1. $x = 5u - v,\ y = u + 3v.$

The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix} = 5(3) - (-1)(1) = 16.$

2. $x = uv,\ y = u/v.$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} v & u \\ 1/v & -u/v^2 \end{vmatrix} = v\left(-\frac{u}{v^2}\right) - u\left(\frac{1}{v}\right) = -\frac{u}{v} - \frac{u}{v} = -\frac{2u}{v}$$

3. $x = e^{-r}\sin\theta,\ y = e^{r}\cos\theta.$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} -e^{-r}\sin\theta & e^{-r}\cos\theta \\ e^{r}\cos\theta & -e^{r}\sin\theta \end{vmatrix} = e^{-r}e^{r}\sin^2\theta - e^{-r}e^{r}\cos^2\theta = \sin^2\theta - \cos^2\theta \text{ or } -\cos 2\theta$$

4. $x = e^{s+t},\ y = e^{s-t}.$

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{vmatrix} = \begin{vmatrix} e^{s+t} & e^{s+t} \\ e^{s-t} & -e^{s-t} \end{vmatrix} = -e^{s+t}e^{s-t} - e^{s+t}e^{s-t} = -2e^{2s}$$

5. $x = u/v,\ y = v/w,\ z = w/u.$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} 1/v & -u/v^2 & 0 \\ 0 & 1/w & -v/w^2 \\ -w/u^2 & 0 & 1/u \end{vmatrix}$$

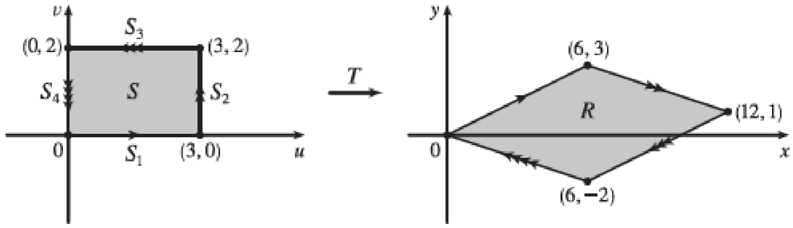
$$= \frac{1}{v}\begin{vmatrix} 1/w & -v/w^2 \\ 0 & 1/u \end{vmatrix} - \left(-\frac{u}{v^2}\right)\begin{vmatrix} 0 & -v/w^2 \\ -w/u^2 & 1/u \end{vmatrix} + 0\begin{vmatrix} 0 & 1/w \\ -w/u^2 & 0 \end{vmatrix}$$

$$= \frac{1}{v}\left(\frac{1}{uw} - 0\right) + \frac{u}{v^2}\left(0 - \frac{v}{u^2 w}\right) + 0 = \frac{1}{uvw} - \frac{1}{uvw} = 0$$

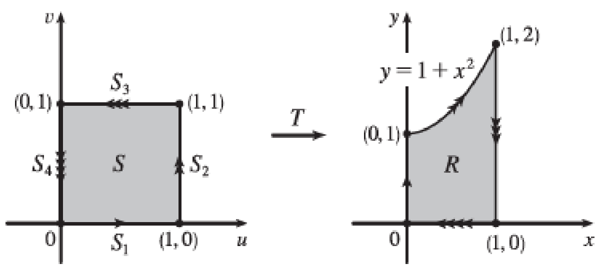
6. $x = v + w^2,\ y = w + u^2,\ z = u + v^2.$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & 1 & 2w \\ 2u & 0 & 1 \\ 1 & 2v & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & 1 \\ 2v & 0 \end{vmatrix} - 1\begin{vmatrix} 2u & 1 \\ 1 & 0 \end{vmatrix} + 2w\begin{vmatrix} 2u & 0 \\ 1 & 2v \end{vmatrix} = 0 - (0 - 1) + 2w\,(4uv - 0) = 1 + 8uvw$$

