Advanced Algebra (1) class note

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Tue. 16th Sep.

3.1 Rings and Homomorphism

Def

A set R is a <u>*ring*</u> if R has two elements 0, 1 and two binary operations $+$, $\cdot$ on R such that
(1a). $(a+b)+c = a+(b+c)$
(1b). $a+b = b+a$
(1c). $a+0 = 0+a = a$
(1d). there exists $-a \in R$ with $a+(-a) = 0 = (-a)+a$
(2a). $(a+\cdot b)\cdot c = a\cdot(b\cdot c)$
(2b). $a\cdot 1 = a = 1\cdot a$
(3a). $a\cdot(b+c) = a\cdot b + a\cdot c$
$a\cdot(b+c) = a\cdot b + a\cdot c$
for any $a, b, c \in R$.

ex

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are rings with usual $0, 1$ and usual $+, \cdot$.

Notation

(1). Write ab for $a\cdot b$, $na = a+a+\cdots+a$ n times, and $a-b = a+(-b)$ for $a, b \in R$ and $n \in \mathbb{N}$.
(2). If $ab = ba$ for all $a, b \in R$, then R is said to be <u>*commutative*</u>.
(3). $C(R) := \{a \in R \mid ab = ba \text{ for all } b \in R\}$ is called the <u>*center*</u> of R.

ex

Let $M_n(\mathbb{R})$ be a set of $n\times n$ matrices over $\mathbb{R}$.
Then $M_n(\mathbb{R})$ is a ring under usual matrix operations $+, \cdot$.

ex

Let $GL_n(\mathbb{R})$ be the set of all $n\times n$ invertible matrices under usual $+, \cdot$.
$GL_n(\mathbb{R})$ is not a ring since $O \notin GL_n(\mathbb{R})$.

ex

$i = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$, $j = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $k = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \in M_2(\mathbb{C})$.
You can check $i^2 = j^2 = k^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $ij = k = -ji, jk = i = -kj, ki = j = -ik$, and $Q := \{aI+bi+cj+dk \mid a,b,c,d \in \mathbb{R}\}$ is a ring (subring of $M_2(\mathbb{C})$), called real quarterian.

Note

$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{R} \right\} \subseteq C(Q)$ and Q is not commutative.

ex

$H = \left\{ aI + bi + cj + dk + e\left(\frac{I}{2} + \frac{i}{2} + \frac{j}{2} + \frac{k}{2}\right) \mid a,b,c,d,e \in \mathbb{Z} \right\}$ is a ring.
Check $\left(\frac{1}{2} + \frac{i}{2} + \frac{j}{2} + \frac{k}{2}\right)\left(\frac{1}{2} - \frac{i}{2} - \frac{j}{2} - \frac{k}{2}\right)$
$= \frac{1}{4}(1+i+j+k)(1-i-j-k)$
$= \frac{1}{4}(1+1+1+1) = 1 \in H.$
H is called Herwitz ring.

Def

If for any nonzero $a \in R$, there exists an element $a^{-1} \in R$ s.t. $aa^{-1} = 1 = a^{-1}a$, then R is said to be a $\underline{divisionring}$.
A field is a commutatinve division ring.

ex

Q is a division ring, and $\mathbb{Q}$ is a field.