3.2 Ideals

Definition: $I \subseteq R$ is called a left ideal (resp. a right ideal) if

- $a - b \in I$
- $ra \in I$ (resp. $ar \in I$)

for $a, b \in I$ and $r \in R$. An ideal is a left ideal and a right ideal.

Example: $\{\begin{pmatrix} 0 & \dots & 0 & a_{1k} & 0 & \dots & 0 \\ 0 & \dots & 0 & a_{2k} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & a_{nk} & 0 & \dots & 0 \end{pmatrix} \mid a_{ik} \in R\} \subseteq M_n(R)$ is a left ideal, but not a right ideal.

Theorem: Let I be an ideal of R. Set $R/I := \{a + I \mid a \in R\}$, where $a + I := \{a + b \mid b \in I\}$. Define $+, \cdot$ on R/I by $(a+I)+(b+I) = a+b+I$ and $(a+I)\cdot(b+I) = ab+I$ for $a, b \in R$. Then R/I is a ring. (The quotient ring of R by I)

Proof. It is straight forward to check every axiom of ring eacept that $+, \cdot$ are well-defined. Suppose $a+I = a'+I$ and $b+I = b'+I$ for $a, b, a', b' \in R$. Then $(a+b)-(a'+b') = (a-a')+(b-b') \in I$ and $ab - a'b' = ab - ab' + ab' - a'b' = a(b-b') + (a-a')b \in I$. Hence $a+b+I = a'+b'+I$ and $ab+I = a'b'+I$. $\square$