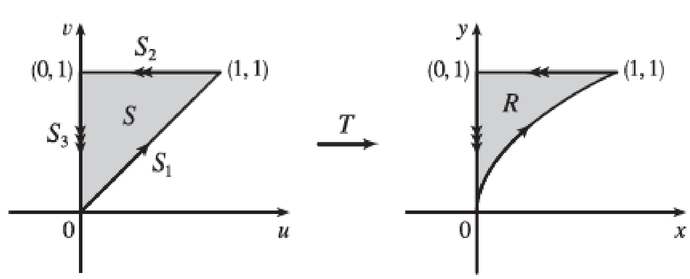
**7.** The transformation maps the boundary of $S$ to the boundary of the image $R$, so we first look at side $S_1$ in the $uv$-plane. $S_1$ is described by $v = 0 \quad [0 \le u \le 3]$, so $x = 2u + 3v = 2u$ and $y = u - v = u$. Eliminating $u$, we have $x = 2y$, $0 \le x \le 6$. $S_2$ is the line segment $u = 3$, $0 \le v \le 2$, so $x = 6 + 3v$ and $y = 3 - v$. Then $v = 3 - y \quad \Rightarrow \quad x = 6 + 3(3 - y) = 15 - 3y$, $6 \le x \le 12$. $S_3$ is the line segment $v = 2$, $0 \le u \le 3$, so $x = 2u + 6$ and $y = u - 2$, giving $u = y + 2 \quad \Rightarrow \quad x = 2y + 10$, $6 \le x \le 12$. Finally, $S_4$ is the segment $u = 0$, $0 \le v \le 2$, so $x = 3v$ and $y = -v \quad \Rightarrow \quad x = -3y$, $0 \le x \le 6$. The image of set $S$ is the region $R$ shown in the $xy$-plane, a parallelogram bounded by these four segments.



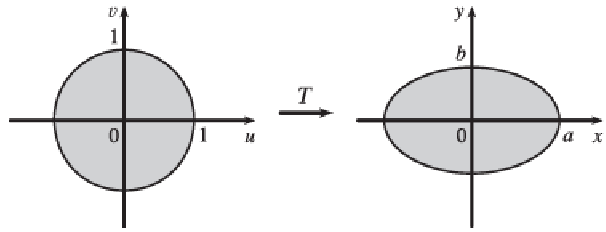

**8.** $S_1$ is the line segment $v = 0$, $0 \le u \le 1$, so $x = v = 0$ and $y = u(1 + v^2) = u$. Since $0 \le u \le 1$, the image is the line segment $x = 0$, $0 \le y \le 1$. $S_2$ is the segment $u = 1$, $0 \le v \le 1$, so $x = v$ and $y = u(1 + v^2) = 1 + x^2$. Thus the image is the portion of the parabola $y = 1 + x^2$ for $0 \le x \le 1$. $S_3$ is the segment $v = 1$, $0 \le u \le 1$, so $x = 1$ and $y = 2u$. The image is the segment $x = 1$, $0 \le y \le 2$. $S_4$ is described by $u = 0$, $0 \le v \le 1$, so $0 \le x = v \le 1$ and $y = u(1 + v^2) = 0$. The image is the line segment $y = 0$, $0 \le x \le 1$. Thus, the image of $S$ is the region $R$ bounded by the parabola $y = 1 + x^2$, the $x$-axis, and the lines $x = 0$, $x = 1$.




**9.** $S_1$ is the line segment $u = v$, $0 \le u \le 1$, so $y = v = u$ and $x = u^2 = y^2$. Since $0 \le u \le 1$, the image is the portion of the parabola $x = y^2$, $0 \le y \le 1$. $S_2$ is the segment $v = 1$, $0 \le u \le 1$, thus $y = v = 1$ and $x = u^2$, so $0 \le x \le 1$. The image is the line segment $y = 1$, $0 \le x \le 1$. $S_3$ is the segment $u = 0$, $0 \le v \le 1$, so $x = u^2 = 0$ and $y = v \quad \Rightarrow \quad 0 \le y \le 1$. The image is the segment $x = 0$, $0 \le y \le 1$. Thus, the image of $S$ is the region $R$ in the first quadrant bounded by the parabola $x = y^2$, the $y$-axis, and the line $y = 1$.



**10.** Substituting $u = \frac{x}{a}$, $v = \frac{y}{b}$ into $u^2 + v^2 \le 1$ gives $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$, so the image of $u^2 + v^2 \le 1$ is the elliptical region $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$.




