4. EXACT SEQUENCE

We always assume that R is commutative.

Definition. A sequence of R-homomorphisms

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} M_3 \cdots \xrightarrow{f_n} M_n$$

is **exact** if $img\ f_i = ker\ f_{i+1}$ for $1 \leq i \leq n-1$.

Note:

1. $f_{i+1}f_i = 0$.
2. Usually we write a sequence of at least 2 homomorphisms in a horizontal or vertical diagram to mean exact e.g.

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \quad \text{or} \quad \begin{array}{c} M_1 \\ \downarrow f \\ M_2 \\ \downarrow g \\ M_3 \end{array}$$

3. If the homomorphisms are canonicalor not important, they are omitted.
4. $0 \to N \xrightarrow{f} M \Leftrightarrow$ f is 1-1.
5. $N \xrightarrow{f} M \to 0 \Leftrightarrow$ f is onto.
6. $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \Rightarrow 0 \to M/_{ker\ f} \xrightarrow{f} M_2 \xrightarrow{g} img\ g \to 0$

Example. $0 \to M \to M \oplus N \to N \to 0$.

Example. $N \subseteq M \Rightarrow 0 \to N \to M \to M/_N \to 0$.

Definition. The diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ h\downarrow & & \downarrow g \\ M_3 & \xrightarrow{k} & M_4 \end{array}$$

of homomorphisms is commutative if $g \circ f = k \circ h$.

Note: Usually we draw

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ h\downarrow & \circlearrowright & \downarrow g \\ M_3 & \xrightarrow{k} & M_4 \end{array}$$

to mean the commutative.