1. *Recall* : (The Short Five Lemma) Suppose we have the following commutative diagram of R-module homomorphisms with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "f_1"'] \arrow[dr, phantom, "\circlearrowleft"] & M_2 \arrow[r] \arrow[d, "f_2"'] \arrow[dr, phantom, "\circlearrowleft"] & M_3 \arrow[r] \arrow[d, "f_3"'] & 0 \\ 0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & 0 \end{tikzcd}$$

   (a) f_1 surjection and f_2, f_4 injection $\Longrightarrow$ f_3 injection.

   (b) f_5 injection and f_2, f_4 surjection $\Longrightarrow$ f_3 surjection.

2. *Definition* : We say the two exact sequence in the above diagram are isomorphic if f_1 , f_2 and f_3 are isomorphism

3. *Theorem* : suppose

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

   Then the following are equivalent

   (a) There exist $h : M_2 \longrightarrow M$ s.t. $gh = \iota_{M_2}$

   (b) There exist $k : M \longrightarrow M_1$ s.t. $kf = \iota_{M_1}$

   (c) $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ and $0 \longrightarrow M_1 \longrightarrow M_1 \oplus M_2 \longrightarrow M_2 \longrightarrow 0$ are isomorphic
   In particular $M \cong M_1 \oplus M_2$

   Proof:(b)$\Longrightarrow$(c)consider

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "f"] \arrow[d, "id"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r, "g"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & M_2 \arrow[r] \arrow[d, "id"'] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M_1 \oplus M_2 \arrow[r] & M_2 \arrow[r] & 0 \end{tikzcd}$$

Define $\phi : M \to M_1 \oplus M_2$ by $\phi(m) = (k(m), g(m))$ for $m \in M$

It is clear that the diagram is commutative.

(c)$\Rightarrow$(b) and (a)$\Leftrightarrow$(c) are Exercise.