**11.** $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ and $x - 3y = (2u+v) - 3(u+2v) = -u - 5v$. To find the region $S$ in the $uv$-plane that corresponds to $R$ we first find the corresponding boundary under the given transformation. The line through $(0,0)$ and $(2,1)$ is $y = \frac{1}{2}x$ which is the image of $u + 2v = \frac{1}{2}(2u+v) \;\Rightarrow\; v = 0$; the line through $(2,1)$ and $(1,2)$ is $x + y = 3$ which is the image of $(2u+v) + (u+2v) = 3 \;\Rightarrow\; u + v = 1$; the line through $(0,0)$ and $(1,2)$ is $y = 2x$ which is the image of $u + 2v = 2(2u+v) \;\Rightarrow\; u = 0$. Thus $S$ is the triangle $0 \le v \le 1-u$, $0 \le u \le 1$ in the $uv$-plane and

$$\iint_R (x-3y)\,dA = \int_0^1\int_0^{1-u} (-u-5v)\,|3|\,dv\,du = -3\int_0^1 \left[uv + \tfrac{5}{2}v^2\right]_{v=0}^{v=1-u} du$$

$$= -3\int_0^1 \left(u - u^2 + \tfrac{5}{2}(1-u)^2\right) du = -3\left[\tfrac{1}{2}u^2 - \tfrac{1}{3}u^3 - \tfrac{5}{6}(1-u)^3\right]_0^1 = -3\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{5}{6}\right) = -3$$

**12.** $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = \frac{1}{4}$, $4x + 8y = 4 \cdot \frac{1}{4}(u+v) + 8 \cdot \frac{1}{4}(v - 3u) = 3v - 5u$. $R$ is a parallelogram bounded by the lines $x - y = -4$, $x - y = 4$, $3x + y = 0$, $3x + y = 8$. Since $u = x - y$ and $v = 3x + y$, $R$ is the image of the rectangle enclosed by the lines $u = -4$, $u = 4$, $v = 0$, and $v = 8$. Thus

$$\iint_R (4x+8y)\,dA = \int_{-4}^4\int_0^8 (3v-5u)\left|\tfrac{1}{4}\right| dv\,du = \tfrac{1}{4}\int_{-4}^4 \left[\tfrac{3}{2}v^2 - 5uv\right]_{v=0}^{v=8} du$$

$$= \tfrac{1}{4}\int_{-4}^4 (96 - 40u)\,du = \tfrac{1}{4}\left[96u - 20u^2\right]_{-4}^4 = 192$$

**13.** $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$, $x^2 = 4u^2$ and the planar ellipse $9x^2 + 4y^2 \le 36$ is the image of the disk $u^2 + v^2 \le 1$. Thus

$$\iint_R x^2\,dA = \iint_{u^2+v^2\le 1} (4u^2)(6)\,du\,dv = \int_0^{2\pi}\int_0^1 (24r^2\cos^2\theta)\,r\,dr\,d\theta = 24\int_0^{2\pi}\cos^2\theta\,d\theta\int_0^1 r^3\,dr$$

$$= 24\left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^1 = 24(\pi)\left(\tfrac{1}{4}\right) = 6\pi$$

**14.** $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = \frac{4}{\sqrt{3}}$, $x^2 - xy + y^2 = 2u^2 + 2v^2$ and the planar ellipse $x^2 - xy + y^2 \le 2$ is the image of the disk $u^2 + v^2 \le 1$. Thus

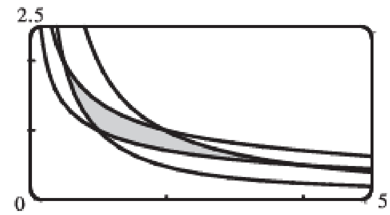
$$\iint_R (x^2 - xy + y^2)\,dA = \iint_{u^2+v^2\le 1} (2u^2 + 2v^2)\left(\tfrac{4}{\sqrt{3}}\,du\,dv\right) = \int_0^{2\pi}\int_0^1 \tfrac{8}{\sqrt{3}}r^3\,dr\,d\theta = \tfrac{4\pi}{\sqrt{3}}$$

**15.** $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$, $xy = u$, $y = x$ is the image of the parabola $v^2 = u$, $y = 3x$ is the image of the parabola $v^2 = 3u$, and the hyperbolas $xy = 1$, $xy = 3$ are the images of the lines $u = 1$ and $u = 3$ respectively. Thus

$$\iint_R xy\,dA = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u\left(\frac{1}{v}\right) dv\,du = \int_1^3 u\left(\ln\sqrt{3u} - \ln\sqrt{u}\right) du = \int_1^3 u \ln\sqrt{3}\,du = 4\ln\sqrt{3} = 2\ln 3.$$

