Algebra

10.30,11.6

Theorem. *R is U.F.D.* $\Rightarrow$ *R[x] is U.F.D..*

Corollary. *R[x][y]=R[x,y] is UFD.*

proof. Let F be the field of quotient of R.

Note $F[x]$ is U.F.D.

Pick $f(x) \in R[x]$. We can assume $degf(x) \geq 1$. Then $f(x) = c(f)f_1(x)$ where $f_1(x) \in R[x]$ is primitive.

Note $f_1(x) = \frac{b}{a}h_1(x)h_2(x)...h_k(x)$ where $h_i(x) \in R[x]$ are irreducible primitive, by the UFD of $F[x]$.

Then $a = c(af_1(x)) = c(bh_1(x)...h_2(x)) = b$.

Hence we assume $f(x) = c(f)h_1(x)h_2(x)...h_k(x)$.

Since $C(f) \in R$, suppose $C(f) = c_1c_2...c_n$, for some irreducible elements $c_i \in R$.

Then $f(x) = c_1c_2...c_nh_1(x)h_2(x)...h_k(x)$ is a product of irreducible elements in $R[x]$.

(Uniqueness)

Suppose $c_1c_2...c_nh_1(x)h_2(x)...h_k(x) = d_1d_2...d_mh'_1(x)h'_2(x)...h'_s(x)$

where $c_i, d_i \in R$ and $h_i(x), h'_i(x) \in R[x]$ have degree ≥ 1,all of them irreducible in $R[x]$.

we assume $h_i(x), h'_i(x)$ are primitive, then

$c_1c_2...c_n = C(c_1c_2...c_nh_1(x)h_2(x)...h_k(x)) = C(d_1d_2...d_mh'_1(x)h'_2(x)...h'_k(x)) = d_1d_2...d_m$

Hence $n = m$ and there exists a bijection σ on $1, 2, ..., k$, s.t $c_i = d_{\sigma(i)}$.
Also we have $h_1(x)h_2(x)...h_k(x) = h'_1(x)h'_2(x)...h'_k(x)$ viewing they are in $F[x]$ and by UFD of $F[x]$, we have $k = s$ and there exists a bijection on $1, 2, ..., k$ such that $h_i(x), h'_i(x)$ are associates in $F[x]$, and then are associates in $R[x]$. $\square$

Theorem. *(Eisenstein's Criterion)*
Let R be a UFD and F its quotient field.
Let $f(x) = \sum_{i=1}^{n} a_i x^i \in R[x]$ have degree ≥ 1.
Let $P \in R$ be an irreducible element s.t $p|a_i$ for all $i \leq n-1$, $p \nmid a_n$ and $p^2 \nmid a_0$.
Then $f(x)$ is irreducible in $F[x]$

ex. *$R = \mathbb{Z}$ and $F = \mathbb{Q}$,*
$f(x) = 2x^2 + 6x + 6 \in Z[x]$
Pick $p = 3$, then $3|6 = a_0 = a_1$,$3 \nmid 2 = a_2$ and $3^2 \nmid 6 = a_0$
Hence $2x^2 + 6x + 6$ is irreducible in $Q[x]$.
Note: $2x^2 + 6x + 6 = 2(x^2 + 3x + 3)$ and 2 is not unit.

proof. Since the content $c(f)$ is a unit in F, we can assume f is primitive in $R[x]$. It suffices to show f is irreducible in $R[x]$ by previous lemma. Suppose $\sum_{i=0}^{n} a_i x^i = \sum_{i=1}^{m} b_i x^i \sum_{i=1}^{k} c_i x^i$. Since $p|a_0 = b_0c_0$ and $p^2 \nmid a_0 = b_0c_0$. we can assume $p|b_0$ and $p \nmid c_0$. (The other case $p|c_0$, $p \nmid b_0$ is similar.) Since $p \nmid a_n, p \nmid b_i$ for some i. Let s be the integer s.t $p|b_i$ for $i < s$ and $p \nmid b_s$. Note $s \leqq m < n$. Then $p|a_s = b_0c_s + b_1c_{s-1} + ... + b_{s-1}c_1 + b_sc_0$, and hence $p|b_sc_0$, a contradiction. $\square$