§ 3.5 Rings of Polynomials and Formal Power Series

Theorem 1 *Let R be a ring, not necessary commutative. Set*
$R[x] := \left\{ \sum a_i x^i \mid a_i \in R, a_i = 0 \ for \ all \ but \ a \ finite \ number \ of \ i, i = 0, 1, \ldots \right\}$.
$R[[x]] := \left\{ \sum a_i x^i \mid a_i \in R, i = 0, 1, \ldots \right\}$.
For $\sum a_i x^i, \sum b_i x^i \in R[[x]]$, *define: 1.* $\sum a_i x^i + \sum b_i x^i = \sum (a_i + b_i) x^i$,
and 2. $\left(\sum a_i x^i\right) \cdot \left(\sum b_i x^i\right) = \sum \left(\sum_{k+l=i} a_k b_l \right) x^i$.

Then (1) $R[x] \subseteq R[[x]]$ are rings,

(2) If R has no zero divisor, then $R[[x]]$ has no zero divisor, and

(3) The map $R \to R[[x]]$ by $r \mapsto r + 0x + 0x^2 + \cdots$ is an injective homomorphism.

Proof. Clearly. □

Notation.

1. For $r \in R$, we write r for $r + 0x + 0x^2 + \cdots$ in $R[[x]]$.
2. We use x^i for $0 + 0x + \cdots + 0x^{i-1} + x + 0x^{i+1} + \cdots$ in $R[[x]]$.
3. For each $f(x) \in R[x]$, $f(x) \neq 0$, there exists $n \in \mathbb{N} \bigcup \{0\}$ s.t. $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. The integer n is called the degree of $f(x)$ and a_n is the leading coefficient.

Note. (1) $rx = xr$, for $r \in R$, $x \in R[[x]]$.
(2) If R is a field, then $R[x]$ is an $Euclideandomain$ with $\mu(f) = deg(f)$, for $0 \neq f \in R[x]$.

Theorem 2 *Let F be a field. Then $F[[x]]$ is a local ring.*

Proof. We claim (x) is the unique maximal ideal.
(1) Clearly, $(x) \neq F[[x]]$ since $1 \notin (x)$. And it's easy to check that (x) is a maximal ideal.
(2) It suffices to show that (x) contains all nonunits. Pick $\sum a_i x^i \in F[[x]] - (x)$, note that $x_0 \neq 0$. Set $b_0 = a_0^{-1}$ and $b_i = -a_0^{-1} \cdot (a_1 b_{i-1} + a_2 b_{i-2} + \ldots + a_i b_0)$ for $i \geq 1$. Hence $\left(\sum a_i x^i\right) \cdot \left(\sum b_i x^i\right) = 1$. Thus all elements not in (x) are units. Then by previous theorem, (x) is the unique maximal ideal. And hence we are done. □

Note. (1) The set of units in $F[x]$ is $F - \{0\}$.
(2) $F[x]$ is ED, and hence PID and UFD.

Definition. $R[x, y] := R[x][y]$, where R is a commutative ring.

Note. (1) $R[x, y] = \left\{ \sum a_{ij} x^i y^j \mid a_{ij} \in R \right\}$.
(2) $F[x, y]$ is not PID. Since (x, y) is not generated by a single element.
(3) (x) is a prime ideal in $F[x, y]$, but not a maximal ideal. Since $(x) \subset (x, y)$.