**16.** Here $y = \dfrac{v}{u}$, $x = \dfrac{u^2}{v}$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u/v & -u^2/v^2 \\ -v/u^2 & 1/u \end{vmatrix} = \dfrac{1}{v}$ and $R$ is the image of the square with vertices $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$. So

$$\iint_R y^2\,dA = \int_1^2 \int_1^2 \frac{v^2}{u^2}\left(\frac{1}{v}\right) du\,dv = \int_1^2 \frac{v}{2}\,dv = \frac{3}{4}$$




**17.** (a) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and since $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$, $w = \dfrac{z}{c}$ the solid enclosed by the ellipsoid is the image of the ball $u^2 + v^2 + w^2 \le 1$. So

$$\iiint_E dV = \iiint_{u^2+v^2+w^2 \le 1} abc\,du\,dv\,dw = (abc)(\text{volume of the ball}) = \tfrac{4}{3}\pi abc$$

(b) If we approximate the surface of the earth by the ellipsoid $\dfrac{x^2}{6378^2} + \dfrac{y^2}{6378^2} + \dfrac{z^2}{6356^2} = 1$, then we can estimate the volume of the earth by finding the volume of the solid $E$ enclosed by the ellipsoid. From part (a), this is $\iiint_E dV = \frac{4}{3}\pi(6378)(6378)(6356) \approx 1.083 \times 10^{12}\ \text{km}^3$.

**18.** The moment of intertia about the $z$-axis is $I_z = \iiint_E \left(x^2 + y^2\right) \rho(x,y,z)\,dV$, where $E$ is the solid enclosed by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. As in Exercise 17(a), we use the transformation $x = au$, $y = bv$, $z = cw$, so $\left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| = abc$ and

$$\begin{aligned}
I_z &= \iiint_E \left(x^2 + y^2\right) k\,dV = \iiint_{u^2+v^2+w^2 \le 1} k(a^2u^2 + b^2v^2)(abc)\,du\,dv\,dw \\
&= abck \int_0^\pi \int_0^{2\pi} \int_0^1 (a^2\rho^2 \sin^2\phi \cos^2\theta + b^2\rho^2 \sin^2\phi \sin^2\theta)\,\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi \\
&= abck \left[a^2 \int_0^\pi \int_0^{2\pi} \int_0^1 (\rho^2 \sin^2\phi \cos^2\theta)\,\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi + b^2 \int_0^\pi \int_0^{2\pi} \int_0^1 (\rho^2 \sin^2\phi \sin^2\theta)\,\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi\right] \\
&= a^3bck \int_0^\pi \sin^3\phi\,d\phi \int_0^{2\pi} \cos^2\theta\,d\theta \int_0^1 \rho^4\,d\rho + ab^3ck \int_0^\pi \sin^3\phi\,d\phi \int_0^{2\pi} \sin^2\theta\,d\theta \int_0^1 \rho^4\,d\rho \\
&= a^3bck \left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^\pi \left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \left[\tfrac{1}{5}\rho^5\right]_0^1 + ab^3ck \left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^\pi \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \left[\tfrac{1}{5}\rho^5\right]_0^1 \\
&= a^3bck\left(\tfrac{4}{3}\right)(\pi)\left(\tfrac{1}{5}\right) + ab^3ck\left(\tfrac{4}{3}\right)(\pi)\left(\tfrac{1}{5}\right) = \tfrac{4}{15}\pi\left(a^2 + b^2\right) abck
\end{aligned}$$

**19.** Letting $u = x - 2y$ and $v = 3x - y$, we have $x = \frac{1}{5}(2v - u)$ and $y = \frac{1}{5}(v - 3u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{vmatrix} = \dfrac{1}{5}$ and $R$ is the image of the rectangle enclosed by the lines $u = 0$, $u = 4$, $v = 1$, and $v = 8$. Thus

$$\iint_R \frac{x - 2y}{3x - y}\, dA = \int_0^4 \int_1^8 \frac{u}{v} \left|\frac{1}{5}\right| dv\, du = \frac{1}{5}\int_0^4 u\, du \int_1^8 \frac{1}{v}\, dv = \tfrac{1}{5}\left[\tfrac{1}{2}u^2\right]_0^4 \left[\ln|v|\right]_1^8 = \tfrac{8}{5}\ln 8.$$

**20.** Letting $u = x + y$ and $v = x - y$, we have $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\dfrac{1}{2}$ and $R$ is the image of the rectangle enclosed by the lines $u = 0$, $u = 3$, $v = 0$, and $v = 2$. Thus

$$\iint_R (x + y)\, e^{x^2 - y^2}\, dA = \int_0^3 \int_0^2 u e^{uv} \left|-\tfrac{1}{2}\right| dv\, du = \tfrac{1}{2}\int_0^3 \left[e^{uv}\right]_{v=0}^{v=2} du = \tfrac{1}{2}\int_0^3 (e^{2u} - 1)\, du$$
$$= \tfrac{1}{2}\left[\tfrac{1}{2}e^{2u} - u\right]_0^3 = \tfrac{1}{2}\left(\tfrac{1}{2}e^6 - 3 - \tfrac{1}{2}\right) = \tfrac{1}{4}(e^6 - 7)$$

**21.** Letting $u = y - x$, $v = y + x$, we have $y = \frac{1}{2}(u + v)$, $x = \frac{1}{2}(v - u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\dfrac{1}{2}$ and $R$ is the image of the trapezoidal region with vertices $(-1, 1)$, $(-2, 2)$, $(2, 2)$, and $(1, 1)$. Thus

$$\iint_R \cos\frac{y - x}{y + x}\, dA = \int_1^2 \int_{-v}^{v} \cos\frac{u}{v} \left|-\frac{1}{2}\right| du\, dv = \frac{1}{2}\int_1^2 \left[v \sin\frac{u}{v}\right]_{u=-v}^{u=v} dv = \frac{1}{2}\int_1^2 2v\sin(1)\, dv = \tfrac{3}{2}\sin 1$$

**22.** Letting $u = 3x$, $v = 2y$, we have $9x^2 + 4y^2 = u^2 + v^2$, $x = \frac{1}{3}u$, and $y = \frac{1}{2}v$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{6}$ and $R$ is the image of the quarter-disk $D$ given by $u^2 + v^2 \le 1$, $u \ge 0$, $v \ge 0$. Thus

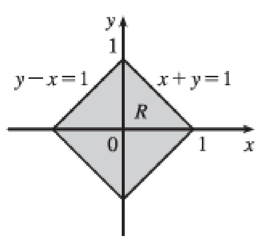
$$\iint_R \sin(9x^2 + 4y^2)\, dA = \iint_D \tfrac{1}{6}\sin(u^2 + v^2)\, du\, dv = \int_0^{\pi/2}\int_0^1 \tfrac{1}{6}\sin(r^2)\, r\, dr\, d\theta = \tfrac{\pi}{12}\left[-\tfrac{1}{2}\cos r^2\right]_0^1 = \tfrac{\pi}{24}(1 - \cos 1)$$

**23.** Let $u = x + y$ and $v = -x + y$. Then $u + v = 2y \Rightarrow y = \frac{1}{2}(u + v)$ and $u - v = 2x \Rightarrow x = \frac{1}{2}(u - v)$.

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$. Now $|u| = |x + y| \le |x| + |y| \le 1 \Rightarrow -1 \le u \le 1$, and

$|v| = |-x + y| \le |x| + |y| \le 1 \Rightarrow -1 \le v \le 1$. $R$ is the image of the square region with vertices $(1, 1)$, $(1, -1)$, $(-1, -1)$, and $(-1, 1)$.

So $\iint_R e^{x+y}\, dA = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^u\, du\, dv = \frac{1}{2}\left[e^u\right]_{-1}^{1}\left[v\right]_{-1}^{1} = e - e^{-1}$.




**24.** Let $u = x + y$ and $v = y$, then $x = u - v$, $y = v$, $\dfrac{\partial(x,y)}{\partial(u,v)} = 1$ and $R$ is the image under $T$ of the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$. Thus

$$\iint_R f(x + y)\, dA = \int_0^1\int_0^u (1)\, f(u)\, dv\, du = \int_0^1 f(u)\left[\, v\, \right]_{v=0}^{v=u} du = \int_0^1 u f(u)\, du \quad \text{as desired.}$$