} + \left[-\tfrac{1}{2}e^{-\rho^2}\right]_0^R\right)$$
$$= 4\pi\lim_{R\to\infty}\left[-\tfrac{1}{2}R^2e^{-R^2} - \tfrac{1}{2}e^{-R^2} + \tfrac{1}{2}\right] = 4\pi\left(\tfrac{1}{2}\right) = 2\pi$$

(Note that $R^2e^{-R^2} \to 0$ as $R \to \infty$ by l'Hospital's Rule.)

45. (a) From the diagram, $z = r\cot\phi_0$ to $z = \sqrt{a^2 - r^2}$, $r = 0$ to $r = a\sin\phi_0$ (or use $a^2 - r^2 = r^2\cot^2\phi_0$). Thus



$$V = \int_0^{2\pi}\int_0^{a\sin\phi_0}\int_{r\cot\phi_0}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$

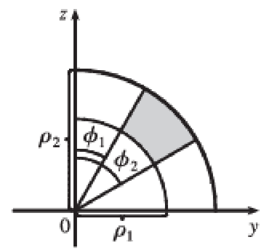
$$= 2\pi\int_0^{a\sin\phi_0}\left(r\sqrt{a^2-r^2} - r^2\cot\phi_0\right)dr$$

$$= \tfrac{2\pi}{3}\left[-(a^2-r^2)^{3/2} - r^3\cot\phi_0\right]_0^{a\sin\phi_0}$$

$$= \tfrac{2\pi}{3}\left[-\left(a^2 - a^2\sin^2\phi_0\right)^{3/2} - a^3\sin^3\phi_0\cot\phi_0 + a^3\right]$$

$$= \tfrac{2}{3}\pi a^3\left[1 - \left(\cos^3\phi_0 + \sin^2\phi_0\cos\phi_0\right)\right] = \tfrac{2}{3}\pi a^3(1 - \cos\phi_0)$$

(b) The wedge in question is the shaded area rotated from $\theta = \theta_1$ to $\theta = \theta_2$. Letting



V_{ij} = volume of the region bounded by the sphere of radius ρ_i and the cone with angle ϕ_j ($\theta = \theta_1$ to θ_2)

and letting V be the volume of the wedge, we have

$$V = (V_{22} - V_{21}) - (V_{12} - V_{11})$$

$$= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\rho_2^3(1-\cos\phi_2) - \rho_2^3(1-\cos\phi_1) - \rho_1^3(1-\cos\phi_2) + \rho_1^3(1-\cos\phi_1)\right]$$

$$= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\left(\rho_2^3 - \rho_1^3\right)(1-\cos\phi_2) - \left(\rho_2^3 - \rho_1^3\right)(1-\cos\phi_1)\right] = \tfrac{1}{3}(\theta_2 - \theta_1)\left[\left(\rho_2^3 - \rho_1^3\right)(\cos\phi_1 - \cos\phi_2)\right]$$

Or: Show that $V = \displaystyle\int_{\theta_1}^{\theta_2}\int_{\rho_1\sin\phi_1}^{\rho_2\sin\phi_2}\int_{r\cot\phi_2}^{r\cot\phi_1} r\,dz\,dr\,d\theta$.

(c) By the Mean Value Theorem with $f(\rho) = \rho^3$ there exists some $\tilde{\rho}$ with $\rho_1 \le \tilde{\rho} \le \rho_2$ such that $f(\rho_2) - f(\rho_1) = f'(\tilde{\rho})(\rho_2 - \rho_1)$ or $\rho_1^3 - \rho_2^3 = 3\tilde{\rho}^2\Delta\rho$. Similarly there exists ϕ with $\phi_1 \le \tilde{\phi} \le \phi_2$ such that $\cos\phi_2 - \cos\phi_1 = \left(-\sin\tilde{\phi}\right)\Delta\phi$. Substituting into the result from (b) gives

$\Delta V = (\tilde{\rho}^2\,\Delta\rho)(\theta_2 - \theta_1)(\sin\tilde{\phi})\,\Delta\phi = \tilde{\rho}^2\sin\tilde{\phi}\,\Delta\rho\,\Delta\phi\,\Delta\theta$.