Advanced Algebra 1 Class Note

3.4 Ring of Quotients and Localization (cont.)

We study the ideals in R and ideals in $S^{-1}R$.

Example

$R = \mathbb{Z}$.

$S = \{3^n \mid n \in \mathbb{N} \cup \{0\}\}$ is multiplicative colsed.

$S^{-1}R = \left\{ \frac{m}{3^n} \mid m \in \mathbb{Z}, n \in \mathbb{N} \right\}$.

$2\mathbb{Z}$(=(2) in $\mathbb{Z}$) is an ideal in $\mathbb{Z}$.

$2S^{-1}\mathbb{Z}$(=(2) in $S^{-1}\mathbb{Z}$) is an ideal in $S^{-1}\mathbb{Z}$.

Note $6S^{-1}\mathbb{Z} = 2S^{-1}\mathbb{Z}$.

Theorem

Let R be a ring and $S \subseteq R$ be multiplicative closed.

Set $\mathbb{U} = \{P \mid P \text{ is a prime ideal in } R \text{ with } P \cap S = \emptyset\}$.

$\mathbb{V} = \{P' \mid P' \text{ is a prime ideal in } S^{-1}R\}$.

Then there exists a bijection

$$f : \mathbb{U} \longmapsto \mathbb{V}$$

$$\begin{aligned} \text{defined by } f(P) &= PS^{-1}R \\ &:= \{ab \mid a \in P, b \in S^{-1}R\} \text{ for } P \in \mathbb{U}. \end{aligned}$$

Proof

We need to check f is well-defined (i.e. $PS^{-1}R \in \mathbb{V}$), 1-1, and onto.

We check f is 1-1 only.

Suppose $f(P) = f(Q)$ for some $P, Q \in \mathbb{U}$.

It suffices to show $P \subseteq Q$.

Pick $b \in P$.

Then $\frac{b}{1} \in f(P) = f(Q)$.

Hence $\frac{b}{1} = \frac{d}{c}$ for some $d \in Q$, $c \in S$.

Thus $(bc - d)\,\rho = 0$ for some $\rho \in S$.

Then $bc\rho = d\rho \in Q$. (Since $d \in Q$)

Note $c\rho \in S$.

Hence $c\rho \notin Q$.

Since Q is a prime, $b \in Q$.

(4) in $\mathbb{Z}$.

(4) in $\mathbb{S}^{-\nVdash}\mathbb{Z}$.

Question

Can you extend both $\mathbb{U}$ and $\mathbb{V}$ in the above theorem?

Definition

Let P be a prime ideal in R.

Then $R_P := (R - P)^{-1}\, R$ is called the localization of R at P.

Example

$R = \mathbb{Z}$.

$3\mathbb{Z}$ is a prime ideal in $\mathbb{Z}$.

$$R_{3\mathbb{Z}} := (R - 3\mathbb{Z})^{-1}\, R$$
$$= \left\{\frac{m}{n} \mid m, n \in \mathbb{Z}, 3 \nmid n\right\} \text{ (localization of } R \text{ at } 3\mathbb{Z}).$$

$\Rightarrow 3R_{3\mathbb{Z}} = \{\frac{n}{m} \mid m, n \in \mathbb{Z}, 3 \mid m, 3 \nmid n\}$ is a unique maximal ideal.

Note all proper ideals of R are contained in $\{\frac{3n}{m} \mid m, n \in \mathbb{Z}, 3 \mid m\}$.

Proof

(Maximal)

$3R_{3\mathbb{Z}}$ is clear an ideal and $3R_{3\mathbb{Z}} \neq R_{3\mathbb{Z}}$.

Suppose $3R_{3\mathbb{Z}} \subsetneq M$, an ideal.

Pick $\dfrac{m}{n} \in M - 3R_{3\mathbb{Z}}$, where $3 \nmid m$ and $3 \nmid n$.

Then $m = \dfrac{m}{n} \times n \in M$.

Hence $1 = am + 3b\uplus \in M$, for some $a, b \in \mathbb{Z}$.

(Since $m \in M$ and $3 \in 3R_{3\mathbb{Z}} \subseteq M$.)

Thus $M = R$, and $3R_{3\mathbb{Z}}$ is a maximal ideal.

(Unique)

Let M_1 be another maximal ideal of R.

If $\dfrac{m}{n} \in M_1 - 3R_{3\mathbb{Z}}$, then $M_1 = R$ by previous argument. $\rightarrow\leftarrow$

Hence $M_1 \subseteq 3R_{3\mathbb{Z}}$.

Thus $M_1 = 3R_{3\mathbb{Z}}$.

Definition

R is a <u>local ring</u> if R has a unique maximal ideal.

Example

A field is a local ring since (0) is the unique maximal ideal.

Example

$\mathbb{Z}_{p^n}$ is a local ring since $p\mathbb{Z}_{p^n}$ is the unique maximal ideal.

Theorem (TFAE)

(1). R is a local ring.

(2). The set of nonunits is contained in a maximal ideal.

(3). The set of nonunits is a maximal ideal.

Proof

(3) $\Rightarrow$ (2) is clear.

(3) $\Rightarrow$ (1)

Note every ideal with a unit is R.

Hence the set of nonunits is the unique maximal ideal.

(2) $\Rightarrow$ (1) is similar to (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3)

Let M be the unique maximal ideal.

Then M is contained in the set of nonunits, since $M \neq R$.

Pick a nonunit $b \in R$.

Then (b) is an ideal and $(b) \neq R$, since b is nonunit.

Then $(b) \subseteq M$, since M is unique maximal ideal.

Hence $b \in M$.

Thus M is the set of all nonunits.