Algebra

2008.10.2 to 2008.10.9

3.4 RINGS OF QUOTIENTS AND LOCALIZATION

We still assume R is commutative.

Definition 1. *A nonempty subset S of a ring R is* ***multiplicative closed*** *if*

(1)$1 \in S$

(2)$a, b \in S \quad \Rightarrow \quad ab \in S$

Example.

Let $P \neq R$ be prime ideal of R. Then $S = R - P$ is muliplicative closed.

Example.

Let $\{P_i \big| i \in Z_+\}$ be a set of proper prime ideals of R. Then $S = R - \bigcup_{i \in Z_+} P_i$ is multiplicative closed.

Proof. Pick $a, b \in S$. Then $a, b \notin P_i$ for each i. Hence $ab \notin P_i$ for each i. Thus $ab \notin S$. □

Example.

$S = \{a \in R \big| ab \neq 0$ for all $b \neq 0$ in $R\}$ is multiplicative closed.

Note. If $a \in R - P$ is a zero divisor, then $ab = 0$ for some $b \in P$.

Definition 2. *Let $S \subseteq R$ be multiplicative closed. Let $\sim$ be a relation on $R \times S$ by*

$$(a, b) \sim (c, d) \Leftrightarrow (ad - bc)\rho = 0 \textit{ for some } \rho \in S$$

Lemma 1. $\sim$ *is an equivalence relation.*

Proof. check

(1)$(a,b) \sim (a,b)$ (reflexive)

(2)$(a,b) \sim (c,d) \Leftrightarrow (c,d) \sim (a,b)$ (symmetric)

(3)$(a,b) \sim (c,d)$ and $(c,d) \sim (e,f) \Rightarrow (a,b) \sim (e,f)$ (transitive)

We prove "transitive" for example.

Suppose $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$.

Then $(ad - bc)\rho = 0$ and $(cf - de)t = 0$ for some ρ,t $\in$ S.

Hence $ad\rho = bc\rho$ and $cft = det$.

Thus $ad\rho ft = bc\rho ft = b\rho det$.

Then $(af - be)d\rho t = 0$ where $d, \rho, t \in S$ and hence $d\rho t \in S$.

Thus $(a,b) \sim (e,f)$. $\square$

Definition 3. *For* $(a,b) \in R \times S$. *Let*

$$\overline{(a,b)} := \{(c,d) \in R \times S \mid (c,d) \sim (a,b)\}$$

Notation.

(1)We use $\frac{a}{b}$ for $\overline{(a,b)}$.

(2)$S^{-1}R := \{\overline{(a,b)} \mid a \in R, b \in S\}$

Lemma 2. *Define* $+$ *and* $\cdot$ *on* $s^{-1}R$ *by*

$$\overline{(a,b)} + \overline{(c,d)} = \overline{(ad+bc, bd)} \text{ and } \overline{(a,b)} \cdot \overline{(c,d)} = \overline{(ac, bd)}.$$

Then $S^{-1}R$ *is a ring with identity* $1 = \overline{(a,a)}$ *and* $0 = \overline{(0,a)}$ *for* $a \in S$

Proof. This is straight forward.

(Don't forget to check well-defined) $\square$

Lemma 3. *Suppose* $S \subseteq \{a \in R \mid a$ *is not a zero divisor* $\}$. *Then the map* $i_s : R \to S^{-1}R$ *defined by* $i_s(a) = \overline{(a,1)}$ *is an injective homomorphism.*

Proof. It is easy to check i_s is homomorphism. To check injective, it suffices to check $ker(i_s) = \{0\}$. Pick $a \in ker(i_s)$. Then $i_s(a) = \overline{(a,1)} = \overline{(0,1)}$. Hence $(a \cdot 1 - 1 \cdot 0)\rho = 0$ for some $\rho \in S$. Thus $a = 0$ by definition of S. $\square$

Lemma 4. *Let R be an integral domain and $S = R - \{0\}$. Then $S^{-1}R$ is a field, called the quotient field of R.*

Proof. Pick $\overline{(a,b)} \in S^{-1}R$ with $\overline{(a,b)} \neq 0$. In particular, $0 \neq a \in S$. Then $\overline{(b,a)} \in S^{-1}R$ and $\overline{(a,b)} \cdot \overline{(b,a)} = \overline{(ab,ab)} = 1$ $\square$

Definition 4. *If $S = \{a \mid a \text{ is not a zero divisor }\}$, then $S^{-1}R$ is called the **complete ring of quotient**.*

Theorem 1. *(Universal Theorem) Let $f : R \to R'$ be a homomorphism of rings and $S \subseteq R$ be multiplicative closed such that $f(a)$ is a unit for $a \in S$. Then there exists a unique map $\hat{f} : S^{-1}R \to R'$ such that $\hat{f}(\overline{(a,1)}) = f(a)$ for $a \in R$. i.e. $\hat{f} \circ i_s = f$.*

Proof. Define $\hat{f} : S^{-1}R \to R'$ by $\hat{f}(\overline{(a,b)}) = f(a)f(b)^{-1}$. It is straight forward to check that $\hat{f}$ is well-defined, homomorphism and $\hat{f} \circ i_s = f$. To prove uniqueness, suppose $\hat{f}' : S^{-1}R \to R'$ such that $\hat{f}' \circ i_s = f$. Note $\hat{f}'(\overline{(1,b)}) \cdot \hat{f}'(\overline{(b,1)}) = \hat{f}'(\overline{(b,b)}) = 1$. Hence $\hat{f}'(\overline{(1,b)}) = (\hat{f}(\overline{(b,1)}))^{-1} = f(b)^{-1}$. Then $\hat{f}'(\overline{(a,b)}) = \hat{f}'(\overline{(a,1)}, \overline{(1,b)}) = f(a) \cdot f(b)^{-1} = \hat{f}(\overline{(a,b)})$ $\square$