Theorem 1 Let $R$ be a ring, not necessary commutative. Set

\[ R[x] := \left\{ \sum a_i x^i \mid a_i \in R, a_i = 0 \text{ for all but a finite number of } i, i = 0, 1, \ldots \right\}. \]

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For $\sum a_ix^i, \sum b_ix^i \in R[[x]]$, define: 1. $\sum a_ix^i + \sum b_ix^i = \sum (a_i + b_i)x^i$,

and 2. $\left( \sum a_ix^i \right) \cdot \left( \sum b_ix^i \right) = \sum \left( \sum a_kb_l \right) x^i$.

Then (1) $R[x] \subseteq R[[x]]$ are rings,

(2) If $R$ has no zero divisor, then $R[[x]]$ has no zero divisor, and

(3) The map $R \to R[[x]]$ by $r \mapsto r + 0x + 0x^2 + \cdots$ is an injective homomorphism.

Proof. Clearly. \(\square\)

Notation.

1. For $r \in R$, we write $r$ for $r + 0x + 0x^2 + \cdots$ in $R[[x]]$.

2. We use $x^i$ for $0 + 0x + \cdots + 0x^{i-1} + x + 0x^{i+1} + \cdots$ in $R[[x]]$.

3. For each $f(x) \in R[x], f(x) \neq 0$, there exists $n \in \mathbb{N} \cup \{0\}$ s.t. $f(x) = a_0 + a_1x + \cdots + a_nx^n$. The integer $n$ is called the degree of $f(x)$ and $a_n$ is the leading coefficient.

Note. (1) $rx = xr$, for $r \in R, x \in R[[x]]$.

(2) If $R$ is a field, then $R[x]$ is an Euclidean domain with $\mu(f) = \text{deg}(f)$, for $0 \neq f \in R[x]$.

Theorem 2 Let $F$ be a field. Then $F[[x]]$ is a local ring.

Proof. We claim $(x)$ is the unique maximal ideal.

(1) Clearly, $(x) \neq F[[x]]$ since $1 \notin (x)$. And it’s easy to check that $(x)$ is a maximal ideal.

(2) It suffices to show that $(x)$ contains all nonunits. Pick $\sum a_ix^i \in F[[x]] - (x)$, note that $x_0 \neq 0$. Set $b_0 = a_0^{-1}$ and $b_i = -a_0^{-1} \cdot (a_1b_{i-1} + a_2b_{i-2} + \cdots + a_ib_0)$ for $i \geq 1$. Hence $\left( \sum a_ix^i \right) \cdot \left( \sum b_ix^i \right) = 1$. Thus all elements not in $(x)$ are units. Then by previous theorem, $(x)$ is the unique maximal ideal. And hence we are done. \(\square\)
Note. (1) The set of units in $F[x]$ is $F - \{0\}$.
   (2) $F[x]$ is $ED$, and hence $PID$ and $UFD$.

Definition. $R[x, y] := R[x][y]$, where $R$ is a commutative ring.

Note. (1) $R[x, y] = \left\{ \sum a_{ij}x^iy^j \mid a_{ij} \in R \right\}$.
   (2) $F[x, y]$ is not $PID$. Since $(x, y)$ is not generated by a single element.
   (3) $(x)$ is a prime ideal in $F[x, y]$, but not a maximal ideal. Since $(x) \subset (x, y)$. 