Algebra

2008.12.11

4.1 EXACT SEQUENCES

Lemma 1. (*Short Five Lemma*) *Suppose*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{g} & M_3 & \longrightarrow & 0 \\
 & & \alpha\Big\downarrow & \circlearrowleft & \beta\Big\downarrow & \circlearrowleft & \gamma\Big\downarrow & & \\
0 & \longrightarrow & N_1 & \xrightarrow{f'} & N_2 & \xrightarrow{g'} & N_3 & \longrightarrow & 0
\end{array} \; .$$

Then

(*i*) α, γ *monomorphisms* $\Rightarrow \beta$ *is a monomorphism;*

(*ii*) α, γ *epimorphisms* $\Rightarrow \beta$ *is a epimorphism;*

Proof. We prove (i) here, (ii) is left as homework. Pick $n_2 \in N_2$ and pick $m_3 \in M_3$ such that $\gamma(m_3) = g'(n_2)$. Since γ is onto, such element m_3 exists. Pick $m_2 \in M_2$ such that $g(m_2) = m_3$, because g is onto, m_2 exists. We have m_2, n_2, then we can derive something from here, that is

$$\begin{aligned}
& g'(\beta(m_2) - n_2) \\
= & \gamma \circ g(m_2) - g'(n_2) \\
= & \gamma \circ g(m_2) - \gamma(m_3) \\
= & \gamma(m_3) - \gamma(m_3) \\
= & 0 \, .
\end{aligned}$$

From above, we know that $\beta(m_2) - n_2 \in \ker g^{'} = \mathrm{Im} f^{'}$. Hence we can choose $n_1 \in N_1$ such that $f^{'}(n_1) = \beta(m_2) - n_2$. Pick $m_1 \in M_1$ such that $\alpha(m_1) = n_1$ (α is onto). Then $\beta(m_2) - n_2 = f^{'}(n_1) = f^{'} \circ \alpha(m_1) = \beta \circ f(m_1)$. Hence $\beta(m_2 - f(m_1)) = n_2$. This shows that for each element y of N_2, there is an element x of M_2 such that $y = \beta(x)$. $